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STRING QUANTIZATION IN ACCELERATED FRAMES AND BLACK HOLES

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A B S T R A C T

We quantize a closed bosonic string in a light-cone gauge in Rindler (uniformly accelerated) space-time and apply it to the Schwarzschild-Kruskal manifold. Inertial and accelerated particle states of the string associated to positive frequency modes with respect to the inertial and Rindler times respectively, are defined. There is a stretching effect of the string due to the presence of an event horizon. We explicitly solve the dynamical constraints leaving as physical degrees of freedom only those transverse to the acceleration. Different mass formulae are introduced depending on whether the centre of mass of the string has uniform speed or uniform acceleration. The expectation value of the Rindler (Schwarzschild) number-mode operator in the string ground state (tachyon) results equal to a thermal spectrum at the Hawking-Unruh temperature $T_s = \alpha/2\pi [\sim M_{p1}(M_{p1}/M)]^{1/D-3}$, where M is the black hole mass]. We find $T_o = M'/2\pi$ where M' is the accelerated ground state string mass and T_o the temperature T_s in dimensionless frequency units. Correlation functions of string co-ordinates and vertex operators and their Fourier transforms in accelerated time (string response functions) are computed and their thermal properties analyzed.

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I. INTRODUCTION

Much of the present interest in string theories comes from the hope that they may provide a finite theory of quantum gravity. It seems natural in this context to investigate the quantization of strings in a curved space-time. As a first step in this programme we study in the present article the quantization of a string in a uniformly accelerated space (D-dimensional Rindler space). Although this is a flat space-time, it possesses a space-time structure including an event horizon, similar to a black-hole manifold. In order to quantize the string properly in this manifold, a horizon regularization is needed. This regularization can be introduced through the definition of the Rindler co-ordinates as follows:

$$\begin{aligned} x_1 \pm x_0 + \varepsilon &= e^{\alpha(X^1 \pm X^0)} \\ x^i &= X^i, \quad i = 2, \dots, D. \end{aligned} \quad (1.1)$$

where (x^1, x^0) and (X^1, X^0) are respectively Minkowskian (Kruskal-like) and Rindler (Schwarzschild-like) co-ordinates. α is a constant defining the proper acceleration of the Rindler observers (α is equal to the surface gravity in the black hole case) and ε is an infinitesimal parameter of the order of the Planck length. The horizon is now at a finite distance $|X^1 \pm X^0| \sim (1/\alpha) \ln 1/\varepsilon$. This regularization reflects the fact that a classical description of the geometry is no longer valid at distance of the order of the Planck length.

We quantize the string in a light-cone gauge where the light-cone variables are $(x^1 \pm x^0)$ and $(X^1 \pm X^0)$. This choice is particularly convenient here since the acceleration points in the x^1 direction. We recall that for Q.F.T. in accelerated frames (in flat and curved space-times), different time-like vector fields lead to inequivalent positive frequency modes. In the light cone gauge, positive frequency modes with respect to the time-like variable (τ) on the string world sheet are physical states when τ is identified with the appropriate time variable in the physical space-time. We associate inertial and accelerated particle states of the string (Sections II and III) to positive frequency modes with respect to x^0 and X^0 respectively. We find that the accelerated frequencies of the accelerated modes differ in a large factor

$$\lambda_0 = \frac{2\pi\alpha}{\ln\left(\frac{2\pi}{\varepsilon} + 1\right)}, \quad (1.2)$$

from the inertial ones. Physically this factor reflects the indefinite increasing of the string length when it approaches the event horizon (Fig. 1). In Section III we develop the string dynamics in Rindler space and write the corresponding constraints. The longitudinal co-ordinates X^0 and X^1 obey non-linear equations of motion but, as in the inertial case, they can be eliminated in terms of the transverse co-ordinates which are the independent dynamical variables. Two possible situations appear depending on whether the centre of mass has a uniform speed or a uniform acceleration. The mass formulae are derived in each case. The Poincaré invariance of the flat Minkowski space-time has a non-linear realization in terms of the Rindler co-ordinates. In Section III we prove that the passage from the inertial to accelerated modes of the string is a canonical transformation both at the classical and quantum levels. We also check that an explicit realization of the Poincaré algebra can be constructed in terms of the accelerated modes, provided (at the quantum level) one sets $D = 26$ and uses symmetric ordering of the operators. In Section IV, the explicit expression of the Bogoliubov coefficients relating the inertial and Rindler modes of the string is found. The expectation value of the Rindler number mode operator in the ground state (tachyon) of the string is computed. This follows a thermal distribution [Eq. (4.13)] with temperature

$$T_s = \frac{\alpha}{2\pi} \quad (1.3)$$

This is the same Hawking-Unruh value that appears in the field-theoretical context. However, if one measures the frequency in dimensionless units (1, 2, ...) instead of multiples of λ_0 [Eq. (1.2)], the temperature of the ground state is a (very large) pure number

$$T_0 = \frac{T_s}{\lambda_0} = \frac{1}{4\pi^2} \ln \left(\frac{2\pi}{\epsilon} + 1 \right) \quad (1.4)$$

We find the expectation value of the accelerated mass M' operator in the (tachyon) ground state; it turns out to be a large positive number

$$M' = 2\pi T_0 \quad (1.5)$$

In Section V we compute string response functions for a Rindler observer. They are obtained by Fourier transforming the correlation functions of the string co-ordinates and vertex operators with respect to Rindler time. The correlation functions turn out to be periodic in the imaginary Rindler time with period $2\pi/\alpha$. We find the response functions equal to a Planckian spectrum with temperature $T_s = \alpha/2\pi$, times form factors characterizing the string (Eqs. (5.8) and (5.9)).

The string response functions fulfil the detailed balance relations. This is another manifestation of the thermal properties discussed above [Eq. (1.3)].

The major features of the string quantization in Rindler space described in Sections II-IV also holds for a string in a Schwarzschild space-time (as is discussed in Section VI). An appropriate light-cone gauge can be introduced for left or right movers in the null Kruskal and Schwarzschild co-ordinates. Now, Eq. (1.2) for λ_0 gives the relation between the Kruskal and Schwarzschild frequencies and T and T_0 [Eqs. (1.3)-(1.4)] give the temperatures characterizing the ground state of the string in the black hole manifold with

$$\varepsilon \sim \left(\frac{M_{pl}}{M} \right)^{\frac{D-2}{D-3}} \quad \text{and} \quad \alpha \equiv K \sim \left(\frac{M_{pl}}{M} \right)^{\frac{D-2}{D-3}}$$

Here M_{pl} and M stand for the Planck mass and the black hole mass respectively. It can be noticed that the Hagedorn temperature in this context ($1/\sqrt{\alpha'} \sim M_{pl}$) is always $\gtrsim T_s \sim M_{pl} (M_{pl}/M)^{1/(D-3)}$ since a basic requirement for the present semi-classical treatment is $M \gtrsim M_{pl}$. The investigations presented in this paper can also be extended to the case of fermionic strings and to more general (non-uniform) accelerations described by analytic (holomorphic) mappings as in the approach of Ref. 1).

II. STRINGS IN RINDLER SPACE

II.1 Canonical Formulation

The string action in a D-dimensional curved space-time reads

$$S = \frac{1}{2\pi\alpha'} \iint d\sigma dz \sqrt{g} g^{\alpha\beta} G_{AB}(X) \partial_\alpha X^A \partial_\beta X^B \quad (2.1)$$

Here $G_{AB}(X)$ ($0 \leq A, B \leq D-1$) stands for the space-time metric. The metric $g_{\alpha\beta}$ ($1 \leq \alpha, \beta \leq 2$) on the world sheet can always be chosen in the conformal gauge as

$$g = \eta(\sigma, z) \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \quad (2.2)$$

$(2\pi\alpha')^{-1}$ is the string tension (we will take here $\alpha' = 1$). In this paper we mainly consider strings in a flat uniformly accelerated space (D-dimensional

Rindler space). Inertial (Minkowskian) co-ordinates (x^A) and accelerated (Rindler) ones (X^A) are related by²⁾

$$\begin{aligned} x^1 - x^0 &= e^{\alpha (X^1 - X^0)} \\ x^1 + x^0 &= e^{\alpha (X^1 + X^0)} \\ x^i &= X^i \end{aligned} \quad , \quad 2 \leq i \leq D-1 \quad (2.3)$$

Here α is a constant defining the (proper) acceleration of the Rindler observers. In D-dimensional Rindler space, the metric and the string action are given by

$$dL^2 = \alpha^2 e^{2\alpha X^1} [(dX^1)^2 - (dX^0)^2] + (dX^i)^2 \quad (2.4)$$

$$S = \frac{1}{2\pi\alpha'} \iint d\sigma d\tau [\alpha^2 e^{\alpha(U+V)} \partial_\beta U \partial^\beta V + \partial_\beta X^i \partial^\beta X^i] \quad (2.5)$$

Here we introduce light-cone variables

$$\begin{aligned} U &= X^1 - X^0 \\ V &= X^1 + X^0 \end{aligned} \quad (2.6)$$

and

$$x_\pm = \sigma \pm \tau$$

The two-dimensional "energy-momentum" tensor, generator of the conformal transformations in the world sheet, reads

$$\begin{aligned} T_{\pm\pm}(x_+, x_-) &= \alpha^2 e^{\alpha(U+V)} \partial_\pm U \partial_\pm V + \partial_\pm X^i \partial_\pm X^i \\ T_{+-} &= 0 \end{aligned} \quad (2.7)$$

The classical equations of motion follow by extremizing the action (2.1) with respect to the world sheet metric $g^{\alpha\beta}$ and the string variables X^A . They read

$$T_{\alpha\beta} = 0 \quad (2.8)$$

i.e.,

$$\begin{aligned} \partial_+ \partial_- U + \alpha \partial_+ U \partial_- U &= 0 \\ \partial_+ \partial_- V + \alpha \partial_+ V \partial_- V &= 0 \end{aligned} \quad (2.9)$$

$$\partial_+ \partial_- X^i = 0, \quad 2 \leq i \leq D-1 \quad (2.10)$$

In the accelerated frame, the string boundary conditions read as in the inertial frames:

$$\begin{aligned} X^A(\sigma+2\pi, \tau) &= X^A(\sigma, \tau) && \text{for closed strings} \\ \frac{\partial X^A}{\partial \sigma}(0, \tau) &= \frac{\partial X^A}{\partial \sigma}(\pi, \tau) = 0 && \text{for open strings} \end{aligned} \quad (2.11)$$

From Eq. (2.5), the canonical momenta are

$$\begin{aligned} \pi_U &= \frac{\partial \mathcal{L}}{\partial \dot{U}} = \frac{\alpha}{\pi} e^{\alpha(U+V)} \dot{V} \\ \pi_V &= \frac{\partial \mathcal{L}}{\partial \dot{V}} = \frac{\alpha}{\pi} e^{\alpha(U+V)} \dot{U} \\ \pi_i &= \frac{\partial \mathcal{L}}{\partial \dot{X}_i} = \frac{1}{\pi} \dot{X}_i \end{aligned} \quad (2.12)$$

and the Hamiltonian writes

$$H = \int d\sigma \left[e^{-\alpha(U+V)} \pi_U \pi_V + \dot{X}_i^2 + X_i'^2 \right] \quad (2.13)$$

As usual, H weakly vanishes taking into account the constraints (2.8)

$$H = \int d\sigma T_{\sigma\sigma} \approx 0 \quad (2.14)$$

The action and the metric [Eqs. (2.4) and (2.5)] are not Poincaré-invariant. They are clearly invariant under the Euclidean group [rotations and translations in the transverse co-ordinates X^i ($2 \leq i \leq D-2$)]. In addition, we have the symmetries

$$\begin{aligned}
 \text{i)} \quad & \begin{aligned} U &\rightarrow \tilde{U} = U + a \\ V &\rightarrow \tilde{V} = V - a \end{aligned} \quad \text{i.e.} \quad \begin{aligned} \frac{X^0}{X^1} &\rightarrow \frac{X^0}{X^1} - a \end{aligned} \\
 \text{ii)} \quad & U \rightarrow \tilde{U} = U + \frac{1}{\alpha} \log [1 + b e^{-\alpha U}] \\
 \text{iii)} \quad & V \rightarrow \tilde{V} = V + \frac{1}{\alpha} \log [1 + c e^{-\alpha V}]
 \end{aligned} \tag{2.15}$$

where a, b, c are arbitrary real parameters. The generators of these invariances are the conserved quantities

$$\text{i)} \quad P = \int d\sigma (\pi_U - \pi_V) \tag{2.16}$$

$$\text{ii)} \quad Q = \int d\sigma e^{-\alpha U} \pi_U \tag{2.17}$$

$$\text{iii)} \quad R = \int d\sigma e^{-\alpha V} \pi_V \tag{2.18}$$

The transformation (i) expresses the stationarity of the Rindler space. The associated conserved quantity P is the energy in this space. The transformations (i)-(iii) correspond to the Lorentz transformations and translations in the (x^0, x^1) Minkowskian plane respectively.

The solution of the equations of motion (2.8) and (2.9) can be written as

$$\begin{aligned}
 \hat{X} &= f_+^{\hat{}}(x_+) + f_-^{\hat{}}(x_-) \\
 U &= \frac{1}{\alpha} \log u, \quad V = \frac{1}{\alpha} \log v
 \end{aligned} \tag{2.19}$$

where

$$u = f(x_+) + g(x_-), \quad v = h(x_+) + j(x_-) \tag{2.20}$$

are the inertial (Minkowskian) light-cone co-ordinates ($u = x^1 - x^0, v = x^1 + x^0$). One can also see from Eqs. (2.16)-(2.18) that Q and R are just the light-cone components of the inertial momentum

$$Q = \int d\sigma \partial_z v = p_+, \quad R = \int d\sigma \partial_z u = p_- \tag{2.21}$$

The solutions of Eq. (2.18)-(2.19) write

$$U = \frac{1}{\alpha} \log \left\{ \mu_0 + \frac{p_- z}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} [u_n e^{in(\sigma-z)} + \tilde{u}_n e^{-in(\sigma+z)}] \right\} \quad (2.22)$$

$$V = \frac{1}{\alpha} \log \left\{ v_0 + \frac{p_+ z}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} [v_n e^{in(\sigma-z)} + \tilde{v}_n e^{-in(\sigma+z)}] \right\} \quad (2.23)$$

$$\bar{X}^i = q^i + \frac{p^i z}{2} + \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{\sqrt{|n|}} [a_n^i e^{in(\sigma-z)} + \bar{a}_n^i e^{-in(\sigma+z)}] \quad (2.24)$$

where Eqs. (2.17)-(2.18) were used. U and V cannot be expressed as linear superpositions of harmonic oscillators. The series expansions on the right hand side of Eqs. (2.22)-(2.23) correspond to the inertial harmonic modes.

The inertial string co-ordinates can be split into left and right movers as³⁾

$$x^\mu(\sigma, \tau) = \bar{x}^\mu(x_+) + \bar{x}^\mu(x_-), \quad (x^\mu = u, v, x^i) \quad (2.25)$$

i.e.,

$$\bar{u}(x_+) = q_- + p_- x_+ + \frac{i}{2} \sum_{n \neq 0} \frac{\bar{u}_n}{n} e^{-in x_+}, \quad p_- = \frac{\mu_0}{2} \quad (2.26)$$

$$\bar{v}(x_+) = q_+ + p_+ x_+ + \frac{i}{2} \sum_{n \neq 0} \frac{\bar{v}_n}{n} e^{-in x_+}, \quad p_+ = \frac{v_0}{2} \quad (2.27)$$

$$\bar{x}^i(x_+) = \frac{q^i}{2} + \frac{p^i}{2} x_+ + \frac{1}{2} \sum_{n \neq 0} \frac{\bar{a}_n^i}{\sqrt{|n|}} e^{-in x_+}$$

$$\bar{p}^i(x_+) = \frac{p^i}{2\pi} - \frac{i}{2\pi} \sum_{n \neq 0} \sqrt{|n|} \operatorname{sg}(n) \bar{a}_n^i e^{-in x_+} \quad (2.28)$$

The coefficients $\bar{u}_n$, $\bar{v}_n$, $\bar{a}_n^i$ are unambiguously defined by Eqs. (2.25)-(2.27) and the requirements

$$\begin{aligned}
 \overleftarrow{\mu}(x_+ + 2\pi) - \overleftarrow{\mu}(x_+) &= 2\pi p_- \\
 \overleftarrow{\nu}(x_+ + 2\pi) - \overleftarrow{\nu}(x_+) &= 2\pi p_+ \\
 \overleftarrow{x}^i(x_+ + 2\pi) - \overleftarrow{x}^i(x_+) &= \pi p^i
 \end{aligned} \tag{2.29}$$

As usual, left and right modes decouple both in the classical and quantum theories. Therefore we start by studying the left movers and ignore for the moment the right modes. As is known, the choice of the conformal gauge (2.2) still allows the reparametrization $x_+ \rightarrow f(x_+)^{3)}$. In this way, one can choose

$$\overleftarrow{\nu} = p_+ x_+ , \tag{2.30}$$

which implies $v_n = 0$ in Eq. (2.27). This choice is particularly convenient here since the axis x^1 is parallel to the acceleration. As we show in Section III the accelerated longitudinal degrees of freedom (X^0 and X^1) can be eliminated as in the inertial case.

As usual, the energy momentum tensor can be Fourier-expanded in terms of the conformal generators L_n :

$$T_{++}(x_+) = \sum_{n \in \mathbb{Z}} \frac{L_n}{2\pi} e^{-in x_+} \tag{2.31}$$

where

$$\begin{aligned}
 L_n &= \frac{1}{2} \sum_m \overleftarrow{\alpha}_m^i \overleftarrow{\alpha}_{n-m}^i + p_+ \overleftarrow{\mu}_n \\
 L_0 &= \frac{1}{2} \sum_m \overleftarrow{\alpha}_m^i \overleftarrow{\alpha}_{n-m}^i + 2 p_+ p_- , \quad p_- = \frac{\mu_0}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{and} \quad \overleftarrow{\alpha}_n^i &= -i \sqrt{|n|} \operatorname{sg}(n) \overleftarrow{a}_n^i \quad n \neq 0 \\
 \overleftarrow{\alpha}_0^i &= p^i
 \end{aligned} \tag{2.32}$$

Let us briefly recall how the constraints (2.8) are explicitly solved in the light-cone gauge^{3),4)}. These constraints imply $L_n \approx 0$, hence Eq. (2.32) yields

$$\overleftarrow{\mu}_m = \frac{1}{2p_+} \sum_n \alpha_m^i \alpha_{n-m}^i \quad (2.33)$$

and
$$\mu_0 = \frac{1}{2p_+} \sum_m \alpha_m^i \alpha_{-m}^i = 2p_-$$

This expresses $\partial \overleftarrow{u} / \partial x_+$ in terms of the transverse components. Only q_- is an independent magnitude. Inserting Eq. (2.33) in (2.26) yields, using Eq. (2.28),

$$\frac{\partial \overleftarrow{\mu}}{\partial x_+} = -\frac{1}{p_+} \left(\frac{\partial x^i}{\partial x_+} \right)^2 \quad (2.34)$$

This formula gives the solution of the constraints (2.8). The mass formula follows from Eq. (2.33)

$$M^2 = p_+ p_- - p_i^2 = \sum_{n=1}^{\infty} n \alpha_n^{i+} \alpha_n^i \quad (2.35)$$

The independent dynamical variables α_n^i ($n \in \mathbb{Z}$), q_i , q_+ and p_+ obey canonical commutation rules

$$\begin{aligned} [\alpha_m^i, \alpha_n^j] &= n \delta_{n+m,0} \delta^{ij} \\ [q^i, \alpha_0^j] &= i \delta^{ij}, \quad [q_-, p_+] = i, \end{aligned} \quad (2.36)$$

all other commutators vanishing. One then finds

$$[\overleftarrow{x}^i(x_+), \overleftarrow{p}^j(y_+)] = \frac{\delta^{ij}}{2} \delta(x_+ - y_+)$$

The 1/2 factor reflects the fact that we only have half of the modes. Order ambiguities are simply avoided by taking symmetric products. For example

$$M^2 = \sum_n \frac{n}{2} (\alpha_n^{i+} \alpha_n^i + \alpha_n^i \alpha_n^{i+}) = -\frac{D-2}{24} + \sum_n n \alpha_n^{i+} \alpha_n^i \quad (2.37)$$

In spite of the fact that the light-cone gauge is not manifestly Lorentz-covariant, it is possible to prove that Poincaré covariance holds⁴⁾, basically using Eq. (2.34) and the symmetric product prescription.

The translation generators read in the light-cone gauge

$$\begin{aligned} P_- &= \frac{1}{2\pi p_+} \int_0^{2\pi} dx_+ \left(\frac{\partial x^i}{\partial x_+} \right)^2 \\ P_+ &= p_+ \quad , \quad P_i = p_i \end{aligned} \quad (2.38)$$

and the Lorentz generators are

$$\begin{aligned} M^{ij} &= \frac{1}{2} \int_0^{2\pi} dx_+ \left(\{x^i, p^j\} - \{x^j, p^i\} \right) \\ M^{+i} &= -p^+ q^i + q^+ p^i = -M^{i+} \\ M^{+-} &= \frac{1}{2} \left(\{q^+, p^-\} - \{q^-, p^+\} \right) \\ M^{-i} &= -M^{i-} = \frac{1}{2} \int dx_+ \left(\{u, p^i\} - \{x^i, p_u\} \right) \end{aligned} \quad (2.39)$$

where $\{a, b\} \equiv ab + ba$.

They obey a Poincaré algebra classically for any D and quantum mechanically only at $D = 26$ ⁴⁾. This follows by replacing the dependent quantity $\partial_+ u$ with the help of Eq. (2.34) and then using Eqs. (2.36).

An analogous treatment can be done for the right movers in the gauge

$$\vec{u} = p_- x_- \quad (2.40)$$

II.2 Horizon regularization in Rindler space

Let us discuss now some features of Rindler space which are important for our study of strings in this space. The transformation (2.3) maps the right-hand wedge $x^1 > |x^0|$ of Minkowski space onto the whole Rindler space $-\infty < X^1, X^0 < +\infty$ (the whole Minkowski space can be covered using four different Rindler patches). As is known, a quantum field in Rindler space is in a thermal state with temperature $T = \alpha/(2\pi)$ ^{1),5)-6)}. In addition, ultra-violet divergences arise in the free energy and entropy of quantum fields from the existence of a horizon at $x^1 = |x^0|$ (i.e. $X^1 = -\infty$) in the space-time. The same problem appears in the case of a four-dimensional black hole⁷⁾. Let us illustrate this phenomenon by

considering a free massive scalar field Ψ . In D-dimensional Rindler space, the positive frequency modes are

$$\Psi = \frac{1}{(2\pi)^{\frac{D-2}{2}}} e^{i(-\lambda X^0 + k^i X^i)} \Phi(X^1), \quad \lambda > 0, \quad k^i \in \mathbb{R} \quad (2.41)$$

where $\Phi(X^1)$ satisfies

$$\left[\frac{d^2}{dX^{12}} + \lambda^2 - (m^2 + k^{i2}) \alpha^2 e^{2\alpha X^1} \right] \Phi(X^1) = 0$$

The total number ($\mathcal{N}$) of wave modes with frequency less than λ can be computed in the semi-classical approximation for $\lambda \gtrsim \alpha$. This is enough to study the ultra-violet behaviour of the quantities interesting us. In the W.K.B. approximation $\mathcal{N}_\lambda$ is given by

$$\pi \mathcal{N}_\lambda = \int_{-H}^{a(\lambda)} dX^1 \int \prod_{i=1}^{D-2} \frac{dk^i}{2\pi} \sqrt{\lambda^2 - (m^2 + k^{i2}) \alpha^2 e^{2\alpha X^1}} \quad (2.42)$$

The integration is taken over the values of k^i and X^1 for which the argument of the square root is positive. Here $a(\lambda)$ is the classical turning point

$$e^{2\alpha a(\lambda)} = \frac{\lambda}{m\alpha},$$

and H is a large cut-off ($H \gtrsim 1/\alpha$) on the negative Rindler co-ordinate X^1 . This shifts the horizon by replacing the light-cone $x^1 = |x_0|$ as a boundary of Rindler space-time by the hyperbola

$$(x^1)^2 - (x^0)^2 = e^{-2\alpha H} \quad (2.43)$$

This regularization takes into account the fact that a classical description of the geometry is no longer valid at distances of order of the Planck length. Thus,

$$e^{-2\alpha H} \sim l_p$$

Evaluating Eq. (2.42) for large H it yields

$$\pi N_\lambda = \frac{1}{4} \frac{(\lambda/\alpha)^{D-1}}{(4\pi)^{(D-3)/2} (D-2)} e^{\alpha H(D-2)} [1 + O(e^{-2\alpha H})] \quad (2.44)$$

Then the free energy and the entropy at temperature T :

$$F = - \int_0^\infty N_\lambda \frac{d\lambda}{(e^{\lambda/T} - 1)} \quad , \quad S = - \frac{\partial F}{\partial T} \quad ,$$

are equal to

$$F = - \frac{\Gamma(D) \zeta(D) T}{(4\pi)^{(D-1)/2} (D-2)} \alpha^{1-D} e^{\alpha H(D-2)} [1 + O(e^{-2\alpha H})] \quad (2.45)$$

$$S = - \frac{D}{T} F > 0$$

In Rindler space, $T = \alpha/2\pi$ and so

$$F = - \frac{\Gamma(D) \zeta(D)}{(4\pi)^{D-\frac{1}{2}} \pi^D (D-2)} \alpha e^{\alpha H(D-2)} \quad (2.46)$$

$$S = \frac{\Gamma(D) \zeta(D)}{(4\pi)^{D-\frac{1}{2}} \pi^D (D-2)} \frac{2\pi D}{\alpha} e^{\alpha H(D-2)} \quad (2.47)$$

We explicitly see that F and S need the ultra-violet cut-off H to be finite. This is equivalent to considering the following mapping defining the accelerated co-ordinates

$$\begin{aligned} x^1 - x^0 + \varepsilon &= e^{\alpha(X^1 - X^0)} \\ x^1 + x^0 + \varepsilon &= e^{\alpha(X^1 + X^0)} \end{aligned} \quad (2.48)$$

where $\varepsilon \approx e^{-\alpha H}$, $-H < X^1 < +\infty$.

III. STRING LIGHT-CONE QUANTIZATION IN RINDLER SPACE

III.1 The inertial string as seen by Rindler observers

In order to define particle states in a given reference frame, one should consider positive frequency modes with respect to the time co-ordinate in that frame^{1),5)-6)}. Since Rindler's frame has a time-like Killing vector ($\partial/\partial X^0$), positive frequency modes are defined with respect to the Rindler time X^0 . In the inertial frame, positive frequency modes are defined with respect to the inertial time (x^0). In the light-cone gauge [Eq. (2.30)], x^0 is proportional to τ (note that the proportionality coefficient p_+ is always positive). Therefore, the modes

$$\overleftarrow{\varphi}_n = \frac{1}{2\sqrt{\pi n}} e^{-in x_+} \quad (3.1)$$

as used in the expansion (2.28), define the inertial particle states of the string.

In order to define the accelerated modes, it is convenient to introduce a new parametrization of the string world sheet such that the new co-ordinate x'_+ coincides with the accelerated co-ordinate V , namely

$$V = x'_+ + \frac{1}{\alpha} \log p_+ \quad (3.2)$$

Therefore, from Eqs. (2.30), (2.48) and (3.2) we have

$$x_+ + \varepsilon = e^{\alpha x'_+} \quad (3.3)$$

As we said in the previous section, we treat separately the left and right movers. Since the values σ and $\sigma + 2\pi$ describe the same point of the string, the wave modes are periodic functions of x_+ and x_- with period 2π . The left modes, which are only functions of x_+ , will be periodic functions of $x'_+ \equiv \sigma' + \tau'$ with period

$$\pi_\varepsilon = \frac{1}{\alpha} \log \left(\frac{2\pi}{\varepsilon} + 1 \right) \quad (3.4)$$

as follows from Eq. (3.3). We shall use the more convenient variable

$$\xi \equiv x'_+ - \frac{1}{\alpha} \ln \varepsilon, \quad ,$$

$0 < \xi < \Pi_\varepsilon$, whereas $(1/\alpha) \ln \varepsilon < x'_+ < (1/\alpha) \ln(2\pi + \varepsilon)$.

Now the accelerated left movers

$$\overleftarrow{\Phi}_n = \frac{1}{2\sqrt{\pi|n|}} e^{-i\lambda_n \xi} \quad (3.5)$$

define the accelerated particle states of the string. The frequencies λ_n are fixed by the periodicity condition (3.4) to be

$$\lambda_n = \frac{2\pi}{\Pi_\varepsilon} n, \quad n = 1, 2, 3, \dots \quad (3.6)$$

We see that the accelerated frequencies differ by a factor

$$\frac{2\pi}{\Pi_\varepsilon} = \frac{2\pi\alpha}{\ln\left(\frac{2\pi}{\varepsilon} + 1\right)} \quad (3.7)$$

from the inertial ones. The frequency spectrum becomes continuum in the $\varepsilon \rightarrow 0$ limit. This is due to the fact that the string length increases indefinitely near the horizon as seen by the accelerated observers ("infinite red shift").

The antiparticle states are given by $\overleftarrow{\Phi}_n^*$. The set $\{\overleftarrow{\Phi}_n, \overleftarrow{\Phi}_n^*\}$ forms a complete orthonormalized basis for left movers with the usual invariant scalar product

$$\langle \overleftarrow{\Phi}_i, \varphi \rangle = -i \int d\Sigma_\mu \overleftarrow{\Phi}_i^* \overrightarrow{j}^\mu \varphi \quad (3.8)$$

The expansion of the string co-ordinates X^i in the basis $\{\overleftarrow{\Phi}_n\}$ defines the accelerated creation and annihilation operators $\overleftarrow{C}_n^i, \overleftarrow{C}_n^{i+}$:

$$\overleftarrow{X}^i(x'_+) = \frac{Q^i}{2} + \frac{P^i}{2} \xi + \sum_{n \neq 0} \overleftarrow{C}_n^i \overleftarrow{\Phi}_n(\xi) \quad (3.9)$$

$$= \frac{Q^i}{2} + \frac{P^i}{2} \xi + \frac{i}{2} \sum_{n \neq 0} \frac{\overleftarrow{\beta}_n^i}{n} e^{-i\lambda_n \xi} \quad (3.10)$$

$$\overleftarrow{P}^i(x'_+) = \frac{P^i}{2\pi} + \frac{1}{\pi_\epsilon} \sum_{n \neq 0} \overleftarrow{\beta}_n^i e^{-i\lambda_n \xi} \quad (3.11)$$

Here $\overleftarrow{P}^i(x'_+)$ is the string canonical momentum and

$$\overleftarrow{C}_n^i = i \operatorname{sg}(n) \frac{\overleftarrow{\beta}_n^i}{\sqrt{|n|}}, \quad \overleftarrow{\beta}_{-n}^i = \overleftarrow{\beta}_n^i$$

Similarly, for the longitudinal co-ordinate $\overleftarrow{u}$:

$$\begin{aligned} \overleftarrow{u}(x'_+) &= u_0 + \frac{P_-}{2} \xi + \frac{i}{2} \sum_{n \neq 0} \frac{\gamma_n}{n} e^{-i\lambda_n \xi} \\ \overleftarrow{P}_u(x'_+) &= \frac{1}{2\pi} P_- + \frac{1}{\pi_\epsilon} \sum_{n \neq 0} \gamma_n e^{-i\lambda_n \xi} = \frac{1}{\pi_\epsilon} \sum_n \gamma_n e^{-i\lambda_n \xi} \end{aligned} \quad (3.12)$$

We recall that the gauge condition [Eqs. (2.30) and (3.3)] yields

$$\overleftarrow{v}(x'_+) = p_+ (e^{\alpha x'_+} - \epsilon) = \epsilon p_+ (e^{\alpha \xi} - 1) \quad (3.13)$$

As in Eq. (2.29), the coefficients $\overleftarrow{\beta}_n^i$ and $\overleftarrow{\gamma}_n$ are unambiguously defined by the expansions (3.9) and (3.12) with the requirement

$$\begin{aligned} \overleftarrow{X}^i(x'_+ + \pi_\epsilon) - \overleftarrow{X}^i(x'_+) &= \frac{P^i}{2} \pi_\epsilon \\ \overleftarrow{u}(x'_+ + \pi_\epsilon) - \overleftarrow{u}(x'_+) &= \frac{P_-}{2} \pi_\epsilon \end{aligned} \quad (3.14)$$

Therefore,

$$\begin{aligned} P^i &= \frac{2\pi}{\pi_\epsilon} p^i \\ P_- &= \frac{2\pi}{\pi_\epsilon} p_- \end{aligned} \quad (3.15)$$

In the accelerated frame, the energy-momentum tensor is traceless and conserved. The left modes provide the T'_{++} component:

$$T'_{++}(x'_+) = \alpha^2 e^{\alpha(U+V)} \partial_+ U \partial_+ V + \partial_+ X^i \partial_+ X^i \quad (3.16)$$

The accelerated conformal generators (L'_n) follow from the Fourier expansion of T'_{++} :

$$T'_{++}(x'_+) = \sum_{-\infty}^{+\infty} \frac{L'_n}{\pi_\epsilon} e^{-i\lambda_n \xi} \quad (3.17)$$

$$L'_n = L'^{\parallel}_n + L'^{\perp}_n$$

where $L'^{\parallel}_n$ and $L'^{\perp}_n$ stand for the longitudinal and transverse parts of the accelerated generators. From Eqs. (3.10), (3.16) and (3.17) we get

$$L'^{\perp}_n = -\frac{1}{\pi_\epsilon} \sum_m \overleftarrow{\beta}_m^i \overleftarrow{\beta}_{m-n}^i \quad (3.18)$$

where

$$\begin{aligned} \beta_0^i &= p^i \\ \gamma_0 &= p_- \end{aligned} \quad (3.19)$$

In particular,

$$L'^{\perp}_0 = \frac{p^{i2}}{\pi_\epsilon} + \frac{2}{\pi_\epsilon} \sum_{m=1}^{\infty} m \zeta_m^{i+} \zeta_m^i \quad (3.20)$$

From Eqs. (3.12), (3.16) and (3.17) we get for the longitudinal part

$$L'^{\parallel}_n = \pi \alpha p_+ \sum_m \frac{\gamma_m}{a + i(n-m)} \quad (3.21)$$

where

$$a = \frac{\alpha}{2\pi} \pi_\epsilon = \frac{1}{2\pi} \ln \left(\frac{2\pi}{\epsilon} + 1 \right) \quad (3.22)$$

The constraints

$$L'^{\parallel}_n + L'^{\perp}_n = 0 \quad (3.23)$$

give the following infinite set of equations relating γ_m to the transverse operators:

$$i \pi \alpha p_+ \sum_{m=-\infty}^{\infty} \frac{\gamma_m}{(n-m-ia)} = L_n^{\perp}, \quad -\infty \leq m \leq +\infty \quad (3.24)$$

These equations have the structure of a discrete Hilbert transform and can easily be solved for γ_n :

$$\gamma_n = \frac{1}{2i\pi^2 \epsilon \left(1 + \frac{\epsilon}{2\pi}\right) \alpha p_+} \sum_{m=-\infty}^{\infty} \frac{L_m^{\perp}}{(n-m+ia)} \quad (3.25)$$

Inserting this formula in Eq. (3.12) and then summing over n gives

$$\frac{\overleftarrow{\partial \mu}}{\partial \xi} = - \frac{e^{-\alpha \xi}}{2\alpha \pi \epsilon p_+} \sum_{m=-\infty}^{\infty} L_m^{\perp} e^{-i\lambda m \xi} \quad (3.26)$$

or

$$\frac{\overleftarrow{\partial \mu}}{\partial \xi} = - \frac{1}{\alpha \epsilon} \frac{e^{-\alpha \xi}}{p_+} \left(\frac{\overleftarrow{\partial X^i}}{\partial \xi} \right)^2 \quad (3.27)$$

We have then solved the constraints $T_{\mu\nu}^{\perp} = 0$ in the accelerated frame. Equation (3.27) can easily be seen to be equivalent to Eq. (2.34) that expressed the solutions of the constraints in the inertial frame. In particular, one derives from Eq. (3.25) for $n = 0$, or integrating Eq. (3.27) in one period, the mass formula

$$M^2 = p_+ p_- - p_i^2 = \sum_{n=1}^{\infty} n a_n^{i\dagger} a_n^i \quad (3.28)$$

It should be noticed that both inertial and accelerated observers get the same mass formula.

Up to now we have treated the dynamics of the string in the accelerated frame classically. It is now straightforward to quantize the string in the light-cone gauge. One must replace Poisson brackets by canonical commutators for the independent dynamical variables. The canonical commutation relations satisfied by u_0 , c_n^i , p_n^i , Q_0^i and p_+ are the following:

$$\begin{aligned}
 [C_m^i, C_m^j] &= \delta_{nm} \delta^{ij} \\
 [\beta_n^i, \beta_m^j] &= n \delta_{n+m,0} \delta^{ij} \\
 [P^i, Q_0^j] &= -\left(\frac{2\pi}{\pi_\xi}\right) i \delta^{ij} \\
 [p_+, \mu_0] &= -i
 \end{aligned} \tag{3.29}$$

The dependent operators r_n are given in terms of these through Eq. (3.29). Order ambiguities are avoided as usual by taking symmetric products. Equation (3.33) guarantees that

$$[\overleftarrow{X}^i(\xi), \overleftarrow{P}^j(\xi')] = \frac{i}{2} \delta(\xi - \xi') \delta^{ij} \tag{3.30}$$

Hence, the passage from inertial to accelerated modes is a canonical transformation.

In Rindler co-ordinates, the Poincaré invariance of flat Minkowski space-time translates into non-linear symmetry transformations having the Euclidean group for the transverse coordinates X^i as a linear subgroup. The Poincaré generators are still given by Eqs. (2.39) but now we must use the Fock representation (3.29) instead of (2.36). We replace the dependent operators $\overleftarrow{\partial u}/\partial \xi$ with the help of (3.27). Since this last equation is identical to its inertial counterpart Eq. (2.34), the expressions of the M^{uv} in terms of the transverse variables $X_i(\xi)$, $P_i(\xi)$ (and q_- and p_+) will be identical in both cases. Moreover, these variables fulfil the same commutation relations in the inertial and accelerated frames, so the Poincaré algebra does hold provided one sets $D = 26$ and uses symmetric ordering.

III.2 Accelerated strings

The explicit solution for the constraints Eq. (3.26) suggests another expansion for $\tilde{u}$ in the accelerated space:

$$\mu = \mu_0 + e^{-\alpha x'_+} \left[\tilde{p}_- + \frac{i}{2} \sum_{n \neq 0} r_n e^{-i\lambda_n \xi} \right] \quad (3.31)$$

In order to find a physical interpretation for this formula one should compare Eqs. (3.13) and (3.31) with the trajectory of a uniformly accelerated point

$$x'_\pm = x^0 + a_\pm = A_\pm e^{\pm \alpha \tau'} \quad (3.32)$$

where τ' is the parameter on the trajectory and $a_\pm$, $A_\pm$ are given constants. One can then interpret Eqs. (3.13) and (3.31) as a string whose centre of mass follows a world line accelerated in the x^1 direction of the type of Eq. (3.32).

Let us now solve the constraints $T_{\mu\nu} \approx 0$ for this case. By using Eqs. (3.13) and (3.31), one finds for T_{++}'

$$T_{++}' = \partial'_+ \mu \partial'_+ \tilde{\nu} = -\alpha^2 p_+ \left[\tilde{p}_- - \frac{1}{2\alpha} \sum_{n \neq 0} r_n (n - ia) e^{-i\lambda_n \xi} \right] \quad (3.33)$$

The use of Eq. (3.17) gives

$$r_n = - \frac{L_n'^\perp}{\pi \alpha (n - ia) p_+}, \quad n \neq 0 \quad (3.34)$$

$$\tilde{p}_- = \frac{L_0'^\perp}{2\pi \alpha a p_+} \quad (3.35)$$

Now it is easy to prove that Eqs. (3.26) - (3.27) also hold in the present case. From Eq. (3.35) it follows that

$$M'^2 = (2\pi \alpha)^2 p_+ \tilde{p}_- - p_i^2 = \sum_1^\infty m C_m^i + \tilde{C}_m^i \quad (3.36)$$

This formula suggests that $(2\pi \alpha)^2 \tilde{p}_-$ can be considered as a u -component of the momentum and M'^2 identified with the mass square operator for an accelerated string.

The quantization here can be done by imposing the commutation rules (3.29) as before. Order ambiguities are avoided by taking symmetric products. Now the quantum mass operator for the accelerated string is defined as

$$M'^2 = \sum_{n=1}^{\infty} \frac{n}{2} (C_n^{i+} C_n^i + C_n^i C_n^{i+}) = \sum_{n=1}^{\infty} n C_n^{i+} C_n^i - \frac{(D-2)}{24} \quad (3.37)$$

This mass operator is different from the inertial one [Eqs. (2.37) or (3.28)] although both have the same eigenvalues.

The treatment given here applies to the left movers. Right movers can be treated in a similar way in the gauge defined by Eq. (2.40) and by introducing a new co-ordinate x'_- which coincides with the accelerated co-ordinate, namely

$$U = x'_- + \frac{1}{\alpha} \log p_-$$

$$\text{ie, } x_- + \varepsilon = e^{\alpha x'_-}$$

IV. THERMAL PROPERTIES OF THE GROUND STATE SPECTRUM

The inertial (a_n^i) and accelerated (c_n^i) operators are related by a Bogoliubov transformation

$$c_m^i = p^i O_m + \sum_{n=1}^{\infty} (A_{mn} a_n^i + B_{mn} a_n^{i+}), \quad m \neq 0 \quad (4.1)$$

where

$$A_{mn} = \langle \phi_{\lambda m}, \psi_n \rangle, \quad B_{mn} = \langle \phi_{\lambda m}, \psi_n^* \rangle \quad (4.2)$$

are the Bogoliubov coefficients and

$$O_m = \left\langle \frac{1}{2} \left(x_+ - \frac{2\pi}{\Pi_\varepsilon} x'_+ \right), \phi_{\lambda m} \right\rangle \quad (4.3)$$

is the contribution of the zero modes. From Eqs. (3.1), (3.5) and (3.8), we have

$$\begin{aligned}
 A_{mn} &= \frac{1}{2\pi} \sqrt{\frac{n}{m}} \int_0^{2\pi} e^{in\sigma - i\frac{\lambda m}{\alpha} \ln(\sigma+\epsilon)} d\sigma \\
 B_{mn} &= \frac{1}{2\pi} \sqrt{\frac{n}{m}} \int_0^{2\pi} e^{in\sigma + i\frac{\lambda m}{\alpha} \ln(\sigma+\epsilon)} d\sigma \quad (4.4) \\
 O_m &= \sqrt{\frac{\pi}{m}} (2\pi)^{-ima} \frac{a^2}{\alpha m(m+ia)}
 \end{aligned}$$

Explicitly,

$$B_{mn}(\epsilon) = \frac{i}{\sqrt{2\pi\pi_\epsilon}} \frac{n}{\sqrt{n}} e^{-i\lambda m/\alpha} e^{-in\epsilon} \frac{e^{-\frac{\pi\lambda m}{2\alpha}}}{\sqrt{\lambda m}} \left[\Gamma\left(\frac{i\lambda m}{\alpha} + 1, -in\epsilon\right) - \Gamma\left(\frac{i\lambda m}{\alpha} + 1, -in\epsilon - 2\pi in\right) \right] \quad (4.5)$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function. In the limit $\epsilon \rightarrow 0$, for $n \lesssim 2\pi/\epsilon$ we have

$$\begin{aligned}
 B_{mn}(\epsilon) &= \frac{i}{2\pi} \frac{n}{\sqrt{n}} e^{-i\lambda m/\alpha} \frac{e^{-\frac{\pi\lambda m}{2\alpha}}}{\sqrt{m}} \gamma\left(\frac{i\lambda m}{\alpha} + 1, -2\pi in\right) \\
 &= \frac{i}{2\pi \sqrt{m}} \left\{ \frac{n}{\sqrt{n}} e^{-i\lambda m/\alpha} e^{-\frac{\pi\lambda m}{2\alpha}} \Gamma\left(1 + \frac{i\lambda m}{\alpha}\right) - \frac{(2\pi)^{i\lambda m/\alpha}}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \right\} \quad (4.6)
 \end{aligned}$$

The $\Gamma(\alpha, z)$ function in Eq. (4.5) takes into account the fact that we have a discrete spectrum linked to the finite length of the string. In the field theoretic case the spectrum is continuum, the second term in Eq. (4.6) is absent and we have^{1), 5)-6)}

$$B_{\lambda k} = \frac{i}{2\pi} \frac{k}{\sqrt{k}} e^{-i\lambda/\alpha} \frac{e^{-\frac{\pi\lambda}{2\alpha}}}{\sqrt{\lambda}} \Gamma\left(\frac{i\lambda}{\alpha} + 1\right) \quad (4.7)$$

where $\lambda, k \in \mathbb{R}$.

It follows that

$$\begin{aligned}
 B_{mn} B_{m'n}^* = & \frac{1}{4\pi^2} \frac{1}{\sqrt{mm'}} \left\{ \Gamma\left(1+i\frac{\lambda_m}{\alpha}\right) \Gamma\left(1-i\frac{\lambda_{m'}}{\alpha}\right) e^{-\frac{\pi(\lambda_m+\lambda_{m'})}{2\alpha}} \frac{n^{-i\frac{(\lambda_m-\lambda_{m'})}{\alpha}}}{n} - \right. \\
 & - \frac{1}{n} \left[e^{-\frac{\pi\lambda_m}{2\alpha}} \Gamma\left(1+i\frac{\lambda_m}{\alpha}\right) (2\pi n)^{-i\frac{\lambda_m}{\alpha}} + e^{-\frac{\pi\lambda_{m'}}{2\alpha}} \Gamma\left(1-i\frac{\lambda_{m'}}{\alpha}\right) (2\pi n)^{i\frac{\lambda_{m'}}{\alpha}} \right] + \\
 & \left. + \frac{(2\pi)}{n} \frac{i}{\alpha} (\lambda_m - \lambda_{m'}) + O\left(\frac{1}{n^2}\right) \right\} \quad (4.8)
 \end{aligned}$$

The ground state of the string is the Minkowski inertial ground state $|0\rangle$,

$$\begin{aligned}
 a_n^i |0\rangle &= 0 \\
 p^i |0\rangle &= 0
 \end{aligned} \quad \forall n > 1 \quad (4.9)$$

This state is not annihilated by the accelerated operators c_n^i . An interesting magnitude to consider is the expectation value of the accelerated number operator in the inertial ground state, i.e.

$$N^i(\lambda_m) = \langle 0 | c_{\lambda_m}^i + c_{\lambda_m}^{i\dagger} | 0 \rangle, \text{ (not sum over } i) \quad (4.10)$$

and from Eq. (4.1) one gets that N^i is i -independent

$$N^i(\lambda_m) = N(\lambda_m) = \sum_{n=1}^{\infty} |B_{mn}|^2 \quad (4.11)$$

Special care is needed in the evaluation of this sum. The short-distance regularization in inertial co-ordinates $|x_+| < \varepsilon$ [Eq. (2.48)] indicates a high energy cut-off

$$|k_m| \lesssim \frac{2\pi}{\varepsilon}$$

Thus, Eq. (4.11) should be replaced by the meaningful definition

$$N(\lambda_m) = \lim_{\varepsilon \rightarrow 0} \sum_{n=1}^{\frac{2\pi/\varepsilon}{\alpha}} |B_{mn}(\varepsilon)|^2 \quad (4.12)$$

Inserting Eq. (4.8) we find

$$N(\lambda_m) = \frac{1}{\left(e^{\frac{2\pi\lambda_m}{\alpha}} - 1\right)} + \frac{\alpha}{2\pi\lambda_m} \quad (4.13)$$

where we used

$$\sum_{n=1}^{\frac{2\pi/\epsilon}{\lambda_m}} \frac{1}{n} \underset{\epsilon \rightarrow 0}{=} \alpha \frac{\pi}{\epsilon} + O(1)$$

The first term in Eq. (4.8) dominates for $m = m'$ and $n \rightarrow \infty$. It is the only term that contributes to $N(\lambda_m)$ in the $\epsilon \rightarrow 0$ limit.

We see that $N(\lambda_m)$ is a Planckian spectrum plus a Rayleigh-Jeans-type term (i.e., a quantum plus a classical thermal distribution). This means that the inertial Minkowski vacuum of the string appears as a thermal state at temperature

$$T_s = \frac{\alpha}{2\pi} \quad (4.14)$$

in the accelerated (Rindler) space since the accelerated particle modes follow a Planckian distribution. The presence of the Rayleigh-Jeans-type term accounts for the string finite length. This term is absent in the field theory case. In the string case, it seems more natural to measure the frequency in dimensionless units (1,2,3,...), thus the ground-state spectrum $N(n)$ reads

$$N(n) = \frac{1}{\left(e^{n/T_0} - 1\right)} + \frac{T_0}{n} \quad (4.15)$$

Here the temperature T_0 is a very large pure number

$$T_0 = \frac{a}{2\pi} = \frac{1}{4\pi^2} \ln\left(\frac{2\pi}{\epsilon}\right) \gg 1 \quad (4.16)$$

i.e., $T_0 = (\pi/\epsilon) T_s$.

T_s [Eq. (4.14)] is the Hawking-Unruh temperature⁴⁾⁻⁶⁾ associated in quantum field theory to the Rindler space. However, the dimensionless temperature T_0 [Eq. (4.16)] seems more natural in the present string context.

We can now evaluate the expectation value of the accelerated M'^2 operator in the inertial ground state. From Eqs. (3.27) and (4.15) we find

$$M'^2 = 24 f(\varepsilon) - 1$$

$$f(\varepsilon) = \sum_{n=1}^{\infty} n N(n) = \sum_{n=1}^{\infty} \left(\frac{n}{e^{n/\tau_0} - 1} + T_0 \right) \quad (4.17)$$

where 24 accounts for the transverse dimensions and -1 for the intercept. This sum can be computed exactly in terms of elliptic functions⁸⁾. We have

$$f(\varepsilon) = \frac{1}{24} \left\{ 1 + \frac{4}{\pi^2} K(k) \left[(2-k^2) K(k) - 3 E(k) \right] - \frac{1}{2\pi^2} \ln \left(\frac{2\pi}{\varepsilon} \right) \right\}$$

where $K(k)$ and $E(k)$ are complete elliptic functions of first and second class respectively; the modulus k is defined by

$$\frac{K(k)}{K'(k)} = \frac{1}{2\pi} \ln \left(\frac{2\pi}{\varepsilon} \right)$$

and we have used

$$\sum_{n=1}^{\infty} 1 = \zeta(-1) = -\frac{1}{12}$$

In the $\varepsilon \rightarrow 0$ limit we get

$$f(\varepsilon) \underset{\varepsilon \rightarrow 0}{=} \frac{1}{24} \left[1 + \frac{1}{4\pi^2} \ln^2 \left(\frac{2\pi}{\varepsilon} \right) - \frac{1}{4\pi^2} \right] + O(\sqrt{\varepsilon}) \quad (4.18)$$

Therefore

$$M'^2 = \frac{1}{4\pi^2} \left[\ln^2 \left(\frac{2\pi}{\varepsilon} \right) - 1 \right] + O(\sqrt{\varepsilon}) \quad (4.19)$$

i.e.

$$M' \simeq \frac{1}{2\pi} \ln \left(\frac{2\pi}{\varepsilon} \right)$$

We see that

$$T_0 = \frac{1}{2\pi} M' \quad (4.20)$$

V. CORRELATIONS FOR THE STRING CO-ORDINATES AND VERTEX OPERATORS

In order to study the string properties in the accelerated space, it is interesting to compute the response function of a Rindler observer^{9),10)}. It is given by the Fourier transform of the correlation function along the world line of the accelerated observer (in D-dimensional space-time). In the present string context we take correlations (for the left movers) in the (light-cone) time variable $V = x'_+ + (1/\alpha) \log p_+$.

Let us start by considering the string in Minkowski space and in the covariant gauge. The two-point correlation function for the string coordinates reads³⁾

$$\begin{aligned} \langle 0 | \overleftarrow{X}_i(x_+) \overleftarrow{X}_j(y_+) | 0 \rangle &= \frac{\delta_{ij}}{4} G(x_+ - y_+) , \\ G(x_+) &= -\log \left[2 \sin \left(\frac{x_+ - i0}{2} \right) \right] \end{aligned} \quad (5.1)$$

The correlation function for the (tachyon) vertex operators is given by³⁾

$$\begin{aligned} \langle 0 | : e^{ik_j \overleftarrow{X}_j(x_+)} : : e^{ik'_j \overleftarrow{X}_j(y_+)} : | 0 \rangle &= \\ &= 2^D \delta^{(D)}(k+k') e^{\frac{k_j^2}{4} G(x_+ - y_+)} \end{aligned} \quad (5.2)$$

where $k_j^2 = 4$ on-shell.

In the inertial case, we have x_+ proportional to v and then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-i\omega x_+} G(x_+ - y_+) dx_+ &= 0, \quad \omega > 0 \\ \text{and} \\ \int_{-\infty}^{\infty} e^{-i\omega x_+} e^{G(x_+ - y_+)} dx_+ &= 0, \quad \omega > 0 \end{aligned} \quad (5.3)$$

The vanishing of these Fourier transforms reflects the obvious fact that the correlation function in the inertial Minkowski vacuum do not contain any positive frequency component with respect to the inertial time. In the theory of linear systems, the response function to an external disturbance is zero before the disturbance is applied as required by causality.

The Rindler correlation functions in the ground state (inertial vacuum) are just obtained by replacing the arguments (x_+, y_+) in the inertial expression (5.1) - (5.2) in terms of the accelerated variables (x'_+, y'_+) , i.e.,

$$x_+ = e^{\alpha x'_+} + \varepsilon$$

Then the accelerated response functions (Fourier transforms with respect to Rindler time) are

$$F_L(\omega) \equiv \int_{-\infty}^{\infty} e^{-i\omega x'_+} G(e^{\alpha x'_+} - e^{\alpha y'_+}) dx'_+$$

and

$$V_L(\omega) \equiv \int_{-\infty}^{\infty} e^{-i\omega x'_+} e^{+G(e^{\alpha x'_+} - e^{\alpha y'_+})} dx'_+ \quad (5.4)$$

where the index "L" stands for left movers. These integrals can be explicitly performed with the result

$$F_L(\omega) = \frac{2\pi}{\omega} \frac{1}{(e^{2\pi\omega/\alpha} - 1)} \rho_L\left(\frac{\omega}{\alpha}\right) \quad (5.5)$$

$$V_L(\omega) = \frac{2\pi}{\alpha} \frac{1}{(e^{2\pi\omega/\alpha} - 1)} \sigma_L\left(\frac{\omega}{\alpha}\right) \quad (5.6)$$

where

$$\rho_L\left(\frac{\omega}{\alpha}\right) = \sum_{p \in \mathbb{Z}} (y_+ + 2\pi p + i0)^{-i\omega/\alpha} \quad (5.7)$$

and we set $k_J^2 = 4$ (tachyon mass-shell). ρ_L and σ_L can be expressed in terms of generalized Riemann zeta functions

$$\xi(\mathbb{Z}, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^{\mathbb{Z}}}$$

as

$$\rho_L\left(\frac{\omega}{\alpha}\right) = (2\pi)^{-i\omega/\alpha} \left\{ \xi\left(\frac{i\omega}{\alpha}, \frac{y_+}{2\pi}\right) + e^{\pi\omega/\alpha} \xi\left(\frac{i\omega}{\alpha}, -\frac{y_+}{2\pi}\right) - \left(\frac{y_+}{2\pi}\right)^{-i\omega/\alpha} \right\} \quad (5.8)$$

and

$$\sigma_L\left(\frac{\omega}{\alpha}\right) = \frac{e^{\frac{\omega\pi}{2\alpha}}}{(4\pi)^{1+\frac{i\omega}{\alpha}}} \left\{ e^{\frac{\omega\pi}{\alpha}} \left[\xi\left(1+\frac{i\omega}{\alpha}, \frac{y_+}{4\pi} + \frac{1}{2}\right) - \xi\left(1+\frac{i\omega}{\alpha}, \frac{y_+}{4\pi}\right) \right] - e^{-\frac{\omega\pi}{\alpha}} \left[\xi\left(1+\frac{i\omega}{\alpha}, -\frac{y_+}{4\pi} - \frac{1}{2}\right) - \xi\left(1+\frac{i\omega}{\alpha}, -\frac{y_+}{4\pi}\right) \right] \right\} \quad (5.9)$$

$\rho_L(\omega/\alpha)$ is the "form factor" of the string. We see that ρ_L is periodic in y_+ with period 2π . The Eq. (5.7) for ρ_L can be interpreted as a sum of images arising from the periodic nature of the closed string space. This is clearly a typical feature of strings absent for other systems. [For a two-dimensional massless field theory in Rindler space, $\rho(\omega/\alpha) = 2\pi/\omega$; for massless D-dimensional fields ρ is a polynomial of degree D-3 in ω^{10} .]

For the right movers of the string $\vec{X}_\mu(x'_-)$, we obtain analogous expressions by considering correlations along a line parallel to the U axis and with $x_- = e^{\alpha x'_-} + \epsilon$. We get

$$F_R(\omega) = \frac{2\pi}{\omega} \frac{1}{(e^{2\pi\omega/\alpha} - 1)} \rho_R\left(\frac{\omega}{\alpha}\right) \quad (5.10)$$

where

$$\rho_R\left(\frac{\omega}{\alpha}\right) = \rho_L\left(-\frac{\omega}{\alpha}\right)$$

It follows that $F_L(-\omega)$ and $F_R(\omega)$ satisfy the relations

$$\begin{aligned} F_L(-\omega) &= e^{\frac{2\pi\omega}{\alpha}} F_R(\omega) \\ F_R(-\omega) &= e^{\frac{2\pi\omega}{\alpha}} F_L(\omega) \end{aligned} \quad (5.11)$$

namely the detailed balance relations. The accelerated correlation functions, Eqs. (5.1) and (5.2), are periodic in the accelerated time with period $i(2\pi/\alpha)$ and Eqs. (5.11) reflect this fact. We see that the response functions $F_L(\omega)$ and $F_R(\omega)$ are proportional to a Planckian spectrum with temperature $\alpha/2\pi$. Therefore, we have found another manifestation of the thermal properties already found in Section IV for $N(\lambda_n)$.

VI. STRINGS NEAR BLACK HOLES

Our investigation of strings in Rindler space-time can be applied to the case of strings in a black hole background. Black hole solutions of Einstein equations exist in D-space-time dimensions ($D > 4$)¹¹⁾. These solutions are asymptotically flat and generalize the Schwarzschild space-time of four dimensions; they have the metric

$$ds^2 = - \left(1 - \frac{C}{R^{D-3}} \right) dT^2 + \frac{dR^2}{\left(1 - \frac{C}{R^{D-3}} \right)} + R^2 d\Omega_{D-2}^2 \quad (6.1)$$

R is the radial co-ordinate, $d\Omega_D^2$ is the line element on the unit D-sphere and the constant C is > 0 . The surface

$$R = C^{1/(D-3)} \equiv R_s$$

is an event horizon (there are both past and future event horizons) and $R = 0$ is a space-like singularity. The horizon radius R_s is related to the black hole mass M by

$$C = 16 \pi G \frac{M}{(D-2) A_{D-2}}$$

where

$$A_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}$$

is the area of a unit (D-2) sphere and G has dimensions of length^{D-2}.

The mass and the surface gravity K of the black hole are related by

$$K = \left(\frac{D-3}{2 R_s} \right) = \frac{(D-3)}{2} \left[\frac{(D-2) A_{D-2}}{16 \pi G M} \right]^{\frac{1}{D-3}} \quad (6.2)$$

For $D = 4$ this yields the standard relations $R_s = 2GM$ and $K = 1/(4GM)$.

The Kruskal extension of this Schwarzschild manifold is given by the mapping

$$r_K \pm t_K = e^{K(R^* \pm T)} \quad (6.3)$$

where

$$R^* = R + R_s^{D-3} \int \frac{dR}{(R^{D-3} - R_s^{D-3})},$$

$$-\infty \leq R^*, T \leq +\infty$$

and K is given by Eq. (6.2).

This is the same exponential mapping as Eq. (2.3) with K instead of α . The Rindler co-ordinates are similar to the Schwarzschild (R^*, T) ones and the Minkowskian co-ordinates are similar to the Kruskal (global) co-ordinates (r_K, t_K) . The event horizon $R = R_s$ corresponds to $R^* = -\infty$. As discussed in Section II, a large cut-off ($H \gtrsim 1/K$) is needed in the negative Schwarzschild co-ordinate R^* . This shifting of the horizon is equivalent to considering a shifting ε in the mapping

$$r_K \pm t_K + \varepsilon = e^{K(R^* \pm T)} \quad (6.4)$$

with

$$\varepsilon = e^{-KH} \sim l_p^2$$

and thus

$$-H \leq R^* \leq +\infty$$

reflecting the fact that a classical description of the geometry is no longer valid at distances of order of the Planck length. We will take

$$\varepsilon = \frac{\lambda_c}{R_s}, \quad \text{where} \quad \lambda_c = \frac{1}{M}$$

is the Compton length of the black hole (here $\hbar = c = 1$). Thus the shifting of the horizon is $H = (1/K) \ln \varepsilon$, with

$$\xi = \pi^{1/2} \left[\frac{(D-2)}{8 \Gamma(\frac{D-1}{2})} \right]^{\frac{1}{D-3}} \left(\frac{M_{pl}}{M} \right)^{\frac{D-2}{D-3}} \quad (6.5)$$

and

$$K = \frac{\pi^{1/2}}{2} (D-3) \left[\frac{(D-2)}{8 \Gamma(\frac{D-1}{2})} \right] \left(\frac{M_{pl}}{M} \right)^{\frac{1}{D-3}} M_{pl} \quad (6.6)$$

Following on the same lines of argument discussed in Sections II and III, for the choice of gauge and parametrizations of the string world sheet [Eqs. (2.30), (3.2) and (3.3)] and considering only left movers, we have

$$\overleftarrow{v} \equiv r_K + t_K = p_+ x_+ \quad (6.7)$$

$$V \equiv R^* + T = x'_+ + \frac{1}{K} \log p_+ \quad (6.8)$$

and

$$x_+ + \xi = e^{K x'_+}$$

(Here the longitudinal direction of the string is in the radial direction.)

Positive frequency modes, $\hat{\psi}_n$ with respect to the Kruskal time t_K and $\hat{\phi}_n$ with respect to the Schwarzschild time T can be defined. The Schwarzschild frequency is equal to

$$\lambda_n = \frac{2\pi}{\pi_\xi} n \quad n = 1, 2, \dots \quad (6.9)$$

where

$$\pi_\xi = \frac{1}{K} \ln \left(\frac{2\pi}{\xi} + 1 \right)$$

K and ϵ are given by Eqs. (6.5) and (6.6). For $\epsilon \ll 1$, the frequency spacing tends to zero reflecting the stretching effect of the string near the horizon as seen by a Schwarzschild external observer. Associated to the modes $\hat{\psi}_n$ and $\hat{\phi}_n$

we will have Kruskal and Schwarzschild operators a_n and C_n respectively and a vacuum state defined by

$$a_n |0_K\rangle = 0, \quad \forall n > 1$$

On the other hand, in order to have a smooth Euclidean manifold from a black hole space-time with topology $R^2 \times S^{D-2}$, the Schwarzschild imaginary time iT must be identified with a period

$$\beta = \frac{2\pi}{K}$$

Then the same periodicity in the imaginary time appears in the correlation functions of string co-ordinates, indicating that the string is in equilibrium with a heat bath at the Hawking temperature

$$T_s = \frac{K}{2\pi} \quad (6.10)$$

The same temperature T is recovered in the function $N(\lambda_m)$, i.e.,

$$N(\lambda_n) = \langle 0_K | C_{\lambda_n}^+ C_{\lambda_n} | 0_K \rangle,$$

which gives a Planckian distribution for the Schwarzschild modes but with a "filter" $|g(\lambda_n)|^2$ equal to the absorption cross-section of the black hole. In the spectrum $N(n)$ in which frequency is measured in dimensionless units $1, 2, \dots$, the temperature of the Planckian distribution is equal to

$$T_0 = \frac{1}{4\pi^2} \ln \left(\frac{2\pi}{\epsilon} \right) \gg 1 \quad (6.11)$$

with ϵ given by Eq. (6.5).

One can consider different higher dimensional black hole space-times, namely a 26-dimensional [Eq. (6.1)] or a four-dimensional black hole with the extra 22 dimensions compactified in a torus¹²⁾. Intermediary situations can also be envisaged but it must be noticed that the qualitative properties of the string quantization will be the same since they depend upon the horizon structure in the two variables R, T (or X^0, X^1 , for Rindler space). We hope to come back to this problem elsewhere.

It can be noticed that the Hagedorn temperature (T_m) in this context is

$$T_m = \frac{1}{\sqrt{\alpha'}} \sim M_{pl}$$

Then, from Eqs. (6.6) and (6.10):

$$\frac{T_s}{T_m} \sim \left(\frac{M_{pl}}{M} \right)^{\frac{1}{D-3}}$$

and we have

$$T_s \lesssim T_m ,$$

since the basic requirement of the present semiclassical treatment is $M \gtrsim M_{pl}$.

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FIGURE CAPTION

Worldsheet of the inertial string in Minkowski co-ordinates. $u = 0$ and $v = 0$ are the horizons of Rindler space.