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Quantum Field Theory and Link Invariants

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Abstract: A skein relation for the expectation values of Wilson line operators in three-dimensional $SU(N)$ Chern-Simons gauge theory is derived. We use a variational method based on the properties of the three-dimensional field theory. The relationship between the above expectation values and the known link invariants is established.

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1. Introduction

In a recent paper [1], Atiyah has suggested new possible applications of quantum field theory in the study of manifolds in three and four dimensions. In particular, he has pointed out that three-dimensional field theories with a pure Chern-Simons action could be relevant for knot theory and might shed new light on the link invariants. A considerable development in this program has been accomplished by Witten [2]. By making use of the deep relationships between the three-dimensional Chern-Simons models and certain two-dimensional conformal field theories, Witten derives a skein relation for the expectation values of the Wilson line operators, $\langle W(L) \rangle$. In this way, he shows that there exists a correspondence between $\langle W(L) \rangle$ and the Jones link invariants [3] and their generalizations. The proof essentially relies on the properties of the two-dimensional Hilbert space which arises as the space of conformal blocks of the underlying conformal field theory.

On the other hand, it would be interesting to derive the skein relation directly from the properties of the three-dimensional field theory. This could give new insights into knot theory. In particular, one could try to associate link invariants to a generic three-dimensional manifold and look for new invariants.

In this paper we shall derive a skein relation for the locally four-valent planar graphs associated with the projections of the knots (or links). We use a variational approach in order to study the behaviour of the expectation values of Wilson line operators computed with the Chern-Simons measure. More precisely, we consider infinitesimal deformations of the paths associated with the Wilson line operators. The fundamental property of the Chern-Simons action Γ , namely

$$\frac{\delta \Gamma}{\delta A_\mu^a(x)} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}^a(x) \quad , \quad (1.1)$$

permits us to derive the basic crossing relations associated with the link diagrams. The values of the coefficients entering into these relations are determined at the first non-trivial order of the coupling constant. From the crossing relations, it follows that the value of $\langle W(L) \rangle$ is not an ambient isotopy *link* invariant but, rather, what is called

a regular isotopy invariant. The latter is equivalent to ambient isotopy invariance of *bands* [4].

The above result is not quite surprising for the following reason. Let us recall that, even if the standard renormalization of the Chern-Simons theory presents no problems, the composite operator $W(L)$ needs anyway a further regularization. The framing of the link L is a sort of regularization preserving the geometric structure of the model. But a framed link actually behaves as a band! It is nice to find out that our variational approach, which apparently deals with non-regularized quantities, leads on general grounds to the same "framed" picture for the links, emerging from perturbation theory. Even more, we will show that the value of the variable measuring the twisting of the band coincides precisely with the variation of $\langle W(L) \rangle$ under a change of framing in the perturbative approach.

We explain in details how $\langle W(L) \rangle$ has to be modified in order to obtain the Jones polynomial and its generalizations (the two variables HOMFLY polynomials [5,6]) associated with the link L . We find agreement with the results of [2].

In our derivation of the skein relation, no use is made of the associated two-dimensional conformal field theory (Wess-Zumino-Witten model). Our construction relies instead on the existence of a Fierz identity for the generators of the group in certain representations. The fact that our coefficients of the skein relation for $SU(N)$ coincide with those found in [2] by means of the $SU(N)$ -WZW model confirms once more the deep relationships between topological three-dimensional models and two-dimensional conformal field theories.

The paper is organized as follows. In sect.2 we introduce the model, we discuss its symmetry properties and recall some basic notions from knot theory. Sect.3 is devoted to the variational derivation of the skein relation for $\langle W(L) \rangle$. Here we describe also the relationship between $\langle W(L) \rangle$ and the link polynomials. A summary and our conclusions are contained in sect.4.

2. The Chern-Simons Model

The simplest field theory with a pure Chern-Simons action is a three-dimensional model for a vector field $A_\mu = A_\mu^a T^a$ with action [2,7,8,9]

$$\Gamma[A] = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho \right) . \quad (2.1)$$

Here T^a are the Hermitian generator matrices of a semisimple group G . We shall consider the case $G = SU(N)$ and T_{ij}^a , ($a = 1, \dots, N^2 - 1$), will be the generators of the fundamental representation normalized as follows

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} . \quad (2.2)$$

Under a gauge transformation $\Omega(x) \in SU(N)$,

$$A_\mu(x) \mapsto A_\mu^\Omega(x) \equiv \Omega^{-1}(x) A_\mu(x) \Omega(x) - i \Omega^{-1}(x) \partial_\mu \Omega(x) , \quad (2.3)$$

the action (2.1) transforms as

$$\Gamma[A^\Omega] = \Gamma[A] + \frac{k}{12\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr}(\Omega^{-1} \partial_\mu \Omega \Omega^{-1} \partial_\nu \Omega \Omega^{-1} \partial_\rho \Omega) . \quad (2.4)$$

The absence of a linear term in $\Omega^{-1} \partial_\mu \Omega$ in eq.(2.4) means that, regardless of the particular value of the coupling constant k , the gauge invariance is preserved to all orders of the perturbative expansion. The action is sensitive only to the global properties of the gauge transformation $\Omega(x)$; in fact, eq.(2.4) shows that $\Gamma[A^\Omega]$ differs from $\Gamma[A]$ by the quantity $2\pi n k$, where n is the topological integer number associated with the map $\Omega(x)$ of the three-dimensional space (that we take to be S^3) into the group $SU(N)$ [10]. When k assumes integer values, the difference between $\Gamma[A^\Omega]$ and $\Gamma[A]$ is an integer times 2π ; in this case, one has a complete (also non-perturbative) gauge invariance because the path integral measure is given by $\exp(i\Gamma)$.

In what follows, we would like to compute the mean values

$$\langle W_{R_1}(C_1) \cdots W_{R_n}(C_n) \rangle = Z^{-1} \int \mathcal{D}A e^{i\Gamma[A]} W(C_1) \cdots W(C_n) \quad (2.5)$$

of the product of Wilson line operators

$$W_{R_i}(C_i) = \text{Tr}_{R_i} \left(P \exp i \oint A_\mu dx^\mu \right) \quad , \quad (2.6)$$

where the path ordering is performed along a given oriented closed path (knot) C_i and the trace is computed in the R_i representation of $SU(N)$. The union of non-intersecting knots C_i is called a link L and, for simplicity, we shall write

$$\langle W_{R_1}(C_1) \cdots W_{R_n}(C_n) \rangle \equiv \langle W(L) \rangle \quad . \quad (2.7)$$

Besides gauge invariance, one of the main features of the model is its general covariance. Indeed, the physics of the system (2.1) is independent of the particular choice of the metric that one can introduce in the three-dimensional space. General covariant quantum field theories are not a novelty in physics; one of the reasons which makes the model (2.1) somewhat special is the fact that general covariance is achieved not by integrating over the space of metrics. The action (2.1) is independent of the metric and, as long as one is interested in the expectation values (2.5), the introduction of a metric is again not required. These features of the model suggest that $\langle W(L) \rangle$ is related to some invariants characterising the link L . In order to discuss this relationship we recall some basic notions from knot theory [4,11].

Consider two links L_1 and L_2 . They are called *ambient isotopic* ($L_1 \sim L_2$) if there exists a smooth deformation through embeddings from one to the other. A practical procedure for verifying ambient isotopy is the following one. We associate with any link L a four-valent planar diagram, namely a diagram obtained by projecting L on a plane in such a way that the projection contains only simple crossing points where the choice of over/under crossing is given. It can be shown that $L_1 \sim L_2$ if and only if there is a finite sequence of Reidemeister moves, shown in Fig.1, which changes the diagram of L_1 into the diagram of L_2 .

It is useful to introduce also the concept of *regular isotopy for diagrams* [4]. Two diagrams are said to be regular isotopic if one can be obtained from the other by a sequence of Reidemeister moves of type II and type III only. Regular isotopy arises for example when, instead of a link, we consider an oriented twisted band. Clearly, the

Reidemeister moves of type I do not relate ambient isotopic bands, see Fig.2. It turns out that ambient isotopy of twisted bands is equivalent to regular isotopy for link (or knot) diagrams.

Now we introduce the HOMFLY polynomials [5,6,12]. To each oriented link there is associated a finite Laurent polynomial $P_L(t, z)$ of two variables with integer coefficients, such that if $L_1 \sim L_2$, then $P_{L_1} = P_{L_2}$. The polynomial P_L can be constructed by means of the *skein relation*

$$t P_{L_+} - t^{-1} P_{L_-} = z P_{L_0} \quad . \quad (2.8)$$

Here, the diagrams associated with L_+ , L_- and L_0 differs only at the site of one crossing as shown in Fig.3. By means of (2.8) and Reidemeister moves of all the three types, the polynomial of any link can be reduced to a sum of polynomials for unknotted links. In this way one can construct recursively P_L for any L starting from the polynomial P_0 of the unknotted knot which is usually normalized to the unity, $P_0 = 1$. It is known [4,6] that the relation (2.8) determines uniquely the link invariant P_L . Unfortunately, it is also known that P_L does not determine uniquely the link L up to ambient isotopy. The link invariant $P_L(t = 1, z)$ is known as the Alexander-Conway polynomial [13], whereas $P_L(t, z = \sqrt{t} - \frac{1}{\sqrt{t}})$ is the Jones polynomial [3].

For discussing the properties of Wilson line operators $\langle W(L) \rangle$, it is necessary to consider the so-called R -polynomial introduced by Kauffman [4,14]. Differently from P_L , R_L is not ambient isotopy invariant but only regular isotopy invariant. R_L is defined by the exchange identity

$$R_{L_+} - R_{L_-} = z R_{L_0} \quad , \quad (2.9)$$

supplemented by

$$R_{\widehat{L}_+} = \alpha R_{\widehat{L}_0} \quad , \quad (2.10)$$

$$R_{\widehat{L}_-} = \alpha^{-1} R_{\widehat{L}_0} \quad , \quad (2.11)$$

where $\widehat{L}_+$, $\widehat{L}_-$ and $\widehat{L}_0$ are displayed on Fig.4. In analogy with the P_L -polynomials, $R_L(\alpha, z)$ can be constructed recursively ($R_0 = 1$); however, differently from the P_L polynomials, in the case $\alpha \neq 1$ one is not allowed to use Reidemeister moves of type I, but instead is bound to use eqs.(2.10,11). Now, the important point to note is that

$R_L(\alpha, z)$ can also be interpreted as an *ambient isotopy invariant for oriented twisted bands* [9], where α is the variable which measures the twisting in the band. As shown by Kauffman, P_L and R_L are closely related. In fact one has

$$P_L(t = \alpha, z) = \alpha^{-w(L)} R_L(\alpha, z) \quad , \quad (2.12)$$

where the *writhe* $w(L)$ is given by

$$w(L) = \sum_p \epsilon(p) \quad . \quad (2.13)$$

The sum in (2.13) is performed over all crossings of the diagram associated with L and

$$\epsilon(L_{\pm}) = \pm 1 \quad . \quad (2.14)$$

One should note that the writhe $w(L)$ is regular isotopy invariant, it is additive and can be computed by analysing locally the diagram associated with L . The eq.(2.12) will be proven, in a more general context, later on.

Summarizing, one can associate a polynomial to every planar diagram; how to construct this polynomial is specified by a set of local rules. The key ingredients are:

- i) the behaviour under Reidemeister moves ;
- ii) the skein relation.

So far, we have discussed two different possibilities. In the first, one associates with a link L the P_L -polynomial. In this case it is assumed that P_L is invariant under all the Reidemeister moves and the skein relation (or exchange identity) is given by eq.(2.8).

In the second possibility, the R_L -polynomial is not invariant under Reidemeister moves of type I, but it is rather "covariant", see eqs.(2.10,11). In addition, one has the exchange identity (2.9).

Comparing eqs.(2.8) and (2.9), we see that the latter can be generalized by introducing a new variable β , which is the analogue of t . We define then the regular isotopy invariant polynomial $S_L(\alpha, \beta, z)$ by means of the "generalized" skein relations

$$S_{\widehat{L}_+} = \alpha S_{\widehat{L}_0} \quad , \quad (2.15)$$

$$S_{\widehat{L}_-} = \alpha^{-1} S_{\widehat{L}_0} \quad , \quad (2.16)$$

$$\beta S_{L_+} - \beta^{-1} S_{L_-} = z S_{L_0} \quad , \quad (2.17)$$

supplemented by the standard normalization condition

$$S_0 = 1 \quad (2.18)$$

for the unknot. As we shall see later, in this form the skein relations are directly applicable to $\langle W(L) \rangle$.

Similarly to the R_L -polynomial, S_L also can be related to P_L . One proceeds as follows. Since both S_L and $w(L)$ are regular isotopy invariants, the same holds for the "new" polynomial

$$N_L(\alpha, \beta, z) \equiv \alpha^{-w(L)} S_L(\alpha, \beta, z) \quad . \quad (2.19)$$

Let us find the generalized skein relations satisfied by N_L . From (2.18) it follows immediately that

$$N_0 = 1 \quad . \quad (2.20)$$

Consider now $N_{\widehat{L}_+}$. From eqs.(2.13,14) one finds

$$w(\widehat{L}_+) = 1 + w(\widehat{L}_0) \quad . \quad (2.21)$$

Therefore, using (2.15) one gets

$$N_{\widehat{L}_+} = \alpha^{-w(\widehat{L}_+)} S_{\widehat{L}_+} = \alpha^{-w(\widehat{L}_0)} S_{\widehat{L}_0} = N_{\widehat{L}_0} \quad . \quad (2.22)$$

Analogously,

$$N_{\widehat{L}_-} = N_{\widehat{L}_0} \quad . \quad (2.23)$$

Eqs.(2.22,23) show that N_L is actually ambient isotopy invariant. Finally, consider the exchange identity,

$$\begin{aligned} & (\alpha\beta) N_{L_+} - (\alpha\beta)^{-1} N_{L_-} = \\ & = (\alpha\beta) \alpha^{-w(L_+)} S_{L_+} - (\alpha\beta)^{-1} \alpha^{-w(L_-)} S_{L_-} = \\ & = \alpha^{-w(L_0)} (\beta S_{L_+} - \beta^{-1} S_{L_-}) = z \alpha^{-w(L_0)} S_{L_0} = z N_{L_0} \quad . \end{aligned} \quad (2.24)$$

Eqs.(2.20,22-24) imply, in view of (2.19), that

$$P_L(t = \alpha\beta, z) = \alpha^{-w(L)} S_L(\alpha, \beta, z) \quad . \quad (2.25)$$

Since the R_L -polynomial is a particular case of the S_L -polynomial (with $\beta = 1$), the above argument proves also eq.(2.12).

Some final comments are in order. On the level of ambient isotopy for links (P_L -polynomial) there is no need for introducing separately α and β . However, as far as $\langle W(L) \rangle$ is concerned, the ambient isotopy of bands (framed links) is relevant. This is the reason why we have introduced the S_L -polynomial and why the distinction between α and β is physically meaningful. If one interprets $\langle W(L) \rangle$ as the quantum mechanical amplitude of a charged particle with fractional statistics moving along L , then α can be understood as the phase change of the wave function under a 2π spin rotation.

3. Variational Derivation of the Skein Relation

In this section, we concentrate on the expectation value $\langle W(L) \rangle$. Our strategy in deriving the associated skein relation is to study the behaviour of $\langle W(L) \rangle$ under infinitesimal deformations of the contour defining L . * A first question one may ask is if $\langle W(L) \rangle$ is ambient isotopic; in other words, is $\langle W(L) \rangle$ invariant under continuous deformations of L ? In what follows we will first answer this question.

Let us focus on the part

$$U(x_1, x_2) = P \exp i \int_{x_1}^{x_2} dx^\mu A_\mu^a R^a \quad (3.1)$$

of the Wilson line operator associated to the oriented path along L connecting two arbitrary points x_1 and x_2 (see Fig.5a). As is well known, an infinitesimal variation of

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the path in x produces the $F_{\mu\nu}$ -insertion (see Fig.5b)

$$U(x_1, x_2) \mapsto U(x_1, x) i \Sigma^{\mu\nu} F_{\mu\nu}^a(x) R^a U(x, x_2) \quad . \quad (3.2)$$

Here there is no summation over μ and ν , $\Sigma^{\mu\nu} = dx^\mu dx^\nu$ is the area element and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c \quad . \quad (3.3)$$

In order to evaluate the effect of the $F_{\mu\nu}$ -insertion on $\langle W(L) \rangle$, we make use of the following remarkable property of the Chern-Simons theory:

$$\begin{aligned} \langle F_{\mu\nu}^a(x) O_1 O_2 \cdots \rangle &\equiv Z^{-1} \int \mathcal{D}A \, e^{i\Gamma} F_{\mu\nu}^a(x) O_1 O_2 \cdots = \\ &= Z^{-1} \int \mathcal{D}A \, \left(-i \frac{4\pi}{k} \right) \epsilon_{\mu\nu\rho} \frac{\delta e^{i\Gamma}}{\delta A_\rho^a(x)} O_1 O_2 \cdots = \\ &= Z^{-1} \int \mathcal{D}A \, e^{i\Gamma} \left(i \frac{4\pi}{k} \right) \epsilon_{\mu\nu\rho} \frac{\delta}{\delta A_\rho^a(x)} (O_1 O_2 \cdots) \quad . \end{aligned} \quad (3.4)$$

In deriving the relation (3.4), eq.(1.1) has been used and an integration by parts has been performed. Combining eqs.(3.2) and (3.4) one gets

$$\begin{aligned} \langle \cdots U(x_1, x_2) \cdots \rangle &\mapsto \\ -i \frac{4\pi}{k} \delta^3(x - y) \epsilon_{\mu\nu\rho} \Sigma^{\mu\nu} dy^\rho \langle \cdots U(x_1, x) \sum_a R^a R^a U(y, x_2) \cdots \rangle & \quad . \end{aligned} \quad (3.5)$$

The differential dy^ρ in the factor

$$v \equiv \delta^3(x - y) \epsilon_{\mu\nu\rho} \Sigma^{\mu\nu} dy^\rho \quad ,$$

is oriented along the tangent to the path. Being the product of a volume element with a δ^3 -function, v is dimensionless and can take the values 0 or ± 1 . If dy^ρ belongs to the plane determined by $\Sigma^{\mu\nu}$, then $v = 0$. Otherwise $v = \pm 1$, depending on the orientation of $\Sigma^{\mu\nu}$ with respect to dy^ρ . (Clearly the orientation of the ambient space is fixed by the choice of $\epsilon_{\mu\nu\rho}$.) Therefore, under infinitesimal deformations of L , one has three possibilities

$$\delta \langle W(L) \rangle = 0 \quad (3.6)$$

or

$$\delta \langle W(L) \rangle = \mp i \frac{4\pi}{k} c_2(R) \langle W(L) \rangle , \quad (3.7)$$

where $c_2(R)$ is the quadratic Casimir operator in the representation R ,

$$c_2(R) \mathbf{1} = \sum_a R^a R^a . \quad (3.8)$$

How should one interpret eqs.(3.6) and (3.7)? First of all eq.(3.7) shows that $\langle W(L) \rangle$ is *not* ambient isotopic. Therefore $\langle W(L) \rangle$ can not be directly associated with a P_L -polynomial. One can hope, however, to relate $\langle W(L) \rangle$ to a S_L -polynomial. In fact, we will show that $\langle W(L) \rangle$ satisfies the generalized skein relations (2.15-17). In order to do this, we would like to translate eqs.(3.6) and (3.7) in terms of diagrammatic relations. There are essentially three ways of assigning diagrammatically a small area to a local deformation of L . The first one is to use the diagram $\widehat{L}_f$ (displayed in Fig.6) in which dy^ρ belongs to the plane determined by $\Sigma^{\mu\nu}$, whereas the remaining possibilities are related with the diagrams $\widehat{L}_+$ and $\widehat{L}_-$ shown in Fig.4.

We propose the following diagrammatic interpretation of (3.6) and (3.7). This interpretation will be subjected later to several checks. Eq.(3.6) means

$$\langle W(\widehat{L}_f) \rangle = \langle W(\widehat{L}_0) \rangle , \quad (3.9)$$

whereas eq.(3.7) is equivalent to

$$\langle W(\widehat{L}_+) \rangle - \langle W(\widehat{L}_-) \rangle = -i \frac{4\pi}{k} c_2(R) \langle W(\widehat{L}_0) \rangle \quad (3.10)$$

in the small area limit of the locks in $\widehat{L}_+$ and $\widehat{L}_-$. Clearly the other choice for the sign on the R.H.S. of (3.10) is equivalent to the exchange $\widehat{L}_+ \leftrightarrow \widehat{L}_-$. To make contact with the S_L -polynomials, eq.(3.10) can be rewritten as

$$\langle W(\widehat{L}_+) \rangle = \alpha \langle W(\widehat{L}_0) \rangle , \quad (3.11)$$

$$\langle W(\widehat{L}_-) \rangle = \alpha^{-1} \langle W(\widehat{L}_0) \rangle , \quad (3.12)$$

where

$$\alpha = 1 - i \frac{2\pi}{k} c_2(R) + O\left(\frac{1}{k^2}\right) . \quad (3.13)$$

Comparing (3.11) and (3.12) with (2.15) and (2.16) we interpret α as the variable measuring the twisting of the band, that is, the framed link. In eq.(3.11,12), the small area limit picks up the first two terms in the expansion (3.13). Because of eq.(1.1), any subsequent variation of the path contributes a factor $\frac{2\pi}{k}$ and by taking into account also the higher dimension operators involved in a finite deformation of the path, in principle one could determine the $O(\frac{1}{k^2})$ -part of α . We shall not elaborate on this point here. Note that for the action (2.1), $\frac{2\pi}{k}$ plays the role of $\hbar$; this allows for a perturbative check of (3.13). Later on, we will verify to the first non trivial order in $\frac{1}{k}$ that the value of α given by eq. (3.13) coincides precisely with the variation of $\langle W(L) \rangle$ under a change of framing in the perturbative approach.

We proceed now by looking for an exchange identity for $\langle W(L) \rangle$ which plays the role of (2.17). A way of relating L_+ and L_- , displayed in Fig.3, is shown in Fig.7. For infinitesimal area of the circle in Fig.7 one can apply once more the variational method used before. One has

$$\langle W(L_+) \rangle = \langle W(L_-) \rangle + \langle \cdots U(1, x) i \Sigma^{\mu\nu} F_{\mu\nu}^a(x) R^a U(x, 2) \cdots U(3, 4) \cdots \rangle \quad . \quad (3.14)$$

At this stage $F_{\mu\nu}^a(x)$ can be replaced by a functional derivative with respect to $A_\rho^a(x)$, which acts on both the paths $1 \rightarrow 2$ and $3 \rightarrow 4$. Since by definition the crossing should not involve any twisting, the functional derivative in A_ρ^a must act trivially on $U(1, 2)$. This means that dy^ρ for the path $1 \rightarrow 2$ belongs to the plane determined by $\Sigma^{\mu\nu}$. In other words, for the path $1 \rightarrow 2$ one should apply equation (3.6) (or eq.(3.9)). So, one finds

$$\begin{aligned} \langle W(L_+) \rangle = \\ \langle W(L_-) \rangle - i \frac{4\pi}{k} \sum_a \langle \cdots U(1, x) R^a U(x, 2) \cdots U(x, 3) R'^a U(x, 4) \cdots \rangle \quad . \quad (3.15) \end{aligned}$$

Note that the paths $1 \rightarrow 2$ and $3 \rightarrow 4$ do not belong necessarily to the same component of L ; in general the representations of G assigned to $1 \rightarrow 2$ and $3 \rightarrow 4$ are different and have been denoted in (3.15) by R and R' respectively. At this point we would like to relate the second term on the R.H.S. of eq.(3.15) with $\langle W(L_0) \rangle$. The simplest case

in which this can be achieved is when a Fierz identity for the sum $\sum_a R_{ij}^a R_{kl}^a$ exists. Taking $G = SU(N)$ and $R^a = R'^a = T^a$ one has

$$\sum_a T_{ij}^a T_{kl}^a = \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2N} \delta_{ij} \delta_{kl} \quad (3.16)$$

and in this case eq.(3.15) takes the form

$$\langle W(L_+) \rangle = \left(1 + i \frac{2\pi}{k} \frac{1}{N} \right) \langle W(L_-) \rangle - i \frac{2\pi}{k} \langle W(L_0) \rangle \quad (3.17)$$

Since $\langle W(L_+) \rangle$ differs from $\langle W(L_-) \rangle$ by $O(\frac{1}{k})$, the eq.(3.17) can be rewritten as

$$\beta \langle W(L_+) \rangle - \beta^{-1} \langle W(L_-) \rangle = z \langle W(L_0) \rangle \quad , \quad (3.18)$$

where

$$\beta = 1 - i \frac{2\pi}{k} \frac{1}{2N} + O\left(\frac{1}{k^2}\right) \quad , \quad (3.19)$$

and

$$z = -i \frac{2\pi}{k} + O\left(\frac{1}{k^2}\right) \quad . \quad (3.20)$$

As before, the $O(\frac{1}{k^2})$ terms in (3.19,20) take into account higher order corrections.

Eqs.(3.11,12) and (3.18) show that the expectation value $\langle W(L) \rangle$ of the Wilson line operator in the fundamental representation of $SU(N)$ obeys the generalized skein relations (2.15-17) of a S_L -polynomial. This is one of the main results of our investigation.

A first consistency check of eqs.(3.11,12) and (3.18) is the following. For Wilson line operators in the fundamental representation of $SU(N)$, eq.(3.18) implies (see Fig.8)

$$\beta \langle W(\widehat{L}_+) \rangle - \beta^{-1} \langle W(\widehat{L}_-) \rangle = z \langle W(\widehat{L}_0) W(C_0) \rangle \quad , \quad (3.21)$$

where C_0 is the unknotted knot. Since the R.H.S. of (3.21) is of order $\frac{1}{k}$ (see eq.(3.20)), we can consistently replace to this order $\langle W(\widehat{L}_0) W(C_0) \rangle$ by $N \langle W(\widehat{L}_0) \rangle$. Eq.(3.21) then takes the form (up to $O(\frac{1}{k^2})$ terms)

$$\left(1 - i \frac{2\pi}{k} \frac{1}{2N} \right) \langle W(\widehat{L}_+) \rangle - \left(1 + i \frac{2\pi}{k} \frac{1}{2N} \right) \langle W(\widehat{L}_-) \rangle = -i \frac{2\pi}{k} N \langle W(\widehat{L}_0) \rangle \quad . \quad (3.22)$$

On the other hand, eqs.(3.11-13) with

$$c_2(T) = \frac{N^2 - 1}{2N} \quad (3.23)$$

give

$$\langle W(\widehat{L}_{\pm}) \rangle = \left[1 \mp i \frac{2\pi}{k} \frac{N^2 - 1}{2N} + O\left(\frac{1}{k^2}\right) \right] \langle W(\widehat{L}_0) \rangle \quad (3.24)$$

By means of (3.24) one finds that eq.(3.22) is indeed satisfied, and this proves, to the first non-trivial order in $\frac{1}{k}$ the internal consistency of the skein relations for $\langle W(L) \rangle$.

A second non-trivial check concerns the value of α . To first order of perturbation theory in $\frac{1}{k}$ one finds for the Wilson line operator associated with the knot C and regularized by framing C with C_f

$$\langle W(C) \rangle_{\text{reg}} = (\dim R) \left[1 - i \frac{2\pi}{k} c_2(R) \ell(C, C_f) + O\left(\frac{1}{k^2}\right) \right] \quad (3.25)$$

Here $\ell(C, C_f)$ is the Gauss (linking) number

$$\ell(C, C_f) = \frac{1}{4\pi} \oint dx^\mu \oint dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (3.26)$$

Under a change of framing $C_f \rightarrow C'_f$ such that $\ell(C, C'_f) - \ell(C, C_f) = 1$ one has

$$\delta_f \langle W(C) \rangle_{\text{reg}} = \left[1 - i \frac{2\pi}{k} c_2(R) + O\left(\frac{1}{k^2}\right) \right] \langle W(C) \rangle_{\text{reg}} \quad (3.27)$$

Eq.(3.27) confirms both the value of α given by eq.(3.13) and its physical interpretation described before.

Summarizing, we have shown that $\langle W(L) \rangle$, suitably normalized, gives rise to an S_L -polynomial. In order to make contact with the skein relation given in [2], one has to note that, by using eq.(2.25), one can associate with $\alpha^{-w(L)} \langle W(L) \rangle$ a HOMFLY polynomial

$$P_L(\alpha\beta, z) = \frac{\alpha^{-w(L)} \langle W(L) \rangle}{\langle W(C_0) \rangle} \quad , \quad (3.28)$$

which satisfies the skein relation (2.8) with

$$t = \alpha\beta = 1 - i \frac{2\pi}{k} \frac{N}{2} + O\left(\frac{1}{k^2}\right) \quad (3.29)$$

and z given by (3.20).

It is quite remarkable that the values of t and z that we have found in the variational approach coincide (to the first non-trivial order in $\frac{1}{k}$) with those suggested by Witten [2] on the basis of conformal field theory arguments.

4. Conclusions and Outlook

In this paper we have studied the expectation values of the Wilson line operators in the three-dimensional Chern-Simons theory. On general grounds we have found that $\langle W(L) \rangle$ behaves as a regular isotopy invariant, that is as a twisted band. In perturbation theory this phenomenon manifests itself in the necessity of introducing a framing of L . At first sight one could believe that the framing is only a technical device for treating divergences in the perturbative expansion, without any impact on the properties of $W(L)$. Our variational approach suggests however that the regular isotopy invariance of $\langle W(L) \rangle$ is an intrinsic feature of the theory.

In the case of $SU(N)$ in the fundamental representation, we have shown that the expectation value $\langle W(L) \rangle$ satisfies the generalized skein relations of a S_L -polynomial. This fact connects the Chern-Simons theory with the known link invariants in knot theory. Another related aspect is that there is a braid group representation realized on the state Hilbert space of the Chern-Simons field theory. Let us briefly describe how this representation can be defined. We shall follow Witten ideas [2], but we shall make no use of the connections with the two-dimensional conformal field theory. Of course, we assume that the quantum Chern-Simons theory makes sense and that $\langle W(L) \rangle$ is finite for any link L .

Let us consider a link L with n components which, for simplicity, are taken to be unknotted. Like any expectation value in field theory, $\langle W(L) \rangle$ can be written as

$$\langle W(L) \rangle = \langle L_d | L_u \rangle, \quad (4.1)$$

where the states $|L_u\rangle$ and $|L_d\rangle$ belong to the Hilbert space $\mathcal{H}$ associated with the plane $t = 0$ shown in Fig.9. As is well known, the wave function corresponding to $|L_u\rangle$ ($|L_d\rangle$) can be obtained, for instance, by performing the path integral over the field configurations defined in the time interval $-\infty \leq t \leq 0$ ($0 \leq t \leq +\infty$) and with given boundary conditions at $t = 0$. Let σ_i be a generator of the braid group B_n . We define the action of σ_i on $|L_u\rangle$ as

$$\hat{\sigma}_i |L_u\rangle = |\sigma_i L_u\rangle, \quad (4.2)$$

where $|\sigma_i L_u\rangle$ is the state entering into eq.(4.1) when the upper part L_u of the link L has been modified according to the canonical action of σ_i on the ordered set of the n oriented legs associated with L_u . An example is shown in Fig.10. It is clear that eq.(4.2) defines a representation ρ of B_n .

It is important to note that, when the value of the coupling constant k is an integer, the Hilbert space $\mathcal{H}$ spanned by $\{|g L_u\rangle : g \in B_n\}$ is *finite dimensional*. This follows [2] from the properties of the Chern-Simons action and has important implications for the representation ρ . In fact, all the matrix elements of $\hat{\sigma}_i$ are determined by a *finite* set of expectation values of Wilson line operators. This means that $\hat{\sigma}_i$ is a bounded operator; clearly the same holds for the representative $\hat{g}$ of any element g of B_n .

There is also a well-defined $*$ -operation defined on $\hat{g}$, namely the Hermitian conjugation with respect to the scalar product in $\mathcal{H}$. Therefore, ρ is actually a (finite dimensional) C^* -algebra representation of B_n on $\mathcal{H}$.

At this level one can relate [2] the Chern-Simons theory with the two-dimensional conformal field theory, whose conformal blocks also realize [15] a representation of the braid group.

It is interesting to note that the existence of a non-trivial C^* -algebra representation of the braid group might give some restrictions on the possible values of the variables entering the link invariants obtained by the Chern-Simons theory. We give one example. For Wilson lines in the fundamental representation of $SU(2)$, the values of t and z that we have found, eqs.(3.20) and (3.29), are consistent (at first order in $\frac{1}{k}$) with the relation $z = \sqrt{t} - \frac{1}{\sqrt{t}}$ defining the Jones polynomial. Suppose now that this relation holds to all orders in $\frac{1}{k}$ so that $\langle W(L) \rangle$ gives rise, at the level of ambient isotopy (see eq.(3.28)), to the Jones polynomial. What are the possible values of t provided by the Chern-Simons theory? The skein relation for the Jones polynomial tells us that the C^* -algebra representation ρ of B_n defines [16] also a C^* -algebra representation of the Hecke algebra $H_n(t)$. If $\hat{\sigma}_i$ are normal matrices ($\hat{\sigma}_i^* \hat{\sigma}_i = \hat{\sigma}_i \hat{\sigma}_i^*$), according to [3,5,17] t must be real non-negative or a root of the unity. In view of eq.(3.29), the second possibility is realized in the Chern-Simons theory. This is also in agreement with the explicit results of [2].

In conclusion, the existence of a "geometric" quantum field theory associated with links suggests new promising developments for the knot theory itself! For instance one has new tools for facing the problem of a complete classification of links. As already mentioned, the HOMFLY polynomials P_L do not determine L uniquely. So the question is if the Chern-Simons theory provides new link invariants. We expect that this is indeed the case. The argument is as follows. The expectation value $\langle W(L) \rangle$ is a regular isotopy invariant for any choice of the representation R_i associated with the component C_i of L . The variational approach shows that for generic R_i , there is no skein relation based only on crossing changes in link diagrams. In looking for new invariants this feature is welcome because, as recently shown [18], any linear recursive relation based on crossing changes in link diagrams leads to a function of P_L . Therefore it is plausible that $\langle W_{R_1}(C_1) \cdots W_{R_n}(C_n) \rangle$ generalize the known link invariants.

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FIGURE CAPTIONS

Fig.1 Reidemeister moves of type I, II and III.

Fig.2 A twisted band is not invariant under Reidemeister moves of type I.

Fig.3 Configurations related by the skein identity.

Fig.4 Configurations related by eqs.(2.10) and (2.11).

Fig.5 An $F_{\mu\nu}$ -insertion at the point x on the Wilson line.

Fig.6 Planar deformation in which the tangent to the path in x belongs to the plane of the deformation.

Fig.7 How to relate overcrossing with undercrossing.

Fig.8 Diagrammatic representation of eq.(3.21).

Fig.9 Cutting the link L by the plane $t = 0$.

Fig.10 Action of σ on L_u .

Fig.1

Fig. 2

L_+

L_-

L_0

Fig. 3

$\hat{L}_+$

$\hat{L}_-$

$\hat{L}_0$

Fig. 4

(a)

(b)

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10