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THE ORDERING OF LOW-LYING BOUND STATES OF THREE IDENTICAL PARTICLES

Jean-Marc RICHARD

Institut des Sciences Nucléaires, 38026 Grenoble Cédex, France

and

Pierre TAXIL

CERN - Division Théorique - CH 1211 Genève 23, Suisse

Abstract

New results are presented on the ordering of bound states of three identical particles, a problem inspired by baryon spectroscopy. We first study the case of a perturbed harmonic oscillator and relate the splitting pattern to the level spacings in the two-body problem. We also obtain much more general results, valid for almost any symmetric potential, not necessarily pairwise. The proof is given in the framework of the hyperspherical formalism.

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1 - Introduction

Many results have been established on the level order and spacing in the two-body problem, with application to quarkonium spectroscopy or other fields [1,2]. In comparison, little is known about the ordering of three-body bound states, even in the relatively simple case of identical particles.

In baryon spectroscopy, various models consisting of three quarks bound by harmonic two-body forces have been used for a long time [3-4]. Anharmonicity corrections were introduced by several authors [4-7] who studied the effect of such perturbations around harmonic oscillator (h.o.) solutions, with application to the phenomenology of excited multiplets, associated with the $N = 1, 2, 3, \dots$ levels of the h.o. Even realistic potentials, like a Coulomb attraction supplemented by linear confinement, are often treated perturbatively around their optimal harmonic approximation [7], a first step towards a complete variational calculation.

In the present paper, we intend to improve and clarify the existing results on anharmonic corrections and to emphasize the link between the splitting pattern of the three-body multiplets and the properties of the two-body binding energy as a function of the angular momentum and radial number. These, in turn, are rigorously related to the convexity properties of the two-body potential.

We also show that the level ordering obtained in the particular h.o. case is in fact valid for a much wider class of potentials, as soon as one disentangles the collective radial excitations (sometimes called breathing modes) from the individual excitations of orbital or radial type.

2. Level order in a nearly harmonic potential

We consider here a system with equal constituent masses $m_1 = m_2 = m_3 = 1$ interacting through the potential

$$V = \sum_{i < j} \frac{2}{3} r_{ij}^2 + v(r_{ij}) \quad (1)$$

where v is a small perturbation. We introduce the Jacobi-coordinates

$$\vec{p} = \vec{r}_2 - \vec{r}_1 \quad \vec{\lambda} = \frac{(2 \vec{r}_3 - \vec{r}_1 - \vec{r}_2)}{\sqrt{3}} \quad (2)$$

and write the unperturbed Hamiltonian as

$$H_0 = \frac{\vec{p}^2}{6} + h(\vec{p}) + h(\vec{\lambda}) \quad (3)$$

where $h(\vec{p}) = \vec{p}_p^2 + \vec{p}^2$ is the three dimensional h.o., whose eigenstates are noted $|n, \ell, m\rangle$ or $|n, \ell\rangle$. We are reminded of the corresponding wave-functions in the Appendix. One may now forget the centre-of-mass motion and obtain by simple factorization the eigenstates of H_0 with increasing number of excited states quanta $N = 2 n_p + \ell_p + 2 n_\lambda + \ell_\lambda$.

a) $N = 0$ and $N = 1$

If a small perturbation $v(r_{ij})$ acts on each quark pair, the ground state with $N = 0$ is shifted by $\epsilon_0 = 3 \delta(0,0)$ where

$$\delta(n, \ell) = \langle n, \ell | v(r) | n, \ell \rangle \quad (4)$$

denotes the matrix elements of v in the h.o. basis.

The $N = 1$ level contains degenerate ρ -excitation with $\ell_p = 1$ and λ -excitations with $\ell_\lambda = 1$. The former receives a shift $\delta(0,1)$ from the (1-2) pair. The contribution of the (2-3) pair, $\langle (0,1) (0,0) | v(r_{23}) | (0,1) (0,0) \rangle$ is calculated by expanding the states in a basis where the Jacobi coordinates are chosen as $\vec{p}' = \vec{r}_3 - \vec{r}_2$ and the corresponding $\vec{\lambda}'$ (this is called a Brody-Moshinsky transformation in Nuclear Physics). The last pair $v(r_{31})$ gives the same shift, and then

$$\epsilon_p = \frac{3}{2} \delta(0,0) + \frac{3}{2} \delta(0,1) \quad (5)$$

The shift of the λ -excitation is also

$$\epsilon_\lambda = \frac{3}{2} \delta(0,0) + \frac{3}{2} \delta(0,1) \quad (6)$$

In fact, the degeneracy of the ($N=1$) levels remains to all orders, as a consequence of permutation symmetry.

b) The $N = 2$ multiplet

This warm-up leads us to the ($N = 2$) level, which contains different types of states. The collective radial excitation, usually associated to the Roper resonances in baryon spectroscopy is

$$| 56', 0^+ > = \sqrt{\frac{1}{2}} ((1,0) (0,0) + (0,0) (1,0)) \quad (7)$$

The orthogonal combination of individual radial excitations,

$$| 70, 0^+ >_\lambda = \sqrt{\frac{1}{2}} ((1,0) (0,0) - (0,0) (1,0)) \quad (8)$$

is a member of a doublet of mixed symmetry, whose partner is obtained by coupling two orbital excitations $\ell_\rho = \ell_\lambda = 1$ to an overall S-wave

$$| 70, 0^+ >_\rho = \sum C(1,m,1,-m; 0,0) (0,1,m) (0,1,-m) \quad (9)$$

where C is the appropriate Clebsh-Gordan coefficient. One also gets an antisymmetric P-wave

$$| 20, 1^+ > = \sum C (1,m,1,m'; 1,m'') (0,1,m) (0,1,m') \quad (10)$$

while the D-wave sector can be separated into a symmetric state

$$| 56, 2^+ > = \sqrt{\frac{1}{2}} ((0,2) (0,0) + (0,0) (0,2)) \quad (11)$$

and a mixed symmetry doublet

$$[70, 2^+ >_\lambda = \sqrt{\frac{1}{2}} ((0,2) (0,0) - (0,0) (0,2)) \quad (12)$$

$$[70, 2^+ >_\rho = \sum C(1,m,1,m'; 1,m'') (0,1,m) (0,1,m') \quad (13)$$

The label 20, 56 or 70 refers to the standard $SU(6)$ classification. The calculation of the first order energy shifts is rather easy if one makes use of the permutation properties, leading to

$$\epsilon(56', 0^+) = \frac{3}{2} [\delta(0,0) + \delta(1,0)]$$

$$\epsilon(56, 2^+) = \frac{3}{2} [\delta(0,0) + \delta(0,2)]$$

$$\epsilon(20, 1^+) = 3 \delta(0,1) \quad (14)$$

$$\epsilon(70, 0^+) = \frac{3}{4} [\delta(0,0) + \delta(1,0)] + \frac{3}{2} \delta(0,1)$$

$$\epsilon(70, 2^+) = \frac{3}{4} [\delta(0,0) + \delta(0,2)] + \frac{3}{2} \delta(0,1)$$

There are, in fact, only 3 independent matrix elements in eq. (14), since, from the polynomial structure of the two-body wave-functions, $\delta(1,0)$ can be expressed in terms of the $\delta(0,\ell)$ with $\ell = 0,1,2$ as per eq. (A 6) of the Appendix. This enables us to rewrite (14) as

$$\epsilon(20, 1^+) = \bar{\epsilon} + \frac{1}{4} \Delta$$

$$\epsilon(70, 2^+) = \bar{\epsilon} + \frac{1}{20} \Delta$$

$$\epsilon(56, 2^+) = \bar{\epsilon} - \frac{3}{20} \Delta \quad (15)$$

$$\epsilon(70, 0^+) = \bar{\epsilon} - \frac{1}{4} \Delta$$

$$\epsilon(56', 0^+) = \bar{\epsilon} - \frac{3}{4} \Delta$$

with $\bar{\epsilon} = \frac{15}{16} \delta(0,0) + \frac{9}{8} \delta(0,1) + \frac{15}{16} \delta(0,2)$. This corresponds to the famous splitting pattern of Fig. 1 [5-7].

Fig. 1 Splitting pattern of the $N = 2$ multiplet due to anharmonic corrections.

The breaking of the degeneracy of the $N = 2$ multiplet has been analyzed by several authors [5-7] who checked that Δ is a positive quantity for simple and plausible choices of the interquark potential such as Coulomb plus linear. Our contribution consists of noting that $\Delta \gtrless 0$ if

$$Y = \frac{d}{dr} \left(\frac{1}{r} \frac{dv}{dr} \right) \gtrless 0 \quad \forall r \quad (16)$$

that is if v is a convex or concave function of r^2 . This results from theorems on the ordering and spacing of the levels for two-body systems. The splitting Δ can, indeed, be written as

$$\Delta = \frac{15}{4} [2\delta(0,1) - \delta(0,0) - \delta(0,2)] \quad (17)$$

$$\Delta = \frac{3}{2} [2\delta(0,1) - \delta(0,0) - \delta(1,0)] \quad (18)$$

$$\Delta = \frac{5}{2} [\delta(0,2) - \delta(1,0)] \quad (19)$$

The relation between the concavity of the potential with respect to r^2 and the sign of Δ in eq. (17), i.e. the concavity of the lowest Regge trajectory $E(\ell)$ has been known for long time. It was proved by Grosse and Martin [8] for perturbations around the harmonic oscillator and a non-perturbative proof has been obtained recently [9].

On the other hand, from eq. (18), Δ measures whether the first orbital excitation is closer to the ground state than the radial excitation. It can be proved, for perturbation around the h.o., that this is related to the sign of Y [2]. The sign of Δ is also immediately understood from eq. (19) since $Y > 0$ implies $\delta(1,0) < \delta(0,2)$ [10], a perturbative result whose non-perturbative generalization exists [11].

c) $N = 3$

The extension to the next h.o. multiplet is straightforward. The various (n_p, ℓ_p) $(n_\lambda, \ell_\lambda)$ states with $H = 2 n_p + \ell_p + 2 n_\lambda + \ell_\lambda = 3$ can be arranged in combinations of well-defined angular momentum and permutation properties. The $(\ell = 3)$ sub-space consists of

$$\begin{aligned}
 |56, 3\rangle &= \frac{1}{2} (0,0) (0,3) - \frac{\sqrt{3}}{2} (0,2) (0,1) \\
 |20, 3\rangle &= \frac{\sqrt{3}}{2} (0,1) (0,2) - \frac{1}{2} (0,3) (0,0) \\
 |70, 3\rangle_\lambda &= \frac{\sqrt{3}}{2} (0,0) (0,3) + \frac{1}{2} (0,2) (0,1) \\
 |70, 3\rangle_\rho &= \frac{1}{2} (0,1) (0,2) + \frac{\sqrt{3}}{2} (0,3) (0,0)
 \end{aligned} \tag{20}$$

The $(\ell = 2)$ states are of mixed symmetry

$$\begin{aligned}
 |70, 2\rangle_\lambda &= [(0,2)(0,1)]_{\ell=2} \\
 |70, 2\rangle_\rho &= [(0,1)(0,2)]_{\ell=2}
 \end{aligned} \tag{21}$$

where the square bracket denotes the appropriate Clebsch-Gordan coupling.

The $(\ell = 1)$ space contains

$$\begin{aligned}
 |56, 1\rangle &= \frac{1}{\sqrt{12}} [-\sqrt{3} (0,0)(1,1) + \sqrt{5}(1,0)(0,1) + 2 [(0,2)(0,1)]_{\ell=1}] \\
 |20, 1\rangle &= \frac{1}{\sqrt{12}} [-\sqrt{3}(1,1)(0,0) + \sqrt{5}(0,1)(1,0) + 2 [(0,1)(0,2)]_{\ell=1}] \\
 |70, 1\rangle'_\lambda &= \frac{1}{2\sqrt{2}} [\sqrt{5} (0,0)(1,1) + \sqrt{3} (1,0)(0,1)] \\
 |70, 1\rangle'_\rho &= \frac{1}{2\sqrt{2}} [\sqrt{5} (1,1)(0,0) + \sqrt{3} (0,1)(1,0)] \\
 |70, 1\rangle''_\lambda &= \frac{1}{\sqrt{24}} [\sqrt{3} (0,0)(1,1) - \sqrt{5} (1,0)(0,1) + 4[(0,2)(0,1)]_{\ell=1}] \\
 |70, 1\rangle''_\rho &= \frac{1}{\sqrt{24}} [\sqrt{3} (1,1)(0,0) - \sqrt{5} (0,1)(1,0) + 4[(0,2)(0,1)]_{\ell=1}]
 \end{aligned} \tag{22}$$

This leads to the shifts (δ_ℓ denotes $\delta(0,\ell)$ defined in eq. 4, and the radial excitations are removed by using eqs A6 and A7) :

$$\begin{aligned}
 \epsilon(56,3^-) &= \frac{3}{4}\delta_0 + \frac{9}{4}\delta_2 = \epsilon + \frac{3}{4}\eta + \frac{\gamma}{2} \\
 \epsilon(20,3^-) &= \frac{9}{4}\delta_1 + \frac{3}{4}\delta_3 = \epsilon + \frac{3}{4}\eta - \frac{\gamma}{2} \\
 \epsilon(70,3^-) &= \frac{9}{8}\delta_0 + \frac{3}{8}\delta_2 + \frac{9}{8}\delta_3 = \epsilon - \frac{5}{4}\eta \\
 \epsilon(70,2^-) &= \frac{3}{2}\delta_1 + \frac{3}{2}\delta_2 = \epsilon + \frac{7}{4}\eta \\
 \epsilon(56,1^-) &= \frac{21}{8}\delta_0 - \frac{15}{4}\delta_1 + \frac{33}{8}\delta_2 = \epsilon - \frac{7}{4}\eta + \frac{7}{4}\gamma \\
 \epsilon(20,1^-) &= \frac{33}{8}\delta_1 - \frac{15}{4}\delta_2 + \frac{21}{8}\delta_3 = \epsilon - \frac{7}{4}\eta - \frac{7}{4}\gamma \\
 \epsilon(70,1^-)' &= \frac{21}{32}\delta_0 + \frac{27}{32}\delta_1 + \frac{27}{32}\delta_2 + \frac{21}{32}\delta_3 = \epsilon \\
 \epsilon(70,1^-)' &= \frac{57}{32}\delta_0 + \frac{39}{32}\delta_1 - \frac{105}{32}\delta_2 + \frac{105}{32}\delta_3 = \epsilon - 5\eta - \gamma \\
 \langle (70,1^-)' | v | (70,1^-)' \rangle &= \frac{3\sqrt{5}}{32} [-\delta_0 + 9\delta_1 - 15\delta_2 + 7\delta_3] = -\frac{3\sqrt{5}}{4}\eta - \frac{\sqrt{5}}{2}\gamma
 \end{aligned} \tag{23}$$

We have introduced an averaged orbital shift :

$$\epsilon = \frac{3}{32} [7\delta_0 + 9\delta_1 + 9\delta_2 + 7\delta_3] \tag{24}$$

and the following quantities

$$\begin{aligned}
 \eta &= \frac{f(1)+f(2)}{2} \\
 \gamma &= f(2) - f(1)
 \end{aligned} \tag{25}$$

where

$$f(\ell) = \frac{3}{4} [2\delta(0,\ell) - \delta(0,\ell+1) - \delta(0,\ell-1)] \tag{26}$$

Their meaning is rather simple : η measures the average convexity of $\delta(0, \ell)$ between $\ell = 0$ and $\ell = 3$ whereas γ is a discretized third derivative in ℓ which indicates the increase or decrease of the convexity.

Degenerate perturbation theory has been used for the mixed symmetry sector with $\ell = 1$. In some cases, one expects $|\gamma| \ll |\eta|$, leading to the approximate eigenvalues

$$\begin{aligned}\lambda_1(70,1^-) &= \varepsilon + 0.510 \eta + 0.538 \gamma \\ \lambda_2(70,1^-) &= \varepsilon - 5.510 \eta - 1.538 \gamma\end{aligned}\tag{27}$$

Otherwise, an exact diagonalisation is needed and the quantities η and γ do not enter linearly in the expression of the shifts. The splitting pattern of $N = 3$ levels is displayed in Fig. 2 for a linear potential $v(r) = r$.

Fig. 2 Splitting of the $N = 3$ multiplet perturbed by a linear potential..

The above results agree with the expressions given by A.J.G. Hey et al [12], even for the $[56, 1^-]$ case, which is correct in the first paper of ref. [12] but is misprinted in the second one. A further test of the calculations in the ($\ell = 1$) sector is the following. Thanks to the permutation properties, one easily derives

$$\frac{1}{3} \varepsilon(56, 1^-) + \frac{1}{3} \varepsilon(20,1^-) + \frac{2}{3} \varepsilon(70,1^-) + \frac{2}{3} \varepsilon(70,1^-)' = \text{Trace } v(r_{12}) \equiv T \tag{28}$$

but the trace can be calculated either in the elaborate basis (22) or in the original basis consisting of $(0,0)(1,1)$; $(1,0)(0,1)$; $[(0,2)(0,1)]_{\ell=1}$; $(1,1)(0,0)$; $(0,1)(1,0)$ and $[(0,1)(0,2)]_{\ell=1}$ leading to the result

$$T = \frac{5}{2} \delta_0 + \frac{3}{2} \delta_1 - \frac{3}{2} \delta_2 + \frac{7}{2} \delta_3 = 2\epsilon - \frac{9}{2} \eta - \frac{2}{3} \gamma \quad (29)$$

The sign of the splittings η and γ can be related to the properties of the two-body potential $v(r)$. First η given in eq. (25) obviously obeys the same behaviour as Δ , η is governed by the quantity Y (eq. 16). Concerning the sign of γ , our result is that $\gamma \leq 0$ if

$$W = \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(\frac{1}{r} \frac{dv}{dr} \right) \right] \geq 0 \quad (30)$$

and $\lim_{r \rightarrow 0} r^3 v = 0$ as $r \rightarrow 0$.

The proof, given in the Appendix, follows the strategy already used to demonstrate a theorem on the spacings of the $\ell = 0$ levels in two-body systems [2]. The starting point is the observation that γ vanishes for $v = \text{const}$, and $v = r^2$, preserving the linearity of Regge trajectories $E(\ell)$, and also for $v = r^4$.

Let us consider for illustration the case of a perturbation $v(r) = g r^\alpha$, with g small. We get

$$Y = g \alpha(\alpha-2) r^{\alpha-3} \quad (31)$$

$$W = g \alpha(\alpha-2) (\alpha-4) r^{\alpha-5} \quad (32)$$

Thus, for a smooth perturbing potential with $0 < \alpha < 2$, as favoured by various phenomenological studies [13], $Y < 0$ and $W > 0$, implying $\eta > 0$ and $\gamma < 0$. We obtain the same result for a potential $v = -\frac{a}{r} + br$ ($a, b > 0$) when it is treated perturbatively around its optimal harmonic approximation.

d) $N \geq 4$

When N increases, the dimension of the h.o. eigenspaces becomes very large, and one has more and more to deal with matrix diagonalizations to compute the energy shifts, so that the link between the 3-body and the 2-body matrix elements is not linear. The main result is that the shifts of the 3-body states with N quanta involve all 2-body shifts $\delta(0, \ell)$ from $\ell = 0$ to $\ell = N$.

3. Generalization

We now extend appreciably the domain of our investigations, restricted so far to pairwise, nearly harmonic, potentials by considering the class of interactions of the form:

$$V(\vec{r}_1, \vec{r}_2, \vec{r}_3) = V_0(\xi) + v(\vec{\rho}, \vec{\lambda}) \quad (33)$$

where $\xi^2 = \frac{2}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)$. The main potential $V_0(\xi)$ is said to be hyperscalar and $v(\vec{\rho}, \vec{\lambda})$, which is symmetric, will be treated to first order. It is worth noting that most symmetric potentials V one can encounter in practice are of this type, in particular the confining potentials used in baryon spectroscopy [13]. Results are in order :

With such a symmetric and nearly hyperscalar potential, the first excitations with positive parity consist of :

- i) a collective excitation of the ground state, noted $[56', 0^+]$
- ii) a multiplet with states whose permutation properties and angular momentum correspond to the notations $[20, 1^+]$, $[70, 2^+]$, $[56, 2^+]$, $[70, 0^+]$ whose successive spacings are in ratio $2 : 2 : 1$, as illustrated in Fig. 3.

Fig. 3 Splitting pattern of the positive energy excitations in a nearly hyperscalar potential.

Now, if $V_0(\xi) = \xi^2$ and $v(\vec{\rho}, \vec{\lambda})$ is a sum of two-body potentials, then $\Delta = \Delta'$ in Fig. (3) and one recovers the results of the previous section.

Similarly, one can state that for a symmetric and nearly hyperscalar potential, the first excitations in the negative parity sector consist of :

- i) a collective excitation of the negative parity ground state, noted as $[70, 1^-]$
- ii) a series of states whose permutation properties and angular momentum correspond to the notations $[56, 3^-]$, $[20, 3^-]$, $[70, 3^-]$, $[70, 2^-]$, $[56, 1^-]$, $[20, 1^-]$ and $[70, 1^-]$, whose

splitting pattern is described in Fig. 4. The shifts of these 7 states can be parametrized in terms of 3 parameters ε , η and γ , as in the first 7 equations of eq. (23).

If $V_0(\xi) = \xi^2$, the $[70, 1^-]$ state is accidentally degenerated with the $[70, 1^-]$ one to zeroth order and a matrix diagonalization has to be performed to compute the first order shifts. The 2 last equations of eq.(23) become then valid.

We will give the proof within the hyperspherical formalism [14]. It involves some technical calculations, whose details will be omitted. Let us first introduce some notations. The relative coordinates of eq.(2) built a six-dimensional vector $\tilde{\mathbf{x}} = (\vec{\rho}, \vec{\lambda})$, whose spherical coordinates are $\tilde{\mathbf{x}} = (\xi, \Omega)$, with

$$\xi^2 = \vec{\lambda}^2 + \vec{\rho}^2, \quad \Omega = (\hat{\lambda}, \hat{\rho}, \text{tg}^{-1} \frac{\rho}{\lambda}) \quad (34)$$

The 3-body Schrödinger equation, rewritten as

$$[-\Delta_6 + V(\tilde{\mathbf{x}}) - E] \Psi(\tilde{\mathbf{x}}) = 0 \quad (35)$$

can be solved by generalized partial-wave expansion

$$\Psi(\vec{\rho}, \vec{\lambda}) = \xi^{-5/2} \sum_{[K]} u_{[K]}(\xi) P_{[K]}(\Omega) \quad (36)$$

where $[K]$ denotes the grand angular momentum K and its associated magnetic numbers and $P_{[K]}(\Omega)$ the orthonormalized hyperspherical harmonics. This leads to a system of coupled equations

$$\left[\frac{d^2}{d\xi^2} - \frac{(K+3/2)(K+5/2)}{\xi^2} + E - V_{[K][K]}(\xi) \right] u_{[K]}(\xi) = \sum_{[K'] \neq [K]} V_{[K][K']}(\xi) u_{[K']}(\xi) \quad (37)$$

which is formally equivalent to (35). The matrix elements are :

$$V_{[K][K']}(\xi) = \int d\Omega P_{[K]}^*(\Omega) V(\xi, \Omega) P_{[K']}(\Omega) \quad (38)$$

For potentials of type (33), to zeroth order, the equations decouple and the effective potential $(K+\frac{3}{2})(K+\frac{5}{2})/\xi^2 + V_0(\xi)$ and hence the unperturbed radial function $u_{[K]}^{(0)}$ depends only on K , not on the magnetic numbers. Apart from when there exists an accidental degeneracy of radial and orbital excitations of $V_0(\xi)$, the coupling potentials $v_{[K][K']}(\xi)$ with $K \neq K'$ will

contribute only to second order. If, furthermore, the zeroth order states of given K have been uniquely identified by their rotation or permutation properties (this is possible for $K \leq 3$) the first order shifts are given by the simple integrals

$$\varepsilon([K], v) = \int d^3\rho \, d^3\lambda \, |\Psi_{[K]}^{(o)}(\vec{\rho}, \vec{\lambda})|^2 v(\vec{\rho}, \vec{\lambda}) = \int |u_{[K]}^{(o)}(\xi)|^2 d\xi v_{[K][K]} \quad (39)$$

The point now is that $v_{[K][K]}$ has a very simple structure as a function of the magnetic numbers. First, we expand $v(\vec{\rho}, \vec{\lambda})$ itself as

$$v(\vec{\rho}, \vec{\lambda}) = \pi^{3/2} [q_0(\Omega) v_0(\xi) + q_4(\Omega) v_4(\xi) + q_6(\Omega) v_6(\xi) + \dots] \quad (40)$$

where $q_0(\Omega) = \pi^{-3/2}$, $q_4(\Omega)$, $q_6(\Omega)$, ... are the hyperspherical harmonics which are fully symmetric under permutations and scalar under ordinary rotations. This leads to

$$v_{[K][K]} = \sum_{K'} v_{K'}(\xi) \langle [K], K', [K] \rangle \quad (41)$$

where we have introduced the so-called 3H coefficients

$$\langle [K], K', [K] \rangle = \pi^{3/2} \int d\Omega P_{[K]}^*(\Omega) q_{K'}(\Omega) P_{[K]}(\Omega) \quad (42)$$

which are purely geometrical in nature and hence independent from the potential. Their computation eventually leads to the following partial-wave potentials $v_{[K][K]}$:

$$\begin{array}{lll} K = 0 & [56, 0^+] & v_0(\xi) \\ K = 1 & [70, 1^-] & v_0(\xi) \end{array} \quad (43)$$

$$\begin{array}{lll} & [20, 1^+] & v_0(\xi) + 5 \frac{\sqrt{3}}{15} v_4(\xi) \\ & [70, 2^+] & v_0(\xi) + \frac{\sqrt{3}}{15} v_4(\xi) \\ K=2 & [56, 2^+] & v_0(\xi) - 3 \frac{\sqrt{3}}{15} v_4(\xi) \\ & [70, 2^+] & v_0(\xi) - 5 \frac{\sqrt{3}}{15} v_4(\xi) \end{array} \quad (44)$$

$$\begin{array}{ll}
[56,3^-] & v_0(\xi) + \frac{\sqrt{3}}{7} v_4(\xi) + \frac{\sqrt{2}}{7} v_6(\xi) \\
[20,3^-] & v_0(\xi) + \frac{\sqrt{3}}{7} v_4(\xi) - \frac{\sqrt{2}}{7} v_6(\xi) \\
[70,3^-] & v_0(\xi) - \frac{5\sqrt{3}}{21} v_4(\xi) \\
K=3 \quad [70,2^-] & v_0(\xi) + \frac{\sqrt{3}}{3} v_4(\xi) \quad (45) \\
[56,1^-] & v_0(\xi) - \frac{\sqrt{3}}{3} v_4(\xi) + \frac{\sqrt{2}}{2} v_6(\xi) \\
[20,1^-] & v_0(\xi) - \frac{\sqrt{3}}{3} v_4(\xi) - \frac{\sqrt{2}}{2} v_6(\xi) \\
[70,1^-] & v_0(\xi)
\end{array}$$

The splitting patterns result directly from the above expressions, when inserted in eq.(39). The 3H coefficients of eq.(42) can be calculated by straightforward angular integration, using the explicit expressions of the hyperspherical harmonics [15]. Alternatively, one can derive the 3H coefficients from the special case of the h.o. perturbed by simple potentials like $\sum r_{ij}^n$, whose multipole content (40) is easily obtained. This is another example of the many links between the harmonic oscillator and the hyperspherical formalisms in few-body problems [16]. A harmonic confinement is, indeed, a special case of hyperscalar potential, and the number of quanta N is given by $N = K + 2n$, where n is the number of nodes in the hyperradial wave- function $u_K(\xi)$. This alternative derivation of the 3H coefficients works as follows.

There is first a triangular rule $0 \leq K' \leq 2K$ constraining the 3H coefficients, which generalizes the usual triangular rule for Clebsh-Gordan coupling. Eq.(43) is a direct consequence of this rule, which also constrains the $K=2$ and $K=3$ partial-wave potentials to be of the type :

$$\begin{array}{ll}
K=2 & \alpha_0 v_0(\xi) + \alpha_4 v_4(\xi) \\
K=3 & \beta_0 v_0(\xi) + \beta_4 v_4(\xi) + \beta_6 v_6(\xi) \quad (46)
\end{array}$$

From the $v = \text{constant}$ or $v = \sum r_{ij}^2$ cases, one checks that $\alpha_0 = \beta_0 = 1$. The coefficients α_4 and β_4 can be deduced from eqs. (14) and (24) applied to the potential

$$\sum r_{ij}^4 = \frac{15}{16} \xi^4 \pi^{3/2} q_0(\Omega) - \frac{\sqrt{3}}{16} \xi^4 \pi^{3/2} q_4(\Omega) \quad (47)$$

and, from the knowledge of the h.o. matrix elements

$$\langle \xi^4 \rangle_{K=2} = 30 \quad \langle \xi^4 \rangle_{K=3} = 42 \quad (48)$$

Similarly, the potential

$$\Sigma r_{ij}^6 = \frac{21}{32} \xi^6 \pi^{3/2} q_0(\Omega) - \frac{3\sqrt{3}}{32} \xi^6 \pi^{3/2} q_4(\Omega) - \frac{3}{32\sqrt{8}} \xi^6 \pi^{3/2} q_6(\Omega) \quad (49)$$

when inserted in eq. (24), gives access to the β_6 coefficients, if one remembers that $\langle \xi^6 \rangle_{K=3} = 336$ when ξ^6 perturbs the hyperradial $V(\xi) = \xi^2$.

For illustration, let us consider three particles with mass $m = 1$ bound by a two-body linear potential $V = \Sigma r_{ij}$. In the hyperscalar approximation $V_0(\xi) = \frac{32}{5\pi} \xi$, one gets the following binding energies $E_{K,n}^{(0)}$ with grand angular K and radial number n

$$\begin{aligned} K=0, n=0,1 \quad E^{(0)} &= 6.1348, 8.4577, \\ K=1, n=0,1 \quad E^{(0)} &= 7.4858, 9.6192, \\ K=2, n=0 \quad E^{(0)} &= 8.7245 \\ K=3, n=0 \quad E^{(0)} &= 9.8810 \end{aligned} \quad (50)$$

by solving numerically the single radial equation

$$\left[\frac{d^2}{d\xi^2} - \frac{(K+3/2)(K+5/2)}{\xi^2} + E - V_0(\xi) \right] u^{(0)}(\xi) = 0 \quad (51)$$

The small difference $v = V - V_0$ can be treated to first order. Its multipoles are $v_0(\xi) = 0$ (by construction), $v_4(\xi) = \frac{32\sqrt{3}}{315\pi} \xi$ and $v_6(\xi) = \frac{64\sqrt{2}}{1155\pi} \xi$. Since, from the virial theorem,

$\langle \xi \rangle_{K,n} = \frac{5\pi}{48} E_{K,n}^{(0)}$, one can easily compute the first order shifts as $E^{(1)} = 0$ for $K = 0$ and $K = 1$; for $K = 2$, $E^{(1)} = 0.0923, 0.0185, -0.0554, -0.0923$, respectively for $[20,1^+]$, $[70,2^+]$, $[56,2^+]$ and $[70,0^+]$, corresponding to $\Delta = 0.369$ and $\Delta' = 0.349$ in Fig. (3). Since the linear potential is not too far from a h.o., Δ' does not differ much from Δ . For a Coulomb potential, one would get a larger effect $\Delta'/\Delta = 59/42$.

Similarly, for the $K = 3$ levels in a linear potential, one gets to zeroth order $E^{(0)} = 9.8810$ and the matrix elements $\langle v_0 \rangle = 0$, $\langle v_4 \rangle = 0.1811$ and $\langle v_6 \rangle = 0.0807$ to be

inserted in eq. (44). This gives the splitting pattern of fig. (4) where the radial excitation of the $K = 1$ multiplet is also shown for comparison. The ordering is the same as in fig. (2), where the linear potential was used only as a perturbation of h.o., but the values of the spacings are quite different.

Fig.4 Negative parity excitations in a symmetric potential. The example corresponds to $V = \Sigma r_{ij}$, treated to first order around its hyperscalar approximation.

4. Conclusion

To summarize, we have first checked the existing results on the splitting pattern of the h.o. spectrum due to pairwise anharmonic corrections. We have also given sufficient conditions on the potential that enable us to predict the sign of the splittings. The h.o. spectrum is perturbed in a way which reflects the concavity properties of the 2-body binding energy as a function of orbital momentum or radial number. This study gave us the opportunity to review the existing literature on 2-body spectra and motivated further investigations which are presented in the next paper [2].

We have also discovered that most of the regularities in the splitting pattern, first derived in the h.o. case, are in fact much more general. Any symmetric potential which is a

plausible candidate to describe the confinement of three quarks consists of a dominant hyperscalar piece supplemented by small multipoles which enter only one after the other when one moves up along the 3-body spectrum. The splittings are given in terms of the same number of basic matrix elements as in the h.o. case, provided one treats separately the hyperradial excitations and the grand orbital ones.

The hyperspherical formalism was already used by Høgaasen and one of us (J.M.R.) [17], who have shown that for any local confining potential, not necessarily pairwise, the first excitation has negative parity. This is a problem for the potential model description of the baryon spectrum, since the Roper resonances, with positive parity, are not higher in mass than the orbital excitations. The result is essentially based on eq. (43) : the leading equation for the $K = 1$ states is governed by the same diagonal potential $V_o(r) + v_o(r)$ as the $K = 0$ ones. The effective angular momentum is $\ell = \frac{5}{2}$ for $K = 1$, instead of $\ell = \frac{3}{2}$ for $K = 0$, and one may use the results on 2-body binding energies with neighbouring angular momenta. This is one more illustration of the link between the 2-body and 3-body problems.

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APPENDIX

a) We first recall the harmonic oscillator wave-functions and some relations between matrix elements of interest. More general formulas can be found in many textbooks.

If $|n, \ell, m\rangle = \varphi_{n\ell}(r) Y_\ell^m(\theta, \phi)$ are the normalized wave-functions of the Hamiltonian

$$h(\vec{r}) = \vec{p}_r^2 + \vec{r}^2 \quad (A1)$$

we have in particular :

$$\varphi_{o\ell}(r) = \frac{u_\ell(r)}{r} = \left(\frac{2^\ell}{3.5 \dots 2\ell+1} \right)^{1/2} r^{\ell-1} u_0(r) \quad (A2)$$

with

$$u_0(r) = 2 r \pi^{-1/4} \exp(-\frac{1}{2} r^2) \quad (A3)$$

and also

$$\varphi_{1\ell}(r) = \left(\frac{2\ell+3}{2}\right)^{1/2} \left(1 - \frac{2}{2\ell+3}r^2\right) \varphi_{0\ell}(r) \quad (\text{A4})$$

In consequence, the matrix elements

$$\delta(n,\ell) = \langle n,\ell | v(r) | n,\ell \rangle \quad (\text{A5})$$

can readily be calculated for $n=0$ and 1, and one easily derives from eq. A2 and A4 the useful relations:

$$\delta(1,0) = \frac{3}{2} \delta(0,0) - 3 \delta(0,1) + \frac{5}{2} \delta(0,2) \quad (\text{A6})$$

$$\delta(1,1) = \frac{5}{2} \delta(0,0) - 5 \delta(0,2) + \frac{7}{2} \delta(0,3) \quad (\text{A7})$$

b) Proof of the theorem, eq. 30)

Rewriting for convenience $\gamma = \frac{3}{4} \tilde{\gamma}$, with

$$\tilde{\gamma} = 2 \delta(0,2) - \delta(0,1) - \delta(0,3) - [2 \delta(0,1) - \delta(0,2) - \delta(0,0)] \quad (\text{A8})$$

we first note that, for $v(r) = g r^\alpha$

$$\tilde{\gamma} = \frac{2\pi^{-1/2}}{105} \Gamma\left(\frac{3+\alpha}{2}\right) \alpha (2 - \alpha) (\alpha - 4) \quad (\text{A9})$$

so it vanishes for $\alpha = 0, 2$ and 4. This is a-priori obvious for a constant or harmonic potential but not in the quartic case.

In general, defining

$$i(r) = 2 u_2^2(r) - u_1^2(r) - u_3^2(r) - [2 u_1^2(r) - u_2^2(r) - u_0^2(r)] \quad (\text{A10})$$

one has

$$\tilde{\gamma} = \int_0^\infty v(r) i(r) dr \quad (\text{A11})$$

Then, we perform a series of integrations by parts (the same strategy is followed in [2] to prove a similar theorem) :

i) First

$$\tilde{\gamma} = \lim_{r \rightarrow 0} v(r) I(r) + \int_0^{\infty} I(r) \frac{dv}{dr} dr \quad (\text{A12})$$

where
$$I(r) = \int_r^{\infty} i(r') dr' \quad (\text{A13})$$

Obviously from the definition (A10), and also because $\tilde{\gamma}$ vanishes for $v = \text{const.}$, one has $I(0) = 0$. Thus, according to the behaviour of $u_\ell^2(r)$ when $r \rightarrow 0$, if

$$\lim_{r \rightarrow 0} r^3 v(r) = 0 \quad (\text{A14})$$

then
$$\tilde{\gamma} = \int_0^{\infty} I(r) \frac{dv}{dr} dr \quad (\text{A15})$$

ii) A second integration by parts gives

$$\tilde{\gamma} = \lim_{r \rightarrow 0} \frac{1}{r} \frac{dv}{dr} J(r) + \int_0^{\infty} J(r) \left(\frac{d}{dr} \frac{1}{r} \frac{dv}{dr} \right) dr \quad (\text{A16})$$

where
$$J(r) = \int_r^{\infty} r' I(r') dr' \quad (\text{A17})$$

Again (A16) and the vanishing of $\tilde{\gamma}$ for a harmonic potential implies that $J(0) = 0$. Consequently if

$$\lim_{r \rightarrow 0} r^4 \frac{dv}{dr} = 0 \quad (\text{A18})$$

then
$$\tilde{\gamma} = \int_0^{\infty} J(r) \left(\frac{d}{dr} \frac{1}{r} \frac{dv}{dr} \right) dr \quad (\text{A19})$$

iii) The last integration gives

$$\tilde{\gamma} = \lim_{r \rightarrow 0} \frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{dv}{dr} K(r) + \int_0^{\infty} K(r) \left(\frac{d}{dr} \frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{dv}{dr} \right) dr \quad (\text{A20})$$

where
$$K(r) = \int_r^\infty r' J(r') dr' \quad (A21)$$

We know that $\tilde{\gamma} = 0$ for $v = r^4$, this implies $K(0) = 0$ and if

$$\lim_{r \rightarrow 0} r^6 \frac{dv}{dr} = 0 \quad (A22)$$

then
$$\tilde{\gamma} = \int_0^\infty K(r) \left(\frac{d}{dr} \frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{dv}{dr} \right) dr \quad (A23)$$

Now, using again the properties $I(0) = J(0) = K(0) = 0$, it is straightforward to calculate explicitly from their definitions $I(r)$, $J(r)$ and $K(r)$ using partial integrations. One can see that $I(r)$ and $J(r)$ oscillate but $K(r)$ is always negative, being given by

$$K(r) = \frac{-4\pi^{-1/2}}{105} r^7 e^{-r^2} < 0 \quad (A24)$$

Now, if we add that one can get rid of conditions (A18) and (A22) from condition (A14) (see the appendix of [2]) then

$$W(r) = \frac{d}{dr} \frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{dv}{dr} \geq 0$$

and

$$\lim_{r \rightarrow 0} r^3 \frac{dv}{dr} = 0$$

implies

$$\tilde{\gamma} \leq 0$$

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