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TWO – DIMENSIONAL CHIRAL GAUGE THEORIES IN THE SCHRÖDINGER REPRESENTATION

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Abstract

We present a quantization, in the Schrödinger representation, of a two – dimensional theory consisting of an Abelian gauge field coupled to an arbitrary number of right – and left – handed fermions. A consistent and unitary quantum theory is obtained. The gauge symmetry is broken but Lorentz invariance survives quantization. The theory contains a series of chiral massless particles and a massive mode, which can be interpreted as collective excitations of the fields that we have in the initial Lagrangian. Our formalism is quite transparent and allows a clear understanding of the consistency of the model.

1. - INTRODUCTION

Chiral gauge theories raise many interesting problems in quantum field theory. Except when the matter content of the theory is carefully adjusted, the gauge symmetry does not survive quantization and the theory is anomalous. As gauge invariance usually guarantees unitarity and renormalizability, the consistency of such an anomalous gauge theory is threatened. For this reason, a gauge theory anomaly is commonly considered as a symptom of pathology of the theory. However, in two dimensions, Jackiw and Rajaraman[1] showed how one can obtain a consistent unitary theory from an anomalous theory. These authors analysed the theory of a fermion chirally coupled to a $U(1)$ gauge field (chiral Schwinger model). Their argument stems from the fact that once gauge invariance has been given up, one can exploit the ambiguity that appears in the regularization of the quantum theory. Since this pioneer work appeared, these models have been studied from many points of view[2–6]. Here we want to analyse these theories in yet another approach: the Schrödinger representation in the Hamiltonian formulation. This approach has been previously followed in ref.[7], from where we have borrowed many of our notations (although our conclusions are quite different, see below). We think that in our analysis a more transparent picture of the physical content of these models is obtained.

In the Schrödinger representation all the operators are taken time-independent and the dependence on time of the theory is carried exclusively by the states. The Hamiltonian of the quantum theory is obtained from its classical counterpart by means of a regularization. As we will see, in momentum space, the problem is reduced to finding the energy levels of a system in non-relativistic quantum mechanics. Once this Hamiltonian is diagonalized, the particle spectrum of the theory is clearly displayed. In the general Abelian case the spectrum will consist of a massive scalar excitation together with a series of massless particles. The physical excitations (i.e. those appearing in the spectrum) can be easily related to the elementary excitations that we had in the starting Hamiltonian(i.e. before diagonalization) : the former can be understood as collective modes of the latter. In this way a transparent description of the Hilbert space of the theory is obtained without introducing any unphysical fields(as is done in other approaches[6,8]). In our method, Lorentz invariance is not manifest.

This is of course the price that we must pay for the clarity of the procedure. We will check, however, that when the theory is appropriately regularized, the spectrum of the theory is perfectly relativistic. This is something that is not obvious from the beginning, and in fact it has been claimed several times in the literature that Lorentz invariance is lost in these kind of theories[7,9,10].

A great advantage of our approach is that it allows us to solve the theory when there is a different number of fermion fields of each chirality in the initial Lagrangian. Up to now these models have been solved using non-chiral bosonization, which requires an equal number of left- and right-handed fermionic species(although the coupling to the gauge field does not need to be the same for each chirality). In particular when only one chirality is coupled to the gauge field, the uncoupled fermions act as spectators in this non-chiral bosonization and it is not clear whether or not they are really essential in the subsequent diagonalization of the theory(i. e. in its consistency). On the contrary we are going to use a chiral bosonization (à la Floreanini-Jackiw[11]) which will reveal that the theory is also consistent when it has a left-right asymmetric matter content. In addition, in our analysis the chiral nature of the massless particles appearing in the spectrum of the theory will become evident.

In this work we are going to study a theory with $N_R(N_L)$ right-handed (left-handed) fermions coupled with charge $e_R(e_L)$ to an Abelian gauge field. The corresponding classical Lagrangian density is :

$$\mathcal{L} = \sum_{f=1}^{N_R} i \bar{\psi}_f (\partial_0 + \partial_1) \psi_f + \sum_{m=1}^{N_L} i \bar{\chi}_m (\partial_0 - \partial_1) \chi_m - e_R (A_0 + A_1) j_R - e_L (A_0 - A_1) j_L - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (1.1)$$

where the right- and left-handed currents are:

$$j_R(x) = \sum_{f=1}^{N_R} \bar{\psi}_f(x) \psi_f(x) \quad j_L(x) = \sum_{m=1}^{N_L} \bar{\chi}_m(x) \chi_m(x) \quad (1.2)$$

The canonical momenta associated to the fields appearing in $\mathcal{L}$ are:

$$\begin{aligned}
\Pi_{\psi_f}(x) &= i \psi_f^\dagger(x) & \Pi_{\chi_m}(x) &= i \chi_m^\dagger(x) \\
\Pi_{A_1}(x) &= \partial_0 A_1 - \partial_1 A_0 = E(x) & \Pi_{A_0}(x) &= 0
\end{aligned}
\tag{1.3}$$

and the corresponding classical Hamiltonian is :

$$H = \int dx \left[-i \sum_{f=1}^{N_R} \psi_f^\dagger \partial_1 \psi_f + i \sum_{m=1}^{N_L} \chi_m^\dagger \partial_1 \chi_m + e_R (A_0 + A_1) \dot{j}_R + e_L (A_0 - A_1) \dot{j}_L + \frac{E^2}{2} + E \partial_1 A_0 \right]
\tag{1.4}$$

As we have said above, we are going to quantize this system in the Schrödinger representation. This means that the operators will be time-independent and all the time-dependence of the theory will be concentrated on the states. The canonical commutation and anticommutation relations are :

$$\{ \psi_f(x), \psi_g^\dagger(y) \} = \delta_{fg} \delta(x-y)
\tag{1.5a}$$

$$\{ \chi_m(x), \chi_n^\dagger(y) \} = \delta_{mn} \delta(x-y)
\tag{1.5b}$$

$$[A_1(x), E(y)] = i \delta(x-y)
\tag{1.5c}$$

In the quantum theory the Hamiltonian H is ill-defined because it involves products of fields at the same point. A careful regularization of H is needed in order to obtain a well-defined operator from the formal expression given in Eq. (1.4). We shall perform this regularization in two steps. In section 2 we shall concentrate ourselves on the free fermionic part of H . In this section we also introduce many of the notations that will be used through the paper. The full Hamiltonian is analysed in section 3, whereas the energy levels of the theory are obtained in section 4. The relation between the physical and elementary excitations is established in section 5. The addition of a Wess-Zumino term and the corresponding solution of the model are studied in section 6. Finally in section 7 we summarize our results and draw some conclusions.

2. - FREE FERMIONS ON A CIRCLE

Let us consider a free fermionic theory, whose Hamiltonian H_0 is given by the first two terms of the left-hand side of Eq. (1.4). For simplicity we are going to work on a circle of length 2π with periodic boundary conditions. Let us first of all expand the fermionic fields $\psi_f(x)$ and $\chi_m(x)$ appearing in (1.4) in terms of its Fourier components :

$$\psi_f(x) = \frac{1}{\sqrt{2\pi}} \sum_{q \in \mathbb{Z}} \alpha_f(q) e^{iqx - |q|\epsilon} \quad (2.1a)$$

$$\chi_m(x) = \frac{1}{\sqrt{2\pi}} \sum_{q \in \mathbb{Z}} \beta_m(q) e^{iqx - |q|\epsilon} \quad (2.1b)$$

where $\epsilon > 0$ is a convergence factor which will be taken equal to zero at the end of any computation.

The fermionic creation and annihilation operators satisfy the anticommutation relations :

$$\{\alpha_f(p), \alpha_g^\dagger(q)\} = \delta_{fg} \delta_{pq} \quad (2.2a)$$

$$\{\beta_m(p), \beta_n^\dagger(q)\} = \delta_{mn} \delta_{pq} \quad (2.2b)$$

Other possible anticommutators between the α 's and β 's vanish. Using Eqs. (2.2) the canonical anticommutation relations (1.5a) and (1.5b) are easily reproduced. The fermionic Fock space is built by acting with the oscillator operators on a certain Fock vacuum. In order to define it, let us observe that e^{iqx} is an energy eigenstate of the free single-particle right-handed Hamiltonian $H^R = -i\partial_1$ (left-handed Hamiltonian $H^L = i\partial_1$) with energy q (respectively $-q$). The Fock vacuum is defined as the state in which all negative(and zero) energy levels are filled. It can be written as a tensor product :

$$|0\rangle = |0\rangle_L \otimes |0\rangle_R \quad (2.3)$$

where $|0\rangle_L$ and $|0\rangle_R$ are the states that satisfy :

$$\begin{aligned} \alpha_f^\dagger(q) |0\rangle_R &= 0 \quad \text{if } q \leq 0, & \alpha_f(q) |0\rangle_R &= 0 \quad \text{if } q > 0 \\ \beta_m^\dagger(q) |0\rangle_L &= 0 \quad \text{if } q \geq 0, & \beta_m(q) |0\rangle_L &= 0 \quad \text{if } q < 0 \end{aligned} \quad (2.4)$$

Our fermionic Fock space will consist of states in which almost all(i.e. all but a finite number) of the negative energy levels are filled and almost all of the positive energy levels are empty. These kinds of states will be called finitely excited states.

Associated with this Fock vacuum we can define a normal order that automatically will subtract from H the infinite energy of the Dirac sea :

$$:\alpha_f^\dagger(p)\alpha_f(p): = \begin{cases} \alpha_f^\dagger(p)\alpha_f(p) & \text{if } p > 0 \\ -\alpha_f(p)\alpha_f^\dagger(p) & \text{if } p \leq 0 \end{cases} \quad : \beta_m^\dagger(p)\beta_m(p) : = \begin{cases} \beta_m^\dagger(p)\beta_m(p) & \text{if } p < 0 \\ -\beta_m(p)\beta_m^\dagger(p) & \text{if } p \geq 0 \end{cases} \quad (2.5)$$

The normal-ordered product of any other pair of oscillator operators is just the ordinary product.

The free fermionic part of H can be now written in normal-ordered form :

$$H_0 = \int dx \left[-i \sum_{f=1}^{N_R} : \psi_f^\dagger \partial_t \psi_f : + i \sum_{m=1}^{N_L} : \chi_m^\dagger \partial_t \chi_m : \right] = \\ = \sum_{p>0} p \left[\sum_{f=1}^{N_R} \alpha_f^\dagger(p) \alpha_f(p) + \sum_{m=1}^{N_L} \beta_m(p) \beta_m^\dagger(p) \right] + \sum_{p<0} |p| \left[\sum_{f=1}^{N_R} \alpha_f(p) \alpha_f^\dagger(p) + \sum_{m=1}^{N_L} \beta_m^\dagger(p) \beta_m(p) \right] \quad (2.6)$$

We are interested in putting H_0 in terms of bosonic operators. With this purpose let us define the operators(for $p > 0$) :

$$a_R^f(p) = \frac{i}{\sqrt{p}} \sum_{q \in \mathbb{Z}} \alpha_f^\dagger(q-p) \alpha_f(q) \quad a_L^m(p) = \frac{i}{\sqrt{2}} \sum_{q \in \mathbb{Z}} \beta_m^\dagger(q+p) \beta_m(q) \quad (2.7)$$

Acting on our fermionic Fock space one can prove[12] that these operators have the following commutation relations :

$$[a_R^f(p), a_R^{g\dagger}(q)] = \delta_{pq} \delta_{fg} \quad [a_L^m(p), a_L^{n\dagger}(q)] = \delta_{pq} \delta_{mn} \quad (p, q > 0) \quad (2.8)$$

It is important to point out that although the definition of the a-operators involves an infinite sum, no regularization is needed in order to establish the commutation relations (2.8). This is so because acting on a finitely excited state the summation appearing in the definition of the a's can be reduced to

a sum over a finite range of integers. Using the anticommutation relations (2.2) the equations (2.8) follow easily. On the other hand the Hamiltonian H_0 can be put in terms of the a -operators as follows[12] :

$$H_0 = \sum_{p>0} \left[\sum_{f=1}^{N_R} a_R^{f\dagger}(p) a_R^f(p) + \sum_{m=1}^{N_L} a_L^{m\dagger}(p) a_L^m(p) \right] \quad (2.9)$$

It will be very useful to work with the Hermitian combinations of the a 's, defined as ($p > 0$) :

$$\begin{aligned} R_1^f &= \frac{i}{\sqrt{2}} (a_R^{f\dagger}(p) - a_R^f(p)) & R_2^f &= \frac{1}{\sqrt{2}} (a_R^{f\dagger}(p) + a_R^f(p)) \\ L_1^m &= \frac{-i}{\sqrt{2}} (a_L^{m\dagger}(p) - a_L^m(p)) & L_2^m &= \frac{1}{\sqrt{2}} (a_L^{m\dagger}(p) + a_L^m(p)) \end{aligned} \quad (2.10)$$

which have the following commutators :

$$[R_i^f, R_j^g] = -i \varepsilon_{ij} \delta_{fg} \quad [L_i^m, L_j^n] = i \varepsilon_{ij} \delta_{mn} \quad (2.11)$$

(observe that in our notation we have suppressed the dependence on p in the L and R operators). It can be immediately checked that :

$$H_0 = \sum_{p>0} \frac{p}{2} \left[\sum_{f=1}^{N_R} (R_i^f)^2 + \sum_{m=1}^{N_L} (L_i^m)^2 \right] \quad (2.12)$$

where we have neglected an infinite constant (zero-point energy). The Hamiltonian H_0 can be also be written in terms of chiral bosonic fields, which are defined as follows :

$$\begin{aligned} \phi_R^f(x, t) &= \frac{1}{\sqrt{2\pi}} \sum_{p>0} \frac{1}{\sqrt{2p}} \left[a_R^f(p) e^{ip(x-t)} + h.c. \right] + \text{zero mode} \\ \phi_L^m(x, t) &= \frac{1}{\sqrt{2\pi}} \sum_{p>0} \frac{1}{\sqrt{2p}} \left[a_L^m(p) e^{-ip(x+t)} + h.c. \right] + \text{zero mode} \end{aligned} \quad (2.13)$$

where we have not explicitly written down the zero-modes of ϕ_L^f and ϕ_R^m (they will not play any relevant role in our analysis). Notice that in Eq.(2.13) we have written the expression of the chiral bosonic fields at any given time, although in our Hamiltonian formulation we shall only need them at $t=0$. In terms of the chiral scalar fields, H_0 can be written as :

$$H_0 = \int dx N_a \left[\sum_{f=1}^{N_R} (\partial_1 \phi_R^f)^2 + \sum_{m=1}^{N_L} (\partial_1 \phi_L^m)^2 \right] \quad (2.14)$$

where we have again neglected zero-point energy terms. In (2.16) N_a means normal ordering with respect to the a -oscillators. This Hamiltonian can be obtained from the Floreanini-Jackiw Lagrangian density[11] :

$$\mathcal{L}_0 = - \sum_{f=1}^{N_R} \partial_1 \phi_R^f (\partial_0 + \partial_1) \phi_R^f + \sum_{m=1}^{N_L} \partial_1 \phi_L^m (\partial_0 - \partial_1) \phi_L^m \quad (2.15)$$

3. - THE QUANTUM HAMILTONIAN

Let us now consider the regularization of the interaction terms in the Lagrangian (1.1) (third and fourth terms). It is clear that in order to regularize these terms we must carefully define the right- and left-handed currents. By normal-ordering the naive expressions (1.2), we obtain for $j_R(x)$ and $j_L(x)$:

$$\begin{aligned} j_R(x) &= \sum_{f=1}^{N_R} : \psi_f^\dagger(x) \psi_f(x) : + \langle 0 | j_R(x) | 0 \rangle \\ j_L(x) &= \sum_{m=1}^{N_L} : \chi_m^\dagger(x) \chi_m(x) : + \langle 0 | j_L(x) | 0 \rangle \end{aligned} \quad (3.1)$$

where we have allowed a non-vanishing expectation values of the currents in our Fock vacuum (2.3).

Using Eqs. (2.5) and (2.7), the normal-ordered parts of $j_R(x)$ and $j_L(x)$ can be put as :

$$\begin{aligned} \sum_{f=1}^{N_R} : \psi_f^\dagger(x) \psi_f(x) : &= \frac{\Omega_R}{2\pi} + \frac{i}{2\pi} \sum_{f=1}^{N_R} \sum_{p \neq 0} \sqrt{p} a_{R(f)}^\dagger e^{-ipx} \\ \sum_{m=1}^{N_L} : \chi_m^\dagger(x) \chi_m(x) : &= \frac{\Omega_L}{2\pi} - \frac{i}{2\pi} \sum_{m=1}^{N_L} \sum_{p \neq 0} \sqrt{p} a_{L(m)}^\dagger e^{-ipx} \end{aligned} \quad (3.2)$$

with :

$$\begin{aligned} \Omega_R &= \sum_{f=1}^{N_R} \left[\sum_{p > 0} \alpha_f^\dagger(p) \alpha_f(p) - \sum_{p < 0} \alpha_f(p) \alpha_f^\dagger(p) \right] \\ \Omega_L &= \sum_{m=1}^{N_L} \left[\sum_{p < 0} \beta_m^\dagger(p) \beta_m(p) - \sum_{p > 0} \beta_m(p) \beta_m^\dagger(p) \right] \end{aligned} \quad (3.3)$$

In order to compute the vacuum expectation values of j_R and j_L , let us point-split the naive definition (1.2) :

$$\begin{aligned}
\langle 0 | j_R(x) | 0 \rangle &= \frac{1}{2} \lim_{x \rightarrow y} \langle 0 | \exp \left[\frac{i e_R}{2} \int_y^x O^R(\xi) d\xi \right] \sum_{f=1}^{N_R} \psi_f^\dagger(x) \psi_f(y) + h.c. | 0 \rangle \\
\langle 0 | j_L(x) | 0 \rangle &= \frac{1}{2} \lim_{x \rightarrow y} \langle 0 | \exp \left[\frac{i e_L}{2} \int_y^x O^L(\xi) d\xi \right] \sum_{m=1}^{N_L} \chi_m^\dagger(x) \chi_m(y) + h.c. | 0 \rangle
\end{aligned}
\tag{3.4}$$

where O^R and O^L are operators that parametrize the regularization ambiguities. From Eqs.(2.1) and the definition of our Fock vacuum (Eqs. (2.4)) we have for $x \sim y$:

$$\begin{aligned}
\langle 0 | \psi_f^\dagger(x) \psi_g(y) | 0 \rangle &= \frac{i}{2n} \frac{\delta_{fg}}{x-y} & \langle 0 | \chi_m^\dagger(x) \chi_n(y) | 0 \rangle &= \frac{-i}{2n} \frac{\delta_{mn}}{x-y}
\end{aligned}
\tag{3.5}$$

and therefore :

$$\langle 0 | j_R(x) | 0 \rangle = - \frac{e_R N_R}{4n} O^R(x) \quad \langle 0 | j_L(x) | 0 \rangle = \frac{e_L N_L}{4n} O^L(x)
\tag{3.6}$$

In principle O^R and O^L are arbitrary operators constructed from the gauge field (which are obviously the only ones that do not act on the fermionic Fock vacuum). However not all the possible elections for these operators give rise to a physically acceptable theory. Let us consider regularizing operators that depend linearly on the components of the gauge field :

$$\begin{aligned}
O^R(x) &= c_0^R A_0(x) + c_1^R A_1(x) & O^L(x) &= c_0^L A_0(x) + c_1^L A_1(x)
\end{aligned}
\tag{3.7}$$

With this election only quadratic terms in the gauge field are induced in the Hamiltonian of the quantum theory. Using the results of sect. 2, we can now write the quantum version of the classical Hamiltonian (1.4). First of all we expand $A_0(x)$, $A_1(x)$ and $E(x)$ in its Fourier components :

$$A_0(x) = \frac{2}{\sqrt{2n}} \sum_{p>0} [\lambda_1 \cos px + \lambda_2 \sin px] + \lambda_0
\tag{3.8a}$$

$$A_1(x) = \frac{2}{\sqrt{2n}} \sum_{p>0} [\gamma_1 \cos px + \gamma_2 \sin px] + \gamma_0 \quad (3.8b)$$

$$E(x) = \frac{1}{\sqrt{2n}} \sum_{p>0} [\pi_1 \cos px + \pi_2 \sin px] + \frac{\pi_0}{2n} \quad (3.8c)$$

Notice that in our notation we do not display the p -dependence of λ_i, γ_i and π_i . In order to have the canonical commutation relation between the A_1 and E (Eq. (1.5c)), the γ and π operators must satisfy [13] :

$$[\gamma_i, \pi_j] = i \delta_{ij} \quad [\gamma_0, \pi_0] = i \quad (3.9)$$

Putting everything together, the quantum Hamiltonian can be written as a sum of two contributions corresponding to the zero and non-zero Fourier modes :

$$H = H_Z + \sum_{p>0} H_{nz}(p) \quad (3.10)$$

where :

$$H_Z = \frac{1}{4n} \pi_0^2 + \frac{e_L^2}{2} N_L (\lambda_0 - \gamma_0) (c_0^L \lambda_0 + c_1^L \gamma_0) - \\ - \frac{e_R^2}{2} N_R (\lambda_0 + \gamma_0) (c_0^R \lambda_0 + c_1^R \gamma_0) + e_R (\lambda_0 + \gamma_0) \Omega^R + e_L (\lambda_0 - \gamma_0) \Omega^L \quad (3.11)$$

and

$$H_{nz}(p) = \sum_{f=1}^{N_R} \left[\frac{p}{2} (R_i^f)^2 + e_R \sqrt{\frac{p}{n}} (\lambda_i + \gamma_i) R_i^f \right] + \sum_{m=1}^{N_L} \left[\frac{p}{2} (L_i^m)^2 + e_L \sqrt{\frac{p}{n}} (\lambda_i - \gamma_i) L_i^m \right] + \\ + \frac{1}{4} \pi_i^2 - p \varepsilon_{ij} \lambda_i \pi_j - \frac{e_R^2 N_R}{2n} [c_0^R \lambda_i^2 + c_1^R \gamma_i^2 + (c_1^R + c_0^R) \lambda_i \gamma_i] + \\ + \frac{e_L^2 N_L}{2n} [c_0^L \lambda_i^2 - c_1^L \gamma_i^2 + (c_1^L - c_0^L) \lambda_i \gamma_i] \quad (3.12)$$

We have reduced now the problem of solving the anomalous gauge theory to the problem in ordinary quantum mechanics of finding the energy levels of the Hamiltonian (3.10). For arbitrary values of the c -coefficients, the spectrum will not, in general, be relativistic invariant. In order to single out in a systematic way the values of the c 's that give rise to a Lorentz invariant theory, one can look at the Poincaré algebra of the Hamiltonian H , linear momentum P and Lorentz-boost generator M

($[M, H] = iP$, $[H, P] = 0$, $[M, P] = iH$). It can be easily checked[14] that this algebra only closes if the following equations are satisfied :

$$c_0^R + c_1^R = -1 \quad c_0^L - c_1^L = 1 \quad (3.13)$$

Furthermore we shall explicitly prove in the next section that a relativistic dispersion relation for the energy levels is obtained when Eqs. (3.13) are fulfilled. As we do not want to give up Lorentz invariance in our theory, we shall restrict ourselves from now on to the case in which the c 's verify Eqs.(3.13). Accordingly, let us introduce the parametrization :

$$\begin{aligned} c_0^R &= -\frac{1-\alpha^R}{2} & c_1^R &= -\frac{1+\alpha^R}{2} \\ c_0^L &= \frac{1-\alpha^L}{2} & c_1^L &= -\frac{1+\alpha^L}{2} \end{aligned} \quad (3.14)$$

Actually only a particular function α of α^L and α^R will appear in our solution. If we define

$$e^2 = e_R^2 N_R + e_L^2 N_L \quad (3.15)$$

then α is given by :

$$\alpha = \frac{e_R^2 N_R \alpha^R + e_L^2 N_L \alpha^L}{e^2} \quad (3.16)$$

The Fourier components of $A_0(x)$ (λ_0 and λ_i) must be interpreted as Lagrange multipliers. Remembering that the canonical conjugate momentum of $A_0(x)$ is zero, we see that the equations of motion of λ_0 and λ_i are :

$$\frac{\partial H_Z}{\partial \lambda_0} = 0 \quad \frac{\partial H_{\eta Z}}{\partial \lambda_i} = 0 \quad (3.17)$$

For $\alpha \neq 1$ Eqs. (3.17) permit the computation of λ_0 and λ_i in terms of the other variables appearing in H :

$$\lambda_0 = \frac{2}{e^2(\alpha-1)} \left[\frac{e_R^2 N_R - e_L^2 N_L}{2} \gamma_0 + e_R \Omega^R + e_L \Omega^L \right] \quad (3.18a)$$

$$\lambda_i = \frac{2n}{e^2(\alpha-1)} \left[\sqrt{\frac{p}{n}} \left(e_R \sum_{f=1}^{N_R} R_i^f + e_L \sum_{m=1}^{N_L} L_i^m \right) + \frac{e_R^2 N_R - e_L^2 N_L}{2n} \gamma_i - p \varepsilon_{ij} \eta_j \right] \quad (3.18b)$$

In this point it is very interesting to compare what we have done up to now with what we would do in order to solve a vector gauge theory. In the vector case, it is possible to maintain gauge invariance at the quantum level and therefore one is allowed to use up all the gauge freedom. In particular one can fix a gauge. Usually the gauge $A_0(x)=0$ (i.e. $\lambda_0=\lambda_1=0$) is chosen. One could also try to follow the same steps in our chiral case, namely one could set $A_0=0$ from the beginning and see what happens. This is the point of view of Ref.[7] where the case $N_L=1$, $N_R=e_R=0$ has been studied. The analysis performed in this reference shows that although one can define a generator for the gauge transformations that commutes with the Hamiltonian and the theory is unitary, Lorentz invariance is lost. So one must definitively abandon gauge invariance if one only accepts relativistic invariant theories. On the other hand in an anomalous theory, one cannot fix a gauge to quantize the theory since in so doing degrees of freedom may be lost. This is why in our case $A_0(x)$ cannot be fixed to an arbitrary value, but instead is given by Eqs.(3.18) (see however section 6).

4. - THE SPECTRUM

In this section we shall obtain the energy levels of the quantum Hamiltonian H , when λ_0 and λ_1 are given by the equations (3.18). Consider first of all the zero-mode Hamiltonian H_z . It can be easily verified that Ω^R and Ω^L commute with everything else in the Hamiltonian. So they can be considered as constants and in fact put equal to zero without loss of generality (they give rise to linear terms in an harmonic oscillator potential, which only produce a global shift of the energy levels). Substituting Eq. (3.18a) (with $\Omega^R=\Omega^L=0$) in H_z , we obtain :

$$H_z = \frac{1}{4\pi} \pi_0^2 + \frac{e^4 \alpha^2 - 4 N_L N_R e_L^2 e_R^2}{4 e^2 (\alpha - 1)} \gamma_0^2 \quad (4.1)$$

This Hamiltonian corresponds to an harmonic oscillator of frequency M , with

$$M^2 = \frac{e^4 \alpha^2 - 4 N_L N_R e_L^2 e_R^2}{4 \pi e^2 (\alpha - 1)} \quad (4.2)$$

We observe that M^2 is positive for $\alpha > 1$. In the following we shall restrict ourselves to this range of the parameter α . Defining

$$C = \frac{1}{\sqrt{2} M} \left[\sqrt{2} \pi i M \gamma_0 - \frac{\pi_0}{\sqrt{2} \pi} \right] \quad (4.3)$$

H_Z can be put as :

$$H_Z = M \left(C^\dagger C + \frac{1}{2} \right) \quad (4.4)$$

We thus conclude that the zero - mode spectrum corresponds to that of a theory with a massive boson of mass M , given by Eq.(4.2). Of course the zero - mode spectrum is insensitive to the massless excitations present in the theory. These (together with the massive one) will appear in the non - zero momentum modes, which now we consider. Using Eq. (3.18b), $H_{nz}(p)$ can be put as :

$$H_{nz}(p) = \sum_{f=1}^{N_R} \left[\frac{p}{2} (R_i^f)^2 + e_R \sqrt{\frac{p}{\pi}} \gamma_i R_i^f \right] + \sum_{m=1}^{N_L} \left[\frac{p}{2} (L_i^m)^2 - e_L \sqrt{\frac{p}{\pi}} \gamma_i L_i^m \right] + \frac{1}{4} \pi_i^2 + \frac{e^2}{4\pi} (1+\alpha) \gamma_i^2 + \frac{e^2(\alpha-1)}{4\pi} \lambda_i^2 \quad (4.5)$$

where λ_i must be understood as the dependent expression given in Eq. (3.18b). The amazing thing about the Hamiltonian (4.5) is that it can be diagonalized by means of a unitary transformation generated by the operator :

$$U = U_1(\lambda) U_2(\beta_L, \beta_R) U_1(\lambda) \quad (4.6)$$

where

$$U_1(\lambda) = \exp \left[\frac{i\lambda}{\sqrt{N_L N_R}} \sum_{f=1}^{N_R} \sum_{m=1}^{N_L} \epsilon_{ij} R_i^f L_j^m \right] \quad (4.7)$$

$$U_2(\beta_L, \beta_R) = \exp \left[\frac{i\beta_L}{\sqrt{N_L}} \sum_{m=1}^{N_L} L_i^m \pi_i + \frac{i\beta_R}{\sqrt{N_R}} \sum_{f=1}^{N_R} R_i^f \pi_i \right]$$

This operator U transforms the variables appearing in H_{nz} in the following way (π_i is not transformed by U):

$$\begin{aligned}\hat{R}_i^f &= U R_i^f U^\dagger = \\ &= R_i^f + \frac{\sinh 2\lambda}{\sqrt{N_L N_R}} \sum_{m=1}^{N_L} L_i^m + \frac{\cosh 2\lambda - 1}{N_R} \sum_{g=1}^{N_R} R_i^g + \frac{\beta_L \sinh \lambda - \beta_R \cosh \lambda}{\sqrt{N_R}} \varepsilon_{ij} \pi_j\end{aligned}\quad (4.8a)$$

$$\begin{aligned}\hat{L}_i^m &= U L_i^m U^\dagger = \\ &= L_i^m + \frac{\sinh 2\lambda}{\sqrt{N_L N_R}} \sum_{f=1}^{N_R} R_i^f + \frac{\cosh 2\lambda - 1}{N_L} \sum_{n=1}^{N_L} L_i^n + \frac{\beta_L \cosh \lambda - \beta_R \sinh \lambda}{\sqrt{N_L}} \varepsilon_{ij} \pi_j\end{aligned}\quad (4.8b)$$

$$\begin{aligned}\hat{\gamma}_i &= U \gamma_i U^\dagger = \\ &= \gamma_i + \frac{\beta_L \cosh \lambda + \beta_R \sinh \lambda}{\sqrt{N_L}} \sum_{m=1}^{N_L} L_i^m + \frac{\beta_R \cosh \lambda + \beta_L \sinh \lambda}{\sqrt{N_R}} \sum_{f=1}^{N_R} R_i^f + \frac{1}{2} (\beta_L^2 - \beta_R^2) \varepsilon_{ij} \pi_j\end{aligned}\quad (4.8c)$$

The values of the parameters λ, β_R and β_L that put H_{nz} in a diagonal form are the following. The parameter λ is given by :

$$\tanh 2\lambda = \frac{2 e_L e_R}{e^2 \alpha} \sqrt{N_L N_R} \quad (4.9)$$

whereas β_R and β_L are :

$$\begin{aligned}\beta_R &= \sqrt{\frac{P}{n}} \frac{2\pi \cosh 2\lambda}{e^2 \alpha} [\sqrt{N_R} e_R \cosh \lambda - \sqrt{N_L} e_L \sinh \lambda] \\ \beta_L &= \sqrt{\frac{P}{n}} \frac{2\pi \cosh 2\lambda}{e^2 \alpha} [\sqrt{N_R} e_R \sinh \lambda - \sqrt{N_L} e_L \cosh \lambda]\end{aligned}\quad (4.10)$$

After a long (and tedious) calculation one can check that H_{nz} can be put as follows in terms of the transformed variables :

$$H_{nz}(p) = \frac{P}{2} \left[\sum_{f=1}^{N_R} (\hat{R}_i^f)^2 + \sum_{m=1}^{N_L} (\hat{L}_i^m)^2 \right] + \frac{1}{4} \left[\frac{P^2}{M^2} + 1 \right] \pi_i^2 + M^2 (\hat{\gamma}_i)^2 \quad (4.11)$$

Here M is the same mass that appeared in the zero-mode Hamiltonian (Eq.(4.2)). In order to see more clearly the spectrum of H_{nz} , let us define the oscillator operators :

$$b_R^f(p) = \frac{1}{\sqrt{2}} (\hat{R}_2^f + i \hat{R}_1^f) \quad b_L^m(p) = \frac{1}{\sqrt{2}} (\hat{L}_2^m - i \hat{L}_1^m) \quad (4.12a)$$

$$c_p = \frac{iM}{\sqrt{2}E_p} (\hat{\gamma}_1 + i \hat{\gamma}_2) - \frac{1}{2M} \sqrt{\frac{E_p}{2}} (\pi_1 + i \pi_2)$$

$$c_{-p} = \frac{iM}{\sqrt{2}E_p} (\hat{\gamma}_1 - i \hat{\gamma}_2) - \frac{1}{2M} \sqrt{\frac{E_p}{2}} (\pi_1 - i \pi_2) \quad (4.12b)$$

where $p > 0$ and $E_p = \sqrt{p^2 + M^2}$. These operators satisfy the commutation relations :

$$[b_R^f(p), b_R^{g\dagger}(q)] = \delta_{pq} \delta_{fg} \quad [b_L^m(p), b_L^{n\dagger}(q)] = \delta_{pq} \delta_{mn} \quad (p, q > 0) \quad (4.13)$$

and

$$[c_p, c_q^\dagger] = \delta_{pq} \quad (p, q \neq 0) \quad (4.14)$$

Other commutators vanish. On the other hand :

$$H_{nz}(p) = p \left[\sum_{f=1}^{N_R} b_R^{f\dagger}(p) b_R^f(p) + \sum_{m=1}^{N_L} b_L^{m\dagger}(p) b_L^m(p) \right] + E_p (c_p^\dagger c_p + c_{-p}^\dagger c_{-p}) \quad (4.15)$$

We see that the $p \neq 0$ spectrum integrates perfectly well with the $p=0$ energy levels. In fact the total Hamiltonian obtained by substituting (4.4) and (4.15) into (3.10) is :

$$H = \sum_{p>0} p \left[\sum_{f=1}^{N_R} b_R^{f\dagger}(p) b_R^f(p) + \sum_{m=1}^{N_L} b_L^{m\dagger}(p) b_L^m(p) \right] + \sum_{p \in \mathbb{Z}} \sqrt{p^2 + M^2} c_p^\dagger c_p \quad (4.16)$$

where we have neglected zero-point energy terms and denoted c as $c_p=0$. We observe that for $\alpha > 1$ we have a perfectly relativistic spectrum consisting of a massive excitation (of mass M) along with N_R (N_L) right-handed (left-handed) massless modes. When one of the charges vanishes (say e_L) and $N_R = 1$, we obtain the same spectrum as in the work of Jackiw and Rajaraman[1]. So our parameter α can be identified with a generalization of theirs. The massive spectrum for $N_L = N_R = 1$ has been also obtained in Refs[6,8,15]. These results agree with equation (4.2) when the same notations are used.

Notice that in the massless mode contribution to H , only the half($p > 0$) of the modes appear, so these modes are really chiral. When $N_L \neq N_R$ they cannot be combined to form a set of non-chiral excitations.

5. - ELEMENTARY AND PHYSICAL EXCITATIONS

One might wonder what the relation is between the creation and annihilation operators of the physical particles(i.e. the ones that appear in the spectrum of the theory) and the operators corresponding to the elementary excitations(i. e. those introduced before the diagonalization of H , defined in Eqs. (2.7)).

Using Eqs. (4.8) and (4.12), it is possible to establish the relations :

$$\begin{aligned} a_R^f(p) &= b_R^f(p) + \frac{\cosh 2\lambda - 1}{N_R} \sum_{g=1}^{N_R} b_R^g(p) - \frac{\sinh 2\lambda}{\sqrt{N_L N_R}} \sum_{m=1}^{N_L} b_L^m(p) + \sqrt{\frac{P}{E_p}} \frac{g_R}{e\sqrt{\alpha-1}} (c_p^\dagger + c_{-p}) \\ a_L^m(p) &= b_L^m(p) + \frac{\cosh 2\lambda - 1}{N_L} \sum_{n=1}^{N_L} b_L^n(p) - \frac{\sinh 2\lambda}{\sqrt{N_L N_R}} \sum_{f=1}^{N_R} b_R^f(p) + \sqrt{\frac{P}{E_p}} \frac{g_L}{e\sqrt{\alpha-1}} (c_p + c_{-p}^\dagger) \end{aligned} \quad (5.1)$$

In (5.1) g_R and g_L are the following linear combinations of the charges :

$$\begin{aligned} g_R &= e_R \cosh 2\lambda - e_L \sqrt{\frac{N_L}{N_R}} \sinh 2\lambda \\ g_L &= e_L \cosh 2\lambda - e_R \sqrt{\frac{N_R}{N_L}} \sinh 2\lambda \end{aligned} \quad (5.2)$$

Let us introduce the fields for the physical excitations. The chiral bosonic fields corresponding to the right - and left - handed massless particles are defined in the same way as in Eq.(2.13) :

$$\eta_R^f(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{p>0} \frac{1}{\sqrt{2p}} \left[b_R^f(p) e^{ip(x-t)} + h.c. \right] + \text{zero mode} \quad (5.3a)$$

$$\eta_L^m(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{p>0} \frac{1}{\sqrt{2p}} \left[b_L^m(p) e^{-ip(x+t)} + h.c. \right] + \text{zero mode} \quad (5.3b)$$

and the massive field is :

$$\rho(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{p \neq 0} \frac{1}{\sqrt{2E_p}} \left[c_p e^{-ipx - iE_p t} + h.c. \right] + \text{zero mode} \quad (5.3c)$$

Observe that in Eqs.(5.3) we have restored the time – dependence of the fields (we'll soon come back to our $t=0$ fields). Treating properly the zero modes of $\rho(x,t)$, it can be shown that H (as given in Eq.(4.16)) can be obtained from the Lagrangian density :

$$\mathcal{L} = - \sum_{f=1}^{N_R} \partial_\mu \eta_R^f (\partial_0 + \partial_1) \eta_R^f + \sum_{m=1}^{N_L} \partial_\mu \eta_L^m (\partial_0 - \partial_1) \eta_L^m + \frac{1}{2} (\partial_\mu \rho)^2 + M^2 \rho^2 \quad (5.4)$$

The relations (5.1) between the elementary and physical creation and annihilation operators can be translated to a relation between the corresponding $t=0$ fields. Inserting Eqs. (5.1) into Eq. (2.13) and taking into account Eqs. (5.3), one easily obtains :

$$\phi_R^f(x) = \eta_R^f(x) + \frac{\cosh 2\lambda - 1}{N_R} \sum_{g=1}^{N_R} \eta_R^g(x) - \frac{\sinh 2\lambda}{\sqrt{N_L N_R}} \sum_{m=1}^{N_L} \eta_L^m(x) + \frac{g_R}{e\sqrt{\alpha-1}} \rho(x) \quad (5.5a)$$

$$\phi_L^m(x) = \eta_L^m(x) + \frac{\cosh 2\lambda - 1}{N_L} \sum_{n=1}^{N_L} \eta_L^n(x) - \frac{\sinh 2\lambda}{\sqrt{N_L N_R}} \sum_{f=1}^{N_R} \eta_R^f(x) + \frac{g_L}{e\sqrt{\alpha-1}} \rho(x) \quad (5.5b)$$

The gauge field $A_\mu(x)$ can also be put as a function of the η and ρ fields. Using Eqs. (4.8c),(4.12),(5.3) and (3.18) one has[16] :

$$A_\mu(x) = \frac{1}{M} \left[\epsilon_{\mu\nu} \partial^\nu \rho + \frac{e_R^2 N_R - e_L^2 N_L}{e^2(\alpha-1)} \partial_\mu \rho + \frac{2}{e\sqrt{\alpha-1}} \left(e_R \sum_{f=1}^{N_R} \partial_\mu \eta_R^f - e_L \sum_{m=1}^{N_L} \partial_\mu \eta_L^m \right) \right] \quad (5.6)$$

In Eq.(5.6) the time derivatives of the fields at $t=0$ are defined through its Fourier expansion, which is immediately obtained from Eqs.(5.3).

Let us finish this section with several applications of the formalism that we have developed. Consider the case $N_L = N_R = 1$ and imagine that we add a mass to the elementary fermions of the Lagrangian (1.1). In other words we add to $\mathcal{L}$ a term :

$$m_0 (\chi^\dagger \psi + \psi^\dagger \chi) \quad (5.7)$$

(as we have now only one flavour for each chirality, our fields will not carry any flavour index). We want to investigate the effect of this mass term in our solution of the theory. Let us first point out that the fermionic fields χ and ψ can be related to the bosonic fields ϕ_R and ϕ_L by means of the Mandelstam equation[17]:

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2\pi}} N_a \left(e^{2i\sqrt{\pi} \phi_R(x)} \right) U_R \\ \chi(x) &= \frac{1}{\sqrt{2\pi}} N_a \left(e^{-2i\sqrt{\pi} \phi_L(x)} \right) U_L \end{aligned} \quad (5.8)$$

where U_R and U_L are Klein factors that ensure the correct anticommutators. The standard bosonization prescription[18] for the fermionic bilinears gives :

$$\chi^\dagger \psi + \psi^\dagger \chi \Rightarrow N_a \left[\cos(2\sqrt{\pi} \phi(x)) \right] \quad (5.9)$$

here $\phi(x) = \phi_R(x) + \phi_L(x)$. Using Eqs. (5.5) it is straightforward to see that the term that must be added to the Lagrangian (5.4) is :

$$m N_{(b,c)} \left\{ \cos \left[2\sqrt{\pi} \sigma \left(\eta(x) + \frac{e_L + e_R}{e\sqrt{\alpha-1}} g(x) \right) \right] \right\} \quad (5.10)$$

with $\eta(x) = \eta_R(x) + \eta_L(x)$ and $\sigma = \cosh 2\lambda - \sinh 2\lambda$. Observe that the mass m gets renormalized with respect to its bare value m_0 . So now the massive and massless fields are no longer uncoupled. The interaction term (5.10) was previously found by using other approaches in refs.[15,19] where its influence on the consistency of the theory was also studied.

Assume now that in this one-flavour case we put $e_R = 0$ (this is the chiral Schwinger model studied in [1]). In this case the above expressions are greatly simplified (now we have $\lambda = g_R = 0$ and $g_L = e_L = e$). In particular :

$$\phi_L(x) = \eta_L(x) + \frac{1}{\sqrt{\alpha-1}} \varphi(x) \quad (5.11)$$

Substituting (5.11) in (5.8), we see that the left-handed fermionic field χ is :

$$\chi(x) = Z_2^{-1/2} \frac{1}{\sqrt{2\pi}} N_b \left(e^{-2i\sqrt{\pi}\eta_L(x)} \right) \mathbb{1}_L N_c \left(e^{-i\sqrt{\frac{4\pi}{\alpha-1}}\varphi(x)} \right) \quad (5.12)$$

where Z_2 is an (infinite) wave function renormalization constant, which appears as a consequence of the difference between the normal-ordering with respect to the a and (b,c) oscillators. Eq.(5.12) can be simplified if one realizes that the field :

$$\tilde{\chi}(x) = \frac{1}{\sqrt{2\pi}} N_b \left(e^{-2i\sqrt{\pi}\eta_L(x)} \right) \mathbb{1}_L \quad (5.13)$$

is the fermionization of the bosonic chiral field $\eta_L(x)$. So one has :

$$Z_2^{1/2} \chi(x) = N_c \left(e^{-i\sqrt{\frac{4\pi}{\alpha-1}}\varphi(x)} \right) \tilde{\chi}(x) \quad (5.14)$$

Eq. (5.14) is very useful when one wants to compute the correlation functions of the theory. To this end let us observe that the true physical vacuum of the theory is defined as the state that satisfies :

$$c_p |phys\rangle = 0 \quad (p \geq 0) \quad , \quad b_L(p) |phys\rangle = b_R(p) |phys\rangle = 0 \quad (p > 0) \quad (5.15)$$

Using this definition and Eq. (5.14), the computation of the two-point correlation function of $\chi(x)$ is a simple application of the Wick theorem. The result is :

$$Z_2 \langle phys | \chi^\dagger(x) \chi(y) | phys \rangle = S_L(x-y) e^{\frac{4\pi}{\alpha-1} \Delta(x-y, M)} \quad (5.16)$$

where $S_L(x)(\Delta(x,M))$ is the two-point function for the free left-handed fermions (free massive scalars, respectively). This result agrees with that of Refs.[3,6].

6. - WESS - ZUMINO TERM

A crucial point in the approach we have followed up to now is the fact that $A_0(x)$ can be eliminated from H by using its equation of motion(Eq. (3.18)). This was of course possible because gauge invariance is broken and the system has second class constraints. However it is well known that we can convert these second class constraints into first class constraints by adding a Wess - Zumino term to the action[4,5,20]. In this formulation gauge invariance is recovered, although it is not the same that we had at the classical level because now it involves an extra field. It is quite instructive to analyse how the theory is solved in this gauge invariant formulation. We shall address this question in the present section, where for simplicity we shall restrict ourselves to the case in which we have a single left-handed fermion(i.e. when $N_L = 1$, $N_R = e_R = 0$ and $e_L = e$).

The Lagrangian density is now[4,5]:

$$\mathcal{L}^{WZ} = \mathcal{L} + \frac{1}{4\pi} \left[\frac{1}{2}(\alpha-1) \partial_\mu \theta \partial^\mu \theta - e \partial_\mu \theta \left((\alpha-1) A^\mu + \epsilon^{\mu\nu} A_\nu \right) \right] \quad (6.1)$$

where $\theta(x)$ is the Wess - Zumino field and $\mathcal{L}$ is given in Eq.(1.1). The canonical momentum of the θ field is :

$$\pi_\theta(x) = \frac{\alpha-1}{4\pi} \partial_0 \theta - \frac{e}{4\pi} \left[(\alpha-1) A_0 + A_1 \right] \quad (6.2)$$

and the canonical commutation relation is :

$$\left[\theta(x), \pi_\theta(y) \right] = i \delta(x-y) \quad (6.3)$$

The classical Hamiltonian is now given by :

$$\begin{aligned} H^{WZ} = H + \int dx \left[\frac{2\pi}{\alpha-1} \pi_\theta^2 + \frac{\alpha-1}{8\pi} (\partial_1 \theta)^2 + e \pi_\theta \left(A_0 + \frac{A_1}{\alpha-1} \right) + \right. \\ \left. + \frac{e^2}{8\pi(\alpha-1)} \left[(\alpha-1) A_0 + A_1 \right]^2 - \frac{e}{4\pi} \partial_1 \theta \left[(\alpha-1) A_1 + A_0 \right] \right] \end{aligned} \quad (6.4)$$

with H given by Eq.(1.4). As we have done in the previous sections, we are going to work in momentum space. Let us represent $\theta(x)$ and $\pi_\theta(x)$ in the following way :

$$\Theta(x) = 2 \sqrt{\frac{2}{\alpha-1}} \sum_{p>0} \frac{1}{\sqrt{p}} [Q_1 \cos px + Q_2 \sin px] + Q_0 \quad (6.5a)$$

$$\Pi_\theta(x) = \frac{1}{2\pi} \sqrt{\frac{\alpha-1}{2}} \sum_{p>0} \sqrt{p} [P_1 \cos px + P_2 \sin px] + \frac{P_0}{2\pi} \quad (6.5b)$$

(the factors appearing in front of the p summation in Eqs. (6.5a) and (6.5b) have been chosen for convenience). In order to fulfil Eq.(6.3), the following commutation relations must be satisfied :

$$[Q_i, P_j] = i \delta_{ij} \quad [Q_0, P_0] = i \quad (6.6)$$

(other commutators vanish). Actually it will be more convenient to use the following combinations of Q_i and P_i :

$$J_i = \frac{1}{2} P_i + \epsilon_{ij} Q_j \quad G_i = \frac{1}{2} P_i - \epsilon_{ij} Q_j \quad (6.7)$$

which have the following commutators :

$$[J_i, J_j] = i \epsilon_{ij} \quad [J_i, G_j] = 0 \quad [G_i, G_j] = -i \epsilon_{ij} \quad (6.8)$$

Regulating H as we did in sections 2 and 3, and adding the new contributions appearing in Eq.(6.4), one obtains :

$$H^{WZ} = H_z^{WZ} + \sum_{p>0} H_{nz}^{WZ}(p) \quad (6.9)$$

where the zero - mode part of H^{WZ} is :

$$H_z^{WZ} = \frac{1}{4\pi} \Pi_0^2 + \frac{e^2 \alpha^2}{4(\alpha-1)} \gamma_0^2 + e(\lambda_0 - \gamma_0) \Omega^L + \frac{P_0^2}{2\pi} + e P_0 \lambda_0 \quad (6.10)$$

and the $p \neq 0$ Hamiltonian is :

$$\begin{aligned}
H_{nz}^{WZ}(p) = & \frac{P}{2} [L_i^2 + J_i^2 + G_i^2] + \frac{1}{4} \pi_i^2 + e \sqrt{\frac{P}{n}} \lambda_i L_i - p \epsilon_{ij} \lambda_i \pi_j + \\
& + \frac{e}{2} \sqrt{\frac{P}{n}} \frac{\alpha-2}{\sqrt{\alpha-1}} J_i (\lambda_i - \gamma_i) + \frac{e}{2} \sqrt{\frac{P}{n}} \frac{\alpha}{\sqrt{\alpha-1}} G_i (\lambda_i + \gamma_i) + \frac{e^2 \alpha^2}{4n(\alpha-1)} \gamma_i^2
\end{aligned} \tag{6.11}$$

Observe that in Eq.(6.10)(Eq.(6.11)) we do not have a λ_0^2 (λ_i^2 , respectively) term. This is due to the fact that the constraints are now first class. The equation of motion of A_0 gives :

$$P_0 = -\Omega^L \tag{6.12a}$$

$$\pi_i = -\frac{e}{\sqrt{n}p} \epsilon_{ij} \left[L_j + \frac{\alpha-2}{2\sqrt{\alpha-1}} J_j + \frac{\alpha}{2\sqrt{\alpha-1}} G_j \right] \tag{6.12b}$$

We see that now we do not obtain A_0 from its equation of motion as in section 3. Instead we have a constraint in the phase space of the other variables appearing in the Hamiltonian : the Gauss' Law constraint. However as we now have gauge invariance, we can fix the gauge $A_0(x)=0$ (i. e. $\lambda_0=\lambda_i=0$). In so doing, we have not exhausted all the gauge freedom that we have in the theory. We can still perform time-independent gauge transformations. In particular, as we are working on a circle, this will allow us to gauge away the non-zero momentum modes γ_i of the spatial component of the gauge field. This is so because any combination of $\cos px$ and $\sin px$ can be put as a spatial derivative. However when taking $\gamma_i=0$ we must be very careful : its canonical conjugate momentum π_i cannot be put equal to zero because we must fulfil the Gauss' Law constraint (6.12b). Putting $\gamma_i=\lambda_i=0$ in Eq.(6.11) only the first four terms survive. Using Eq.(6.12b) we obtain :

$$\begin{aligned}
H_{nz}^{WZ}(p) = & \left(\frac{P}{2} + \frac{e^2}{4np} \right) L_i^2 + \left(\frac{P}{2} + \frac{e^2(\alpha-2)^2}{16np(\alpha-1)} \right) J_i^2 + \left(\frac{P}{2} + \frac{e^2\alpha^2}{16np(\alpha-1)} \right) G_i^2 + \\
& + \frac{e^2}{4np\sqrt{\alpha-1}} \left[(\alpha-2) J_i + \alpha G_i \right] L_i + \frac{e^2}{8np} \frac{\alpha(\alpha-2)}{\alpha-1} J_i G_i
\end{aligned} \tag{6.13}$$

Let us now look at the spectrum. Consider first the zero-mode Hamiltonian. As P_0 and Ω^L commute with everything else in H_Z^{WZ} and its contribution is linear in γ_0 , they can be put equal to zero without loss of generality (see section 4). Doing that, the remaining expression for H_Z^{WZ} is just a harmonic oscillator Hamiltonian with frequency M , being $M^2 = e^2 \alpha^2 / 4\pi(\alpha - 1)$. This is precisely the value of the mass given by Eq.(4.2) for the case we are considering.

The $p \neq 0$ Hamiltonian (6.13) can be diagonalized by means of the unitary transformation :

$$T(\delta, \Delta) = e^{i\delta \varepsilon_{ij} G_i J_j} e^{i\Delta \varepsilon_{ij} L_i J_j} \quad (6.14)$$

which transforms L_i , J_i and G_i as follows :

$$\hat{L}_i = T L_i T^\dagger = \cos \Delta L_i - \sin \Delta \cosh \delta J_i - \sin \Delta \sinh \delta G_i \quad (6.15a)$$

$$\hat{J}_i = T J_i T^\dagger = \cos \Delta \cosh \delta J_i + \cos \Delta \sinh \delta G_i + \sin \Delta L_i \quad (6.15b)$$

$$\hat{G}_i = T G_i T^\dagger = \cosh \delta G_i + \sinh \delta J_i \quad (6.15c)$$

(6.15c)

when

$$\cos \Delta = \frac{\alpha - 2}{\alpha} \quad \tanh 2\delta = \frac{M^2}{M^2 + 2p^2} \quad (6.16)$$

the Hamiltonian (6.13) is expressed in terms of the transformed operators as :

$$H_{nz}^{WZ}(p) = \frac{p^2}{2} \hat{L}_i^2 + \frac{\sqrt{p^2 + M^2}}{2} [\hat{J}_i^2 + \hat{G}_i^2] \quad (6.17)$$

If we now define :

$$c_p = \frac{1}{\sqrt{2}} (\hat{J}_2 - i \hat{J}_1) \quad c_{-p} = \frac{1}{\sqrt{2}} (\hat{G}_2 + i \hat{G}_1) \quad (c_p > 0) \quad (6.18)$$

then H^{WZ} is given by the left-hand side of Eq.(4.16) (for $N_L = 1$ and $N_R = 0$), where $b_L(p)$ is defined as in (4.12). We thus see that the spectrum of the theory with Wess-Zumino term coincides with the one obtained in section 4. Observe that now we have introduced a new field but also a gauge invariance that did not exist before and therefore the number of independent degrees of freedom does not change. In fact, in the gauge that we have chosen, the degrees of freedom that are eaten by the gauge invariance are transferred to the Wess-Zumino field. We could also relate in this formulation the fields appearing in the initial Lagrangian with those corresponding to the physical particles. This relation now depends on the gauge chosen and in fact the formulation without Wess-Zumino term can be regarded as the gauge-invariant formulation in the gauge in which $\theta(x) = 0$ [5].

7. - SUMMARY AND CONCLUSIONS

We have rigorously quantized in the Schrödinger representation a general two-dimensional chiral gauge theory with an arbitrary number of right and left movers. We have found that the theory can be regulated in such a way that, although gauge invariance is broken, one has a sensible, Lorentz-invariant, quantum field theory. The spectrum of the model contains a series of chiral massless modes along with a massive scalar particle. We have related these excitations appearing in the spectrum to the particles that we had in our initial Lagrangian and we have worked out several applications of our formalism. We have also solved the theory when a Wess-Zumino term is added to the original action in order to restore gauge invariance. We have checked that the spectrum of the theory is not modified with the introduction of the Wess-Zumino term.

The approach we have followed to solve the theory is clearly understood in physical terms when the theory is formulated as a system of coupled oscillators. Indeed, our quantum Hamiltonian (Eq.(3.10)) can be written in terms of interacting bosonic oscillator operators, constructed from

the matter fields (defined in Eq.(2.7)) and the gauge field. The latter can be defined from the Fourier components of $A_1(x)$ and $E(x)$ as we did in Eq.(4.3) (for the zero modes) and (4.12b) (changing $\hat{\gamma}_1$ by γ_1). The election of a set of annihilation operators defines a vacuum and a set of excitations, obtained by acting on it with the corresponding creation operators. What happens in our case is that the vacuum that we start with is not the true vacuum of the theory and its elementary excitations are not the eigenstates of the Hamiltonian. Formulated in these terms the problem is very similar to the one that appears in some condensed matter systems and in fact the unitary transformation U that diagonalizes the Hamiltonian(Eq.4.6) is nothing but a Bogoliubov transformation (see Eq.(5.1)). In this sense, the true vacuum of the theory (defined in Eq.(5.15)) and the physical excitations can be understood as collective modes of the elementary oscillators. This same type of interpretation holds when a Wess – Zumino term is added to the action, but now the analysis depends on the gauge chosen.

Finally, let us point out that in principle our method could be extended to the theory with non – Abelian gauge group. However in this case the gauge covariant derivative will induce non – quadratic terms in the action, which give rise to anharmonic terms when the theory is formulated in terms of oscillators. For this reason we do not expect that the model could be solved exactly and one would need to appeal to the standard perturbative tools in order to shed light on the consistency of the theory.

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References

- [1] R. Jackiw and R. Rajaraman— Phys. Rev. Lett. 54(1985) 1219.
- [2] R. Rajaraman— Phys. Lett. B154(1985)305; Phys. Lett. 162B(1985)148.
- [3] H. Girotti, H.J. Rothe and K.D. Rothe— Phys. Rev. D33(1986)514; Phys. Rev. D34(1986)592.
- [4] K. Harada and I. Tsutsui— Phys. Lett. B183(1987)311.
- [5] N.K. Falck and G.Kramer— Ann. Phys.(NY)176(1987)330.
- [6] D. Boyanovsky— Nucl. Phys. B294(1987)223; D. Boyanovsky, I. Schmidt and M.L. Goltzman— Ann. Phys(NY)185(1988)111.
- [7] M. B. Paranjape— Nucl. Phys. B307(1988)649.
- [8] S. Miyake and K. Shizuya— Phys. Rev. D36(1987)3781; Phys. Rev. D37(1988)2282.
- [9] J. G. Halliday, E. Rabinovici, A. Schwimmer and M. Chanowitz— Nucl. Phys. B268(1986)413.
- [10] A. Niemi and G. Semenoff— Phys. Lett. B175(1986)439.
- [11] R. Floreanini and R. Jackiw— Phys. Rev. Lett. 59(1987)1873.
- [12] N.S. Manton— Ann. Phys.(NY)159(1985)220.
- [13] In fact π_i and π_0 could be represented as $-id/dy_i$ and $-id/dy_0$ respectively, but we shall not need to use any particular representation for these operators.
- [14] P. Mitra and R. Rajaraman— Phys. Rev. D37(1988)448.
- [15] D. Sen— Phys. Rev. Lett. 61(1988)791; Phys. Lett. B212(1988)191.
- [16] we use the convention $\epsilon^{01} = g^{00} = -g^{11} = 1$
- [17] S. Mandelstam— Phys. Rev. D11(1975)3026.
- [18] S. Coleman— Phys. Rev. D11(1975)2088.
- [19] A.V. Ramallo— Phys. Lett. B211(1988)440.
- [20] L. D. Faddeev and S. L. Shatashvili— Phys. Lett. B176(1986)225.