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A BRIEF REVIEW OF THE GROUP THEORETIC APPROACH TO STRING THEORY

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A B S T R A C T

A short review of the group theoretic approach to string theory is given. In particular, it is explained how the method provides a powerful tool with which to compute perturbative string effects.

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1. INTRODUCTION

Despite the evident promise that string theories would provide a framework in which all interactions would ultimately become unified in a consistent way¹⁾, it was also manifest from the earliest days that our basic understanding of these theories was far from sufficient to enable us to decide on such an issue. To remedy this situation, covariant second quantized string field theories²⁾⁻⁵⁾ were developed, in order to apply the usual powerful methods of ordinary Lagrangian theory of gauge fields. A Lagrangian theory can potentially address non-perturbative questions like the choice of the vacuum, and perturbative non-renormalization and finiteness theorems. Covariant string field theory has enabled us to clarify the gauge structure responsible for the absence of string states of negative norm and has in particular revealed the importance of the first quantized BRST operator Q in the second quantized BRST theory⁶⁾ and more surprisingly, the second quantized gauge invariant theory $\langle \chi | Q | \chi \rangle$ ^{4,7)}. However, it has not yet enabled us to solve the non-perturbative problems mentioned above. Although perturbation theory results, which were in essence known almost since 1971⁸⁾, can be recovered from string field theory it is a rather cumbersome process. This may be only due to our present technical inability to handle the rather complicated form of the interaction terms. It is also conceivable that this approach has been too conservative in copying too closely the successful second quantized Yang-Mills theories, and that some more basically string-like features, and deeper mathematical concepts remain to be discovered and combined to efficiently address non-perturbative questions.

To this aim, two new approaches to string theory have been developed in the last year. One goes under the name "group theoretic approach to string theory"⁹⁾⁻¹⁷⁾. The other is based on the mathematics of Grassmannians¹⁹⁾⁻²³⁾.

The group theoretic approach was first discovered within the context of gauge covariant string theory²⁾⁻⁵⁾, which, at least in Refs. 2) and 5), was constructed using this method. It was realized, however, that the techniques used were not restricted to gauge covariant string theory and could be used to provide a powerful method of computing string perturbation theory. Indeed, the group theoretic method generalizes the specifically string-like features of the known string perturbative results and it possesses many of the advantages, although it overcomes the disadvantages of the old dual model⁸⁾.

In the old dual model, one found oscillator vertices which, when integrated over their dependence on Koba-Nielsen co-ordinates²⁴⁾ and moduli²⁵⁾ gave the scattering of arbitrary string states. At tree level, such vertices were deduced^{26),27)} by factorizing the well-known formula for the scattering of N tachyons²⁸⁾. By sewing such tree vertices one was able to deduce vertices

and their integration measure for multiloop string scattering²⁹⁾. Although not technically correct above one loop, due to the presence of negative norm propagating in the loops, these scattering amplitudes had many of the correct generic features. The one-loop exception was due to the ability to insert the Brink-Olive projector³⁰⁾.

However, what was not shown for these vertices was that they decoupled zero norm physical states. In QED, these states are of the form $\epsilon^\mu = k^\mu \epsilon(k)$, while in string theory they are $L_{-1}|\Omega\rangle$, $(L_{-2} + 3/2 L_{-1}^2)|\Omega\rangle$ ³¹⁾, etc., or, in BRST language, $Q|\Lambda\rangle$ ³²⁾. The decoupling of these states requires a knowledge of how string vertices behaved under the conformal group. Of course, it was known almost from the beginning of string theory that the operators, say in the bosonic string, corresponding to the scattering of two arbitrary states and one fixed string state, were operators of conformal weight 1, for example, the Fubini-Veneziano operator e^{ikQ} for tachyon emission³³⁾. (The corresponding fermion emission vertex in string theory had its superconformal properties computed in Ref. 34). The three-string vertex, however, summarized in one expression all such vertices and only in the paper of Ref. 35) were its conformal identities for the special case of $L_0, L_{\pm 1}$ discussed.

In the group theoretic approach⁹⁾⁻¹⁸⁾, the central objects are string vertices which depend on Koba-Nielsen variables²⁵⁾ and moduli²⁶⁾. The objects are in principle defined in the abstract, but one convenient representation is found by using oscillators. The scattering amplitude is found by integrating the vertex over the variables upon which it depends with an appropriate measure saturating the result against the desired physical states. The vertices satisfy certain equations called overlap equations that in particular determine them in a rather straightforward way.

These equations relate via conformal transformations the action of any conformal operator acting on one external string state to its action on the other external string states. The measure is determined by demanding that zero-norm physical states decouple in the amplitude. This method of computing string scattering vertices reduces to an exercise in conformal group theory and it allows one to deduce general properties of amplitudes such as their conformal properties without having to know their explicit form nor perform any tedious algebra.

The general understanding of the conformal properties of string vertices came from the group theoretic approach to string theory. Taking the overlap equation which involves the "conformal operator" $L(z) = \sum_n L_{-n} z^n$, where L_n are the Virasoro operators, one finds conformal transformations which preserve the string vertices. The remaining transformation lead to changes in the

Koba-Nielsen variables and a moduli of the vertex. What change is induced by what transformation is found by also examining the effect of such changes on the overlap equations. The latter are essential for the determination of the measure which results of ensuring that zero-norm physical states decouple. Indeed one finds that the measure obeys a set of first-order differential equations which determine it.

One can implement the group theoretic method with or without the addition of the ghost oscillators and the associated formalism. One advantage of their inclusion is the simplicity of the zero-norm physical states.

In an independent development, it was realized by Belinson³⁶⁾ as a result of a generalization of a theorem of Kodaira and Spencer, that the moduli space of Riemann surfaces could be viewed as coset manifolds. This approach has been implemented in the Grassmannian approach¹⁸⁾⁻²²⁾ and used in the works of Refs. 37) and 38).

Another recent development of interest has been the incorporation of ghosts into the old dual model. One still builds all contributions out of a propagator and a three-string vertex, but these two quantities are now ghost extended. The new three-vertex was given in Ref. 5) while the propagator was given in Ref. 39). The decoupling of negative norm states is achieved through the simple BRST transformation properties of the ghost extended three-vertex⁵⁾ and propagator³⁹⁾. Sewing these objects together ensures the correct decoupling of negative norm states within multiloop contributions⁴⁰⁾.

In this lecture we give a brief review of the theoretical ideas behind the group theoretic approach and present one simple example of how it works in practice. The reader is referred to an already existing review¹⁸⁾ which also reviews the method, but in addition explains a number of topics in previously-known string theories which are relevant to the method. Since the review of Ref. 18) was written, there have been a number of developments and these are to some extent incorporated into this lecture.

We now review the relevant aspects of the group theoretic approach⁹⁾⁻¹⁸⁾, which assumes that the scattering of N physical states is given by

$$W = \int \prod_i' dz_i \prod_r d\sigma_r \bar{f}(z_i, \sigma_r) |\chi\rangle_1 \dots \dots |\chi\rangle_N. \quad (1)$$

where $|\chi\rangle_i$ are physical states and z_i ($i = 1, \dots, N$) and v_r ($r = 1, \dots, 3(g-1) \equiv g_1$, $g > 2$) are the Koba-Nielsen co-ordinates and are the moduli respectively. The vertex V is determined by requiring that it satisfy overlap equations; to

be precise, if $R(z)$ is a conformal operator of dimension d , then V obeys the equation^{9)-18), 41)-43)}

$$V \frac{R^i(\bar{z}^i)}{(\bar{z}^i)^d} = V \frac{R^j(\bar{z}^j)}{(\bar{z}^j)^d} \left(\frac{d\bar{z}^j}{d\bar{z}^i} \right)^d \quad (2)$$

where ξ_i is any analytic function of co-ordinate z which vanishes at z_i , and similarly for ξ_j . We can therefore write

$$\bar{z}^i = (V^i)^{-1} z = \sum_{n=1}^{\infty} \hat{a}_n^i (z - z_i)^n, \quad \hat{a}_1 \neq 0 \quad (3)$$

In terms of the conformal transformations L_n , we may write

$$V^i = \exp \{ z_i L_1^i \} \exp \left\{ - \sum_{n=0}^{\infty} a_n^i L_n^{(i)} \right\} \quad (4)$$

The ξ^i and ξ^j are co-ordinate patches around z_i and z_j and the co-ordinate z is a co-ordinate patch for which both sides of Eq. (3) are defined. This equation then states that when the action of two conformal operators on V are defined in an overlap region then each operator gives the same result when referred to the same co-ordinate system. The superscript indices i and j tell us on which leg the conformal operator acts. We recall that, by definition, a conformal operator $R(\xi)$ of weight d , under a conformal transformation from ξ_i to ξ_j , transforms as

$$R(\bar{z}^i) \rightarrow \left[\frac{\bar{z}^i}{\bar{z}^j} \frac{d\bar{z}^j}{d\bar{z}^i} \right]^d R(\bar{z}^j). \quad (5)$$

For each loop we have the overlap conditions

$$V \frac{R^i(\bar{z}^i)}{(\bar{z}^i)^d} = V \frac{R^i(P_n^i \bar{z}^i)}{(P_n^i \bar{z}^i)^d} \left[\frac{d(P_n^i \bar{z}^i)}{d\bar{z}^i} \right]^d \quad (6)$$

Using Eq. (2) to change the indices in Eq. (6), we find that $P_n^i = (V^i)^{-1} P_n (V^i)$. We take P_n , $n = 1, \dots, g$ to be an arbitrary but given element of $SU(1,1)$ for reasons that will become apparent below. The moduli and Koba-Nielsen dependence of V is defined through Eqs. (2) and (6), there being three parameters in each

P_n and one z_j in each $(V_j)^{-1}$. The vertex also depends on a_n^j , $n = 0, 1, \dots, \infty$, $j = 1, \dots, N$. However, as we shall see, the physical result is independent of these later parameters. It is clear from Eq. (3) that these additional parameters correspond to the freedom to choose another co-ordinate system whose origin is still at z_j but is related by an analytic transformation to the simplest choice $\xi_j = z - z_j$.

The above equations are defined on the Riemann sphere, but for loops Eq. (5) tells us that one may work on only the fundamental domain corresponding to the group generated by the P_n , $n = 1, \dots, g$, that is, the region outside all the isometric circles of P_n . We recognize here the Schottky representation of Riemann surfaces (see Fig. 1). One can further cut the Schottky picture and consider the Fuchsian representation. Corresponding to this further dissection, one inherits one extra overlap condition for each loop.

Equations (2) and (6) hold for any conformal operator. For example, for the bosonic string we may use $Q^\mu(z)$, $P^\mu(z)$, $L(z)$ and the ghosts $\beta(z)$, $\bar{\beta}(z)$ which have conformal weights -1 and $+2$ respectively, and are more usually denoted by $c(z)$ and $b(z)$ in the literature. In the case of loops, some care must be taken where Q^μ and $L(z)$ are concerned, and the reader may consult Refs. 9)-18). These equations uniquely determine V up to a multiplicative function of the moduli and Koba-Nielsen variables. For explicit expressions for V , see Refs. 9)-18). The decoupling of zero norm physical states determines the measure $\bar{f}(z_i, v_r)$, given a choice of the mentioned multiplicative function, the final result for string scattering being independent of the choice.

One can integrate Eq. (2) to obtain what is called an integrated overlap equation of the form^{9)-18), 41), 42)}

$$V \left\{ \sum_{j=1}^N \oint_{\xi_j=0} d\bar{\xi}_j \, e^{(1-d)} \frac{R^j(\bar{\xi}^j)}{(\bar{\xi}^j)^d} \left(\frac{d\bar{\xi}^j}{d\bar{\xi}_i} \right)^{d-1} \right\} = 0 \quad (7)$$

An alternative, although subsequent, derivation of certain integrated overlap equations from the Polyakov path integral is given in Ref. 21), and they were also discussed in Ref. 43). We may identify $\phi^{(1-d)}$ as a tensor of rank $(1-d)$. The above equation is valid provided ϕ is defined on the Riemann sphere for trees and also satisfies the relevant automorphic conditions for loops, i.e.,

$$e^{(1-d)}(\bar{\xi}^i) = e^{(1-d)}(P_n^i \bar{\xi}^i) \left(\frac{dP_n^i \bar{\xi}^i}{d\bar{\xi}^i} \right)^{1-d}. \quad (8)$$

In addition, ϕ can have poles only at Koba-Nielsen points, but is analytic elsewhere. These requirements are an obvious consequence of this integration procedure, which is discussed further in Refs. 9)-18), 41), 42). In certain cases, such as when $R = P^\mu(z) = -z(d/dz)Q^\mu$, one can use functions more general than single-valued automorphic functions¹¹.

Thus the number of identities are in one-to-one correspondence with the number of tensors of rank $(1-d)$ which are defined on the surface and are analytic except at Koba-Nielsen points where they may have poles. For example, for the tree case, we consider the surface is the Riemann sphere with no further identifications.

Let us cover the sphere by two co-ordinate patches ξ_i and ζ_i , where

$$\bar{\zeta}_i = \frac{1}{\bar{\xi}_i} \quad (9)$$

in such a way that one Koba-Nielsen point is at $\xi_i = 0$, and at $\xi_i = (\zeta_i)^{-1} = \infty$ there are no Koba-Nielsen points. Then, if $\phi^{(1-d)} = (\xi_i)^n$, then the behaviour at $\zeta_i \rightarrow 0$ is $\zeta_i^{-n-2(1-d)}$ which is analytic if $n < 2(d-1)$. We note, in particular, that there are no analytic tensors on the sphere for $d < 0$ while for $d > 1$ there are $2d-1$ such tensors.

As the vertex V is determined, up to the above-mentioned function, by the overlaps, so consequently are all of its properties. In particular, it is instructive to write down explicitly some of its conformal properties. Let us consider the action of a conformal transformation $\mathcal{H}^i$, where i refers to the leg of V upon which it acts:

$$V \rightarrow \hat{V} = V \prod_{i=1}^N \mathcal{H}^i \quad (10)$$

Using Eqs. (2) and (6), one can easily see that one can change the moduli v_r , the Koba-Nielsen variables z_j and the a_n^j by

$$V(\hat{v}_r, \hat{z}_j, \hat{a}_n^j) = V(v_r, z_j, a_n^j) \prod_{i=1}^N \mathcal{H}^i \quad (11)$$

provided 9)-18)

$$(v^{i+1}) \mathcal{H}^{i+1} (\hat{v}^{i+1})^{-1} = v^i \mathcal{H}^i (\hat{v}^i)^{-1} \tau_i \quad (12)$$

$$(p_n^i)^{-1} \mathcal{H}^i \hat{p}_n^i = \mathcal{H}^i \tau_{i,n} \quad (13)$$

The first of these equations is solved by taking

$$\mathcal{H}^i = (V^i)^{-1} \mathcal{H}(\hat{V}^i) \quad (14)$$

where

$$\hat{V}^i = V^i(\hat{z}_j, \hat{a}_n^j) \quad (15)$$

while the second equation now reads

$$P_n^{-1} \mathcal{H} \hat{P}_n = \mathcal{H}. \quad (16)$$

where $\hat{P}_n = P_n(\hat{v}_r)$.

One can in particular consider such conformal transformations which are infinitesimal. Those which leave the moduli v_r inert but make an infinitesimal change $z_j = z_j + \epsilon f_j$, $a_n^j = a_n^j + \epsilon k_j$ are given by

$$V(v_r, \hat{z}_j, \hat{a}_n^j) = V(v_r, z_j, a_n^j) \prod_{i=1}^N (V^i)^{-1} \hat{V}^i \quad (17)$$

On the other hand, if we do not change z_j and a_n^j , then $V_j = \hat{V}_j$, but we change the moduli $v_r \rightarrow \hat{v}_r = v_r + \epsilon h_r(v)$. Then Eq. (16) reads

$$P_n(z) + \epsilon f(z) \frac{dP_n(z)}{dz} = \hat{P}_n(z) + \epsilon f(P_n z) \quad (18)$$

where

$$\mathcal{H}(z) = z + \epsilon f(z).$$

Implemented on the vertex, this equation becomes

$$\sum_{r=1}^3 h_r(v) \frac{\partial V}{\partial v_r} = V \left\{ \sum_{j=1}^N \oint_{\tilde{z}_j=0} d\tilde{z}_j \frac{L^j(\tilde{z}^j)}{(\tilde{z}^j)^2} f^j(\tilde{z}^j) + c_f \right\} \equiv V(L_f + c_f) \quad (19)$$

where c_f is a constant which is easily determined by taking a suitable state on either side of this equation. It corresponds to the presence of a multiplier which has not been explicitly written in Eq. (11). Clearly, the moduli-changing equations are sensitive to the multiplicative function, discussed above, which is not determined by the overlap equations. The value of c_f is fixed relative to the choice of this function. We recall, however, that to find the actual scattering amplitude, we multiply by the measure $\bar{f}$ and integrate over all the Koba-Nielsen variables and moduli as in Eq. (11). However, the measure $\bar{f}$ is

closely related to the constants c_f and the final result is independent of what choice of multiplicative factor for V one takes. We will discuss this point further below.

Of importance are those transformations which leave the vertex inert, i.e., which change neither v_r , z_j , nor a_n^j . For the case of a tree graph with N external legs, these are given by $\mathcal{H}^i$'s which satisfy

$$\mathcal{H}^{i+1} = (v^{i+1})^{-1} v^i \mathcal{H}^i (v^i)^{-1} v^{i+1} \quad (20)$$

while, if the contribution has loops, they should also obey

$$P_n^{-1} \mathcal{H} P_n = \mathcal{H} \quad (21)$$

For infinitesimal transformations, we take Eq. (18) with $P_n = \hat{P}_n$, which means that $f(z)$ transforms as a vector field (i.e., a tensor of rank -1) and is defined on the Riemann surface itself. In fact, we have already met this group since we may take $R(z) = L(z)$, which has conformal weight 2 in Eq. (6), and we find

$$V \left\{ \sum_{j=1}^N \oint_{\xi_j=0} d\bar{\xi}_j \varphi^{(-1)} \frac{L^j(\bar{\xi}_j)}{(\bar{\xi}_j)^2} \frac{d\bar{\xi}_j}{d\bar{\xi}^i} + c \right\} = 0 \quad (22)$$

that is, an equation of the same type. Examining the previous equations which define conformal transformations that preserve the vertex, we find that they are the same as those given by Eq. (22). Clearly, the set of such transformations forms a group which, for a vertex with g loops and N external lines, we denote by H_N^g . The explicit computation of H_N^g can be found in Ref. 16).

The transformation properties of the vertex can be summarized by writing the dependence of the vertex V on v_r and z_j , a_n^j through P_n and V^j , i.e., $V(v_r, z_j, a_n^j) = V(P_n, V^j)$. We recall that the vertex $V(P_n, V^j)$ is determined, up to a multiplicative function, by the overlaps which involve V^j and P_n . As such, V depends not on N V^j 's, but only on $(V^j)^{-1} V^i$ for all i, j ; that is, only $N-1$ conformal transformations. [We note that $(V^j)^{-1} V^k = (V^j)^{-1} V^i (V^i)^{-1} V^k$.] We also observe that it is $SU(1,1)$ -invariant in the sense that if $S \in SU(1,1)$, then

$$V(S P_n S^{-1}, S V^j) = V(P_n, V^j) \quad (23)$$

This explains the 3 in the count of the number of moduli.

The conformal transformation property of the vertex embodied in Eqs. (12) and (13) may be written as

$$V(p_n, v^j) \prod_{i=1}^N \mathcal{O}_i = V(\mathcal{O}^{-1} p_n \mathcal{O}, \hat{v}^j) \quad (24)$$

where $\mathcal{O}^j$ is split up according to Eq. (14), i.e., $\hat{v}_j$ is of the required form of Eq. (3) and $\mathcal{O}^{-1} p_n \mathcal{O} \in \text{SU}(1,1)$. We have again omitted the possible scaling of V by a function of v_r and z_j .

Applying two such transformations successively, we find

$$\begin{aligned} V(p_n, v^j) \prod_{i=1}^N \mathcal{O}_i^1 \mathcal{O}_i^2 \\ = V(\mathcal{O}_2^{-1} \mathcal{O}_1^{-1} p_n \mathcal{O}_1 \mathcal{O}_2, v_2^j) \end{aligned} \quad (25)$$

where

$$\begin{aligned} \mathcal{O}_1^j &= (v^j)^{-1} \mathcal{O}_1 v_1^j \\ \mathcal{O}_2^j &= (v_1^j)^{-1} \mathcal{O}_2 v_2^j \end{aligned} \quad (26)$$

One can verify that the order of the factors is correct by carrying out these transformations on the overlap equations. One can recognize¹⁶⁾, in these equations, that a string vertex carries an induced representation of the group of N copies of the Virasoro group $V \times \dots \times V$ in which the little group H_N^8 is trivially realized. We also note that the vertex including ghost oscillators obeys the equation $V \sum_{j=1}^N Q^j = 0$, where Q is the BRST charge as Q is the integral of the BRST current which has conformal weight one, this is an example of an integrated overlap equation.

As stated above, the measure is determined by ensuring that zero-norm physical states decouple⁹⁾⁻¹⁸⁾. These states are of the form

$$L_1 |\Omega\rangle, \quad (L_2 + 3/2 L_1^2) |\Omega\rangle$$

where

$$\begin{aligned} L_n |\Omega\rangle &= L_n |\Omega'\rangle = 0 \quad n \geq 0 \\ L_0 |\Omega\rangle &= (L_0 + 1) |\Omega'\rangle = 0. \end{aligned}$$

etc. We may consider these as being infinitesimal conformal transformations applied to the vertex and as such they may be identified as transformations which preserve the vertex or change its Koba-Nielsen variables or moduli. Consequently, we find generically that

$$\begin{aligned} W |x\rangle_1 \dots |x\rangle_N &= \int \prod_i' dz_i \prod_v d\sigma_v \bar{f} \\ &\left(\sum_v h_v \frac{\partial V}{\partial \sigma_v} + \sum_j g_j \frac{\partial V}{\partial z_j} - C_{n,g} V \right) \\ &|x\rangle_1 \dots |x\rangle_i \dots |x\rangle_N = 0 \end{aligned} \quad (27)$$

where all the states $|x\rangle_i$ ¹⁶⁾ are physical and so obey $(L_n - \delta_{n,0}) |x\rangle = 0$, $n > 0$ and

$$\begin{aligned} \sigma_v &\rightarrow \sigma_v + h_v(\sigma_v, z_j) \\ z_j &\rightarrow z_j + g_j(z_j, \sigma_v). \end{aligned}$$

are the changes that the infinitesimal conformal transformation induces on the vertex. We find, by integrating by parts that

$$0 = \sum_v \frac{\partial}{\partial \sigma_v} (h_v \ln \bar{f}) + \sum_j \frac{\partial}{\partial z_j} (g_j \ln \bar{f}) + C_{n,g}. \quad (28)$$

in other words a first-order differential equation for the measure, $\bar{f}$. In fact, one has as many differential equations as one has Koba-Nielsen variables and moduli and so the measure is determined by them.

Of course, in this process we have neglected the boundary term which in general is not zero and is in fact a BRST anomaly. Indeed, all open strings, except for the gauge group $SO(2^{13})$, have such an anomaly^{10),44)}.

The process of decoupling without ghosts becomes at least at first sight more and more difficult to implement at higher loops due to the complicated structure of the zero-norm physical states. This difficulty is not present when one incorporates ghosts as these states are of the form $Q|\Lambda\rangle$. However, the correct vertex to use is no longer V , when ghost extended, as this is annihilated by Q and so does not lead to any constraints on the measure. In fact, one must multiply the vertex by a ghost oscillator prefactor. This is done explicitly in Refs. 14) and 15) and one finds that the measure obeys the differential equations

$$\frac{\partial \ln \bar{f}}{\partial z_j} = 0 = \frac{\partial \ln \bar{f}}{\partial v_r} + c_r. \quad (29)$$

where c_r are the constants in Eq. (19). One can, of course, redefine one's vertex by multiplying it by $\bar{f}$ and then c_r is zero in Eq. (19) and the corresponding measure is one.

To illustrate how to apply the method, we compute bosonic string scattering at the tree level. We begin by deriving the N string vertex $V^{(N)}$. The overlap equation (2) for the conformal operator

$$\varphi^\mu(z) = + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu z^{-n}}{n} - \alpha_0^\mu \ln z - \frac{\partial}{\partial \alpha_0^\mu} \quad (30)$$

is given by

$$V^{(N)} \left\{ \varphi^{\mu i}(z - z_i) - \varphi^{\mu j}(z - z_j) \right\} = 0 \quad (31)$$

where we have chosen the simplest choice of local co-ordinates (or cycling transformation) namely $\xi_i(z) = (z - z_i)$. The integrated overlap of Eq. (6) becomes for $R(z) = P^\mu(z)$:

$$V^{(N)} \left\{ \sum_{j=1}^N \oint \frac{d\bar{z}_j}{\bar{z}_j} \varphi^{(0)} P^{\mu j}(\bar{z}_j) \right\} = 0 \quad (32)$$

We can take $\phi^{(0)}$ to be of the form

$$\begin{aligned} \varphi^{(0)} &= \frac{1}{(\bar{z}_i)^n} = \frac{1}{(\bar{z}_j + z_j - z_i)^n} \\ &= \sum_{p=0}^{\infty} \binom{-n}{p} \frac{(\bar{z}_j)^p}{(z_j - z_i)^{n+p}} \quad \forall j \neq i \end{aligned} \quad (33)$$

In fact, this set of $\phi^{(0)}$'s form a complete set of all the allowed ϕ 's. Substituting the $\phi^{(0)}$'s into Eq. (32) and using the relation

$$P^\mu(z) = \sum_{n=-\infty}^{\infty} \alpha_{-n}^\mu z^n.$$

we find the identity

$$0 = V^{(N)} \left\{ \alpha_{-n}^{\mu i} + \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{p=0}^{\infty} \binom{-n}{p} \frac{\alpha_p^{\mu j}}{(z_j - z_i)^{n+p}} \right\} \quad (34)$$

The solution to this equation being

$$V = \left\{ \prod_{i=1}^N \langle 0 | \right\} \exp - \sum_{1 \leq i < j \leq N} \left\{ \sum_{n,m=1}^{\infty} \alpha_n^{\mu i} \frac{(n+m-1)! (-1)^m}{m! n! (z_j - z_i)^{n+m}} \alpha_m^{\mu j} + \sum_{n=1}^{\infty} \frac{\alpha_n^{\mu i} \alpha_0^{\mu j}}{n (z_j - z_i)^n} + \frac{\alpha_n^{\mu j} \alpha_0^{\mu i}}{n (z_i - z_j)^n} + \alpha_0^{\mu i} \alpha_0^{\mu j} \mathcal{L}^{ij} \right\} \quad (35)$$

where $\mathcal{L}^{ij}$ is not determined by the integrated overlap and we have arbitrarily chosen the normalization of $V^{(N)}$. To find $\mathcal{L}^{ij}$, one uses the unintegrated Q^μ overlap of Eq. (31) and one finds the result to be

$$\mathcal{L}^{ij} = \ln (z_j - z_i). \quad (36)$$

One could, from the beginning, have determined $V^{(N)}$ completely Q^μ overlap, from the beginning, however, it is technically simpler to first carry out the partial determination given above with the integrated P^μ overlap. It is straightforward to evaluate the vertex for any choice of $(V^1)^{-1}$.

We must now determine the measure. Using Eq. (17) we find that a small change, $z_j \rightarrow z_j + \epsilon$, $z_i \rightarrow z_i$ for $i \neq j$ is induced by

$$\frac{\partial V}{\partial z_j} = V L_{-1}^j \quad (37)$$

Applying the state $L_{-1}^j |\Omega\rangle_j$ to the scattering amplitude

$$W = \int \prod_{j=1}^{N-3} dz_j V \bar{f}. \quad (38)$$

we find that

$$W |x\rangle_1 \dots |x\rangle_N = \int \prod_{k=1}^{N-3} dz_k \bar{f} \frac{\partial V}{\partial z_j}$$

$$|x\rangle_1 \dots |x\rangle_j \dots |x\rangle_N \quad ; \quad j=1, \dots, N-3.$$

(39)

which implies that $\partial \bar{f} / \partial z^j = 0$ or that $\bar{f}$ is a constant. One must also show that the state $L_{-1}^j |\Omega\rangle$ decouples for $j = N-2, N-1, N$. This is achieved by also using the three L_n preserving integrated overlaps of Eq. (22) which follow from taking $\phi^{(-1)}$ to be the three analytic vector fields. This is related to the $SU(1,1)$ invariance of the result which is discussed using these identities in Ref. 13). To complete the discussion of decoupling, one must also show that states of the form $(L_{-2} + 3/2 L_{-1}^2) |\Omega'\rangle$ decouple and that the boundary terms disregarded for these states and those above actually vanish. The former results from use of the preserving integrated overlaps containing L_{-1}^j on one leg only while the boundary terms vanish by standard arguments involving analytic continuation in the external momenta.

Thus we have determined $V^{(N)}$ and the measure $\bar{f}$. To find the scattering amplitude for any desired physical states, one simply saturates W with them.

Finally, we give a guide to where in the literature some of the points not discussed in this brief review can be found. The N bosonic string vertex for the multiloop scattering was given in Ref. 11), the derivation being similar in spirit to that for the tree case. The measure at one loop was found without the incorporation of ghosts in the Fuchsian and Shottky representations of Riemann surfaces in Refs. 10), 11) and 13) and the one-loop BRST anomaly for the open strings except $SO(2^{13})$ in Ref. 10). The incorporation of ghosts and their corresponding vertex contributions are found for the tree case in Ref. 14) and the multiloop case in Ref. 15). The corresponding first-order differential equations for the measure of Eq. (29) are found in Ref. 15).

Unlike in a second-quantized field theory one must explicitly check that the amplitudes computed in this approach are factorizable or, equivalently, that they may be sewn together. Given two vertices one must show how one can join them together with a propagator to form a third vertex of the desired

type. This is an exercise in showing that the unintegrated identities satisfied by the ingredient vertices imply the unintegrated overlaps for the composite vertex, see Ref. 13). One can also carry out the computation of Neveu-Schwarz scattering at the tree and loop level in the way described above^{12),13),17)}.

As mentioned above, string vertices carry an induced representation of N copies of the Virasoro group. One can extend the Virasoro group to include the ghosts of conformal weight two and the BRST charge, i.e., consider the group $U = \{L_n, \bar{\beta}_n, n = 0, \pm 1, \dots, Q\}$. The string vertices with ghosts carry an induced representation of $U \times \dots \times U$ with the isotropy group being given by the overlap identities for these operators. Integrating over the "relevant" part of the coset explains the integration over moduli and Koba-Nielsen variables in a natural way as well as providing automatically the correct ghost insertions which come from integrating over the anticommuting parts of the coset. These points are discussed in Ref. 16).

The extension of the method to the Ramond sector is the subject of Ref. 17) where the idea that string vertices are induced representations plays an important rôle. Explicit computations of fermionic string scattering are given in Ref. 45).

The group theoretical method can be applied to any string theory, all that is required are the set of conformal operators with which to implement the overlaps. From the point of view of perturbation theory, the method is complete. It would be interesting to extend it to non-perturbative effects.

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Figure 1