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On the unitary representations of a class of $N = 4$ superconformal algebras

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ABSTRACT

We derive properties of unitary highest weight representations of the $N = 4$ superconformal algebras in 2 dimensions, characterized by two $\widehat{SU}(2)$ Kač-Moody subalgebras. For the range $3 \leq c < 6$ we can prove that we have constructed all unitary highest weight representations. These reduce in $N = 2$ to the discrete series. For other values of the parameters we derive bounds on the values of h , and we conjecture that these are complete. This is motivated by explicit constructions of representations for various values of the parameters. For the so-called massless representations we do indeed prove that our treatment is complete.

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1 Introduction.

Superconformal algebras appear to play a very interesting role in different contexts of physics. For instance, the description of strings compactified on manifolds of $SU(3)$ or $SU(2)$ holonomy is intimately related to the representation theory of the $N = 2$ [1,2,3] and of the $N = 4$ [4] superconformal algebras introduced by Ademollo et al. [5]. Until recently, it was believed that the only conventional $N = 4$ algebras were precisely the ones introduced by Ademollo et al. By conventional we mean that the algebra is a Lie superalgebra (LSA), contains only one operator of dimension 2 (the energy-momentum tensor), has $N = 4$ dimension $3/2$ supersymmetry operators which square to the energy-momentum tensor, and possesses no operator of dimension smaller than or equal to zero. However, as was pointed out in [6], there exists a one-parameter class of conventional $N = 4$ algebras $\mathcal{A}_\gamma$, closely related to algebras with a non primary field [7] and giving rise to a class of algebras $\tilde{\mathcal{A}}_\gamma$ with no dimension $1/2$ operators [8] but instead, composite operators in the operator product expansions (OPE).¹ This type of algebra was first investigated by Knizhnik [9].

These new $N = 4$ algebras $\mathcal{A}_\gamma$ contain, among other substructures, two commuting $\widehat{SU}(2)$ Kač-Moody (KM) algebras and a finite dimensional subsuperalgebra $D(2, 1, \frac{\gamma}{1-\gamma})$, where $\gamma \in \mathbb{R}$. They are parametrized by the central charge of the Virasoro algebra c and the parameter γ , or, equivalently, by the levels k^+ and k^- of the two $\widehat{SU}(2)$ KM algebras. These parameters are related through

$$c = 6 \frac{k^+ k^-}{k}, \quad (1.1)$$

where we define

$$k = k^+ + k^- = \frac{c}{6\gamma(1-\gamma)}, \quad (1.2)$$

and k^+, k^- are positive integers in unitary representations.

We will first (section 2) derive some immediate results about possible values of the parameters by some simple arguments. Then (section 3) we consider the reduction of the algebra $\mathcal{A}_\gamma$ to $N = 2$ and use known results about $N = 2$ unitary representations to put restrictions on possible unitary highest weight state representations (uhwr) of $\mathcal{A}_\gamma$. These turn out to be very powerful and determine nearly completely the structure of the uhwr of $N = 4$. In fact, an attempt (subsection 3.3) to get more restrictions on h (conformal dimension of the highest weight state (hws) of the unitary representation) by requiring positivity of the square of the norm of some states of the representation, did not give any new bounds. In section 4

¹ We always go from operators to modes using for any field $\phi(z) = \sum_n \phi_n z^{-n-d_\phi}$ where d_ϕ is its dimension.

we shall explicitly construct unitary representations when $k^- = 1$. In that case we can prove that the uhwr which we construct give the complete set. This case corresponds to central charges in the range $3 \leq c < 6$. In the reduction to $N = 2$ they give rise to the discrete series. The reduction of the realization given in [6] naturally provides the realizations for these $N = 2$ models found in [2]. The general case ($k^- > 1$) is illustrated in section 5 for the value $k^- = 2$ and completed for massless representations for arbitrary values of $k^\pm$. We start from the realizations given in [10,11] and discuss if they provide the complete set of unitary representations. This analysis is done for the Neveu-Schwarz moding, but we use in section 6 the existence of an isomorphism [12] within the $\mathcal{A}_\gamma$ algebra to obtain the results for the other modings. The conclusions are summarized in section 7.

Let us now fix some notations and recall some fundamental facts. The uhwr we want to construct are defined, in the Neveu-Schwarz sector, by highest weight states $|hws\rangle$ satisfying (see [6] and Appendix A for notations)

$$\begin{aligned} L_n|hws\rangle &= A_n^{\pm i}|hws\rangle = U_n|hws\rangle = 0 \quad \text{for } n = 1, 2, \dots, \quad i = 1, 2, 3, \\ G_r^a|hws\rangle &= Q_r^a|hws\rangle = 0 \quad \text{for } r = \frac{1}{2}, \frac{3}{2}, \dots, \quad a = 1, 2, 3, 4, \\ A_0^{\pm i}|hws\rangle &= 0, \end{aligned} \tag{1.3}$$

where the operators $A^{\pm i}, U$ are the generators of the two $\widehat{SU}(2)$ and the $\widehat{U}(1)$ KM algebras, the G 's are the supersymmetry generators and the Q 's are four dimension 1/2 generators. The hws are characterized by their eigenvalues under the following zero modes

$$\begin{aligned} L_0|hws\rangle &= h|hws\rangle, \\ iU_0|hws\rangle &= u|hws\rangle, \\ -(A^{+i}A_i^+)_{|0}|hws\rangle &= \ell^+(\ell^+ + 1)|hws\rangle, \\ -(A^{-i}A_i^-)_{|0}|hws\rangle &= \ell^-(\ell^- + 1)|hws\rangle, \end{aligned} \tag{1.4}$$

(- signs and i occur because the operators in [6] were antihermitian, see Appendix A).

As mentioned above, the LSA's $\mathcal{A}_\gamma$ in [6] give rise to the algebras $\tilde{\mathcal{A}}_\gamma$ described in ref. [8]. There it was remarked that the following operators decouple completely from Q and U (in the notations of ref. [6]; normal ordering with respect to the modes will always be tacitly assumed)

$$\begin{aligned} \tilde{L} &= L + \frac{1}{k}(UU + \partial Q^a Q_a), \\ \tilde{G}_a &= G_a + \frac{2}{k}UQ_a - \frac{2}{3k^2}\epsilon_{abcd}Q^bQ^cQ^d + \frac{4}{k}Q^b(\alpha_{ba}^{+i}\tilde{A}_i^+ - \alpha_{ba}^{-i}\tilde{A}_i^-), \\ \tilde{A}^{\pm i} &= A^{\pm i} - \frac{1}{k}\alpha_{ab}^{\pm i}Q^aQ^b. \end{aligned} \tag{1.5}$$

The epsilon symbol is defined in eq.(A.5). The operator $\tilde{L}$ satisfies a Virasoro algebra with central charge $\tilde{c} = c - 3$, and $\tilde{G}$ and $\tilde{A}$ are primary operators of dimension $3/2$ and 1 with respect to it. The dimension 1 operators are again $\widehat{SU}(2)$ KM generators with levels

$$\tilde{k}^{\pm} = k^{\pm} - 1. \quad (1.6)$$

The remaining OPE's are given in (A.7).

The representations of the $\mathcal{A}_\gamma$ and the $\tilde{\mathcal{A}}_\gamma$ algebras are related. Indeed it is clear that the above definitions give a representation of the $\tilde{\mathcal{A}}_\gamma$ algebra as soon as one has a representation of $\mathcal{A}_\gamma$ [6]. But the converse, as remarked in [8], is also true : for any representation of the $\tilde{\mathcal{A}}_\gamma$ algebra, one can add a free boson U and four free fermions Q_a , and provide a representation for $\mathcal{A}_\gamma$ by the inverse of eq.(1.5). The $\tilde{\mathcal{A}}_\gamma$ algebras are also characterized by $k^{\pm}$ (or $\tilde{k}^{\pm}$). For fixed $\tilde{k}^{\pm}$, the uhwr in the Neveu-Schwarz sector are characterized by the third component of isospin $\ell^{\pm}$ ($= \tilde{\ell}^{\pm}$) of the hws and

$$\tilde{h} = h - \frac{u^2}{k}. \quad (1.7)$$

The equation eq.(1.7) relates an infinite class of representations of $\mathcal{A}_\gamma$ (obtained by choosing u arbitrarily in the set of real numbers) to one representation of the $\tilde{\mathcal{A}}_\gamma$ algebra.

2 First results

We will give here the first immediate results which are important. From the general theory of KM algebras it follows that for the $\widehat{SU}(2)$ subalgebras with levels $k^{\pm}$, $\ell^{\pm}$ can be at most $\frac{1}{2}k^{\pm}$. However in $\mathcal{A}_\gamma$, we can show that $\ell^{\pm}$ cannot be equal to the upper bound $k^{\pm}/2$. Suppose indeed that it is equal to its upper bound, then $A_{-1}^{\pm+}|hws\rangle$ has zero norm and must be zero in a unitary representation. From the following commutation relation of the $\mathcal{A}_\gamma$ algebra (see [6])

$$[Q_{\frac{1}{2}-}, A_{-1}^{\pm+}] = iQ_{-\frac{1}{2}\pm K}, \quad (2.1)$$

it follows that

$$Q_{\frac{1}{2}-}A_{-1}^{\pm+}|hws\rangle = iQ_{-\frac{1}{2}\pm K}|hws\rangle = 0. \quad (2.2)$$

But the algebra implies that these states have $\frac{1}{2}k > 0$ as square of their norms, and we have a contradiction. We then conclude that the allowed values of $\ell^{\pm}$ are

$$\ell^{\pm} = 0, \frac{1}{2}, 1, \dots, \frac{k^{\pm} - 1}{2} = \frac{\tilde{k}^{\pm}}{2}. \quad (2.3)$$

Actually this is indeed the expected list given the fact that we have a representation of the $\tilde{\mathcal{A}}_\gamma$ algebra including $\widehat{SU}(2)$ KM algebras of levels $\tilde{k}^{\pm}$.

Now we can still go a step further. We can argue that

$$2\ell^+ = \tilde{k}^+ \text{ or } 2\ell^- = \tilde{k}^- \Rightarrow \tilde{G}_{-\frac{1}{2},+}|hws\rangle = 0. \quad (2.4)$$

Indeed, it follows from these values of ℓ^+ and ℓ^- that in the $\widehat{SU(2)}$ subalgebra with levels $\tilde{k}^\pm$, the state $\tilde{A}_{-1}^{\pm+}|hws\rangle$ has zero norm and must be zero in a unitary representation. From the commutation relation

$$[\tilde{G}_{\frac{1}{2},\mp K}, \tilde{A}_{-1}^{\pm+}] = -i\tilde{G}_{-\frac{1}{2},+}, \quad (2.5)$$

we get in both cases

$$\tilde{G}_{-\frac{1}{2},+}|hws\rangle = 0. \quad (2.6)$$

Representations with this property will be called "massless". By calculating the norm of these states we get for massless representations

$$k\tilde{h} = H_{\frac{1}{2}} \equiv (\ell^+ - \ell^-)^2 + k^-\ell^+ + k^+\ell^-. \quad (2.7)$$

In particular, if $k^- = 1$ we already know that ℓ^- can only be zero, and all representations are massless. We can thus only have representations with

$$k\tilde{h} = \ell^+(\ell^+ + 1). \quad (2.8)$$

In section 4 we shall show that all these values are indeed possible by explicitly constructing unitary representations with these quantum numbers. In section 5, a different existence proof is provided for all the values eq.(2.7).

3 Reduction from $N = 4$ to $N = 2$.

In this section we consider the case where one has an $N=2$ algebra and in addition a complex free boson and free fermion representing the algebra. We construct from this a new $N=2$ realization (an extended Sugawara construction), and the difference between the two realizations gives a third one. We use the $N = 2$ conventions with

$$\{G_r, \overline{G}_s\} = 2L_{r+s} + (r-s)T_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} \quad (3.1)$$

3.1 Reduction of the algebra

We introduce a 'matter multiplet' (with A and $\overline{A}$ being of dimension 1 and Q and $\overline{Q}$ of dimension $\frac{1}{2}$) and we repeat the nontrivial OPE's involving the operators $A, \overline{A}, Q, \overline{Q}$:

$$\begin{aligned} A(z)\overline{A}(w) &= -2(z-w)^{-2} + \dots \\ Q(z)\overline{Q}(w) &= -(z-w)^{-1} + \dots \end{aligned}$$

$$\begin{aligned}
A(z)G(w) &= 2Q(w)(z-w)^{-2} + \dots \\
\bar{A}(z)\bar{G}(w) &= 2\bar{Q}(w)(z-w)^{-2} + \dots \\
Q(z)\bar{G}(w) &= A(w)(z-w)^{-1} + \dots \\
\bar{Q}(z)G(w) &= \bar{A}(w)(z-w)^{-1} + \dots \\
T(z)Q(w) &= Q(w)(z-w)^{-1} + \dots \\
T(z)\bar{Q}(w) &= -\bar{Q}(w)(z-w)^{-1} + \dots
\end{aligned} \tag{3.2}$$

Together with T and G they form a "spectrum generating algebra" for the usual $N = 2$ algebra, by which we mean that we can form the $N = 2$ algebra in terms of them (sort of a generalized Sugawara construction) :

$$\begin{aligned}
L_Q &= -\frac{1}{2}(A\bar{A} + \partial Q\bar{Q} + \partial\bar{Q}Q), \\
G_Q &= -\bar{A}Q, \\
\bar{G}_Q &= -A\bar{Q}, \\
T_Q &= -Q\bar{Q}.
\end{aligned} \tag{3.3}$$

These operators realize an $N = 2$ algebra with $c_Q = 3$. Moreover one can check that the hat operators

$$\begin{aligned}
\hat{L} &= L - L_Q, \\
\hat{G} &= G - G_Q, \\
\hat{\bar{G}} &= \bar{G} - \bar{G}_Q, \\
\hat{T} &= T - T_Q,
\end{aligned} \tag{3.4}$$

also form an $N = 2$ algebra with central charge $\hat{c} = c - 3$.

By reducing $\mathcal{A}_\gamma$ to $N = 2$ we get exactly this situation, i.e. we have a realization of the usual $N = 2$ algebra + a realization of a 'matter multiplet'.

Indeed, starting from the general $\mathcal{A}_\gamma$ algebra, we can identify (see [6] and Appendix)

$$\begin{aligned}
G &= \sqrt{2}G_{+K} = \frac{1}{\sqrt{2}}(G^3 + iG^0), \\
\bar{G} &= \sqrt{2}G_{-K} = \frac{1}{\sqrt{2}}(G^3 - iG^0), \\
T &= 2i(\gamma A_3^+ + (\gamma - 1)A_3^-), \\
A &= \sqrt{\frac{2}{k}}(U - i(A_3^+ + A_3^-)), \\
\bar{A} &= \sqrt{\frac{2}{k}}(U + i(A_3^+ + A_3^-)), \\
Q &= \frac{2}{\sqrt{k}}Q_{+K} = \frac{1}{\sqrt{k}}(Q^3 + iQ^0), \\
\bar{Q} &= \frac{2}{\sqrt{k}}Q_{-K} = \frac{1}{\sqrt{k}}(Q^3 - iQ^0),
\end{aligned} \tag{3.5}$$

or alternatively,

$$\begin{aligned}
G &= \sqrt{2}G_+ = \frac{1}{\sqrt{2}}(G^1 + iG^2), \\
\bar{G} &= \sqrt{2}G_- = \frac{1}{\sqrt{2}}(G^1 - iG^2), \\
T &= 2i(\gamma A_3^+ + (1 - \gamma)A_3^-), \\
A &= \sqrt{\frac{2}{k}}(U - i(A_3^+ - A_3^-)), \\
\bar{A} &= \sqrt{\frac{2}{k}}(U + i(A_3^+ - A_3^-)), \\
Q &= \frac{2}{\sqrt{k}}Q_+ = \frac{1}{\sqrt{k}}(Q^1 + iQ^2), \\
\bar{Q} &= \frac{2}{\sqrt{k}}Q_- = \frac{1}{\sqrt{k}}(Q^1 - iQ^2).
\end{aligned} \tag{3.6}$$

We also define

$$Z = \frac{1}{2}(\bar{A} - A) = i\sqrt{\frac{2}{k}}(A_3^+ + A_3^-) \tag{3.7}$$

if we use the first identification. Following the previous results, we can split the L , G , and T in this representation of $N = 2$ in L_Q, G_Q, T_Q (with $c_Q = 3$) and $\hat{L}, \hat{G}, \hat{T}$. Note that the operators $\hat{G}$ are closely related to the operators $\tilde{G}$ of the $\tilde{\mathcal{A}}_\gamma$ algebra (two from each possibility of reduction). In fact in the reduction to $N = 2$ of these algebras, the terms cubic in Q in the expression for $\tilde{G}$ in eq.(1.5) disappear, and the sum over i in the last term gets restricted to $i = 3$. The hat part has central charge

$$\hat{c} = \frac{6}{k}(k^+k^-) - 3. \tag{3.8}$$

If $k^+, k^- \geq 2$ then $\hat{c}$ is in the continuous range ($\hat{c} \geq 3$). It is interesting to study the case when one of the levels $k^\pm$ is 1 because $\hat{c}$ then takes a value in the discrete $N=2$ series. We first analyze this case and start with a realization of $\mathcal{A}_\gamma$ obtained from a σ -model whose target manifold is the group $SU(2) \otimes U(1)$ (as given in [6]). It has

$$k^- = 1 \quad k^+ = n + 1 \quad \Rightarrow \quad \gamma = \frac{1}{n + 2} \quad (n \in \mathbb{N}) \tag{3.9}$$

and the central charge is

$$c = 6\frac{n + 1}{n + 2} = 3\left(1 + \frac{n}{n + 2}\right). \tag{3.10}$$

These realizations are the only ones of $\mathcal{A}_\gamma$ which have $3 \leq c < 6$. The operators A , $\bar{A}$, Q , $\bar{Q}$ give the contribution $c_Q = 3$, and the remaining

part can be isolated to give a representation for the special series of $N = 2$ with $\hat{c} = 3\frac{n}{n+2}$ [13,1,2] (Construction à la Goddard-Kent-Olive [14]).

We identify (using eq.(3.5))²

$$\begin{aligned} U &= \sqrt{n+2}J^0, \\ Z &= i\sqrt{\frac{2}{n+2}}(J^3 + 2\psi^1\psi^2), \\ Q &= -(\psi^3 + i\psi^0), \\ \bar{Q} &= -(\psi^3 - i\psi^0), \end{aligned} \quad (3.11)$$

and put this in eq.(3.3) to obtain

$$\begin{aligned} L_Q &= -J^0J^0 - \frac{1}{n+2}(J^3J^3 + 4J^3\psi^1\psi^2 + 2\partial\psi^1\psi^1 + 2\partial\psi^2\psi^2), \\ &\quad -\partial\psi^0\psi^0 - \partial\psi^3\psi^3, \\ G_Q &= \sqrt{2}(\psi^3 + i\psi^0) \left(J^0 + \frac{i}{\sqrt{n+2}}(J^3 + 2\psi^1\psi^2) \right), \\ \bar{G}_Q &= \sqrt{2}(\psi^3 - i\psi^0) \left(J^0 - \frac{i}{\sqrt{n+2}}(J^3 + 2\psi^1\psi^2) \right), \\ T_Q &= 2i\psi^3\psi^0. \end{aligned} \quad (3.12)$$

We compare these with the expressions obtained from reducing $\mathcal{A}_\gamma$ to $N = 2$ eq.(3.5). This gives :

$$\begin{aligned} L &= -J^0J^0 - \frac{1}{n+2}J^iJ^i - \partial\psi^a\psi^a, \\ G &= \sqrt{2}(\psi^3 + i\psi^0) \left(J^0 + \frac{i}{\sqrt{n+2}}(J^3 + 2\psi^1\psi^2) \right) \\ &\quad + \sqrt{\frac{2}{n+2}}i(J^1 + iJ^2)(\psi^1 - i\psi^2), \\ T &= \frac{2i}{n+2}(J^3 - n\psi^1\psi^2) + 2i\psi^3\psi^0, \end{aligned} \quad (3.13)$$

and the hat part is eq.(3.4). We find

$$\begin{aligned} \hat{L} &= -\frac{1}{n+2} \left(J^1J^1 + J^2J^2 - 4J^3\psi^1\psi^2 + n(\partial\psi^1\psi^1 + \partial\psi^2\psi^2) \right), \\ \hat{G} &= \sqrt{\frac{2}{n+2}}i(J^1 + iJ^2)(\psi^1 - i\psi^2), \\ \hat{T} &= \frac{2i}{n+2}(J^3 - n\psi^1\psi^2). \end{aligned} \quad (3.14)$$

So all Q operators commute with all $\hat{\cdot}$ operators, and they separately represent the $N = 2$ algebra with central charges respectively 3 and $3\frac{n}{n+2}$.

²There is a misprint in [6] : $Q = -\sqrt{n+2}\psi$.

To get the latter values we may choose three bosonic operators J^1, J^2, J^3 realizing an $\widehat{SU}(2)$ KM algebra with level n , and two dimension $\frac{1}{2}$ fields ψ^1, ψ^2 . This realization has central charge $3\frac{n}{n+2} + 1$. However the operator $\hat{L}$ should commute with the dimension 1 operator Z in eq.(3.11), so it can be written as

$$\hat{L} = -\frac{1}{n+2}(J^1 J^1 + J^2 J^2 + J^3 J^3) - \partial\psi^1 \psi^1 - \partial\psi^2 \psi^2 - \frac{1}{2}ZZ. \quad (3.15)$$

The realizations which we get here are those found in [2].

These are not the only realizations of $\mathcal{A}_\gamma$ with $3 \leq c < 6$. We can also get (see [10]) $k^+ = 1$ by using a Wolf space $\otimes SU(2) \otimes U(1)$, and take $n = 0$, which means essentially that one discards the non-abelian bosonic operators. One obtains then, after subtracting the "Sugawara" construction, $\hat{c} = 3\frac{\tilde{g}-2}{\tilde{g}}$ where $\tilde{g}$ is the dual Coxeter number of the group one has used. The simplest example of the latter kind is to use eight fermions to realize $\widehat{SU}(3)$. After subtraction of a $c_Q = 3$ part, one ends up with $\hat{c} = 1$.

3.2 Representation restrictions from $N = 2$

In general the states with lowest value of h in a uhwr of $\mathcal{A}_\gamma$ have third component of isospin $\ell_3^\pm$ (eigenvalue of $iA_0^{\pm 3}$) between $-\ell^\pm$ and $+\ell^\pm$. They are all hws of representations of $N = 2$. These $N = 2$ representations have the same conformal dimension h and charges q (eigenvalue of T_0)

$$q = \frac{2}{k}(k^-\ell_3^+ + k^+\ell_3^-) \quad (3.16)$$

where we use the identification of eq.(3.6).

As explained above the $N = 2$ reduction splits in two, and so do the representations. On one hand, since Q_r for $r > 0$ should annihilate the vacuum, we have

$$\begin{aligned} kh_Q &= u^2 + (\ell_3^+ - \ell_3^-)^2, \\ q_Q &= 0. \end{aligned} \quad (3.17)$$

On the other hand, $\hat{c}$ is given by eq.(3.8). When $k^- = 1$, ℓ^- can only be zero according to eq.(2.3) and the hat part is a representation in the discrete series of $N = 2$ [13,1]. One could consider $k^+ = 1$ but we assume the reader keeps the symmetry $+\leftrightarrow -$ in mind. It follows then from

$$\begin{aligned} \hat{c} &= 3\frac{k-2}{k}, \\ \hat{q} &= q = \frac{2}{k}\ell_3^+, \end{aligned} \quad (3.18)$$

that

$$k\hat{h} = j(j+1) - (\ell_3^+)^2, \quad (3.19)$$

where j is an integer or half integer in the range $|j| \leq \frac{n}{2}$. So we have

$$kh = kh_Q + k\hat{h} = u^2 + j(j+1). \quad (3.20)$$

We will show later (section 4) that $\ell^+ = j$. When $k^- > 1$, one has, according to eq.(2.3),

$$|\hat{q}| = |q| \leq \frac{\hat{c}}{3}. \quad (3.21)$$

So we use the results on $N = 2$ representations [1,3] in the continuous range of $\hat{c}$, but in this case, the $U(1)$ charge $\hat{q}$ is bounded. The bound on $\hat{h}$ in general depends on the value of $\hat{q}$. To determine the unitarity domain of the $(\hat{h}, \hat{q})$ -plane for these representations, one analyzes the bounds obtained from requiring that the following states have positive norm square

$$\hat{G}_{-m+\frac{1}{2}} \dots \hat{G}_{-\frac{1}{2}} \hat{G}_{-\frac{1}{2}} |hws\rangle, \quad (3.22)$$

$$\bar{\hat{G}}_{-m+\frac{1}{2}} \dots \bar{\hat{G}}_{-\frac{1}{2}} \bar{\hat{G}}_{-\frac{1}{2}} |hws\rangle, \quad (3.23)$$

$$\left((1-q)\hat{L}_{-1} + (2\hat{h}+1)\hat{T}_{-1} - \hat{G}_{-\frac{1}{2}}\bar{\hat{G}}_{-\frac{1}{2}} \right) |hws\rangle. \quad (3.24)$$

The conditions for positivity are ($m \in \mathbb{Z}$)

$$g_{m-\frac{1}{2}} = 2\hat{h} - 2(m - \frac{1}{2})\hat{q} + ((m - \frac{1}{2})^2 - \frac{1}{4})2\frac{k^+k^- - k}{k} \geq 0, \quad (3.25)$$

$$f_{1,2} = 2\hat{h}(\frac{\hat{c}}{3} - 1) - \hat{q}^2 + \frac{\hat{c}}{3} \geq 0. \quad (3.26)$$

In the case where $\hat{q}$ is unrestricted, they lead to a unitarity domain similar to the one given in [1]. However, in the present situation, $\hat{q}_{\min} = -\frac{\hat{c}}{3}$ and $\hat{q}_{\max} = \frac{\hat{c}}{3}$ (according to eq.(3.21)), and we wish to know which lines $g_{m-\frac{1}{2}} = 0$ give the strongest bounds on $\hat{h}$. Let us derive the result for $\hat{q} \geq 0$ as $\hat{q} \leq 0$ is similar and we can get all bounds by looking at $\ell_3 > 0$ anyway. Because the intersection of $g_{m-\frac{1}{2}} = 0$ and $g_{m+\frac{1}{2}} = 0$ occurs, for $m \geq 1$, at

$$\hat{q} = m(\frac{\hat{c}}{3} - 1), \quad (3.27)$$

we need to determine the smallest value of m for which

$$\hat{q}_{\max} \leq m(\frac{\hat{c}}{3} - 1) \quad (3.28)$$

in order to conclude that the unitary domain is determined by $g_{\frac{1}{2}}, g_{\frac{3}{2}}, \dots, g_{m+\frac{1}{2}}$. One can see quite easily that

$$\begin{aligned} m = \frac{3q}{\hat{c} - 3} &= \frac{k^- \ell_3^+ + k^+ \ell_3^-}{k^+ k^- - k} \\ &= 1 + \frac{k^- (2\ell_3^+ - \tilde{k}^+ + 1) + k^+ (2\ell_3^- - \tilde{k}^- + 1)}{2(k^+ k^- - k)} \end{aligned} \quad (3.29)$$

is larger than 1 only if $2\ell_3^+ = \tilde{k}^+$ or $2\ell_3^- = \tilde{k}^-$. In that case however, $g_{\frac{1}{2}} = 0$ as we shall see in section 3.3. The only restriction comes from eq.(3.26) which reduces then to eq.(3.21). The unitarity domain is shown in Figure 1. If $2\ell_3^+ < \tilde{k}^+$ and $2\ell_3^- < \tilde{k}^-$, then $m \leq 1$, which implies that the strongest bound comes from $g_{\frac{1}{2}}$. Therefore the limits on h are obtained by adding the bounds on $\hat{h}$ and h_Q . In conclusion, the bound on h for the cases $k^\pm \geq 2$ and $2\ell^\pm < \tilde{k}^\pm$ is thus

$$k\tilde{h} \geq H_{\frac{1}{2}} = (\ell^+ - \ell^-)^2 + k^- \ell^+ + k^+ \ell^-, \quad (3.30)$$

the limit being reached for "massless representations".

There are 2 special cases. First for the $N = 4$ algebra with only one $SU(2)$, the limit $k^- \rightarrow \infty$ in eq.(3.16) gives $q = 2\ell_3^+$ and the only limit comes from eq.(3.30) in agreement with [4]. In the special case $k^+ = k^- = 2$ ($\hat{c} = 3$), $2\ell^\pm < k^\pm$ implies $\ell^\pm = 0$ and we get no information from the $N = 2$ bounds.

We only have given here a part of the $N = 2$ representations which occur in the reduction from an $\mathcal{A}_\gamma$ representation, namely those which have the same value for h . It would be interesting to find all the other $N = 2$ uhwr within each uhwr of $\mathcal{A}_\gamma$ since we could then add the known characters of the $N = 2$ representations to obtain those of $\mathcal{A}_\gamma$.

3.3 The norm of some other states.

The lower bounds on h we have established so far come from our knowledge of the unitary representation theory of $N = 2$, and we will investigate in this section whether the above bounds can be improved by using the operators $\tilde{G}$ from the $\tilde{\mathcal{A}}_\gamma$ algebra [8] (see eq.(1.5)) which are quite similar to the $\hat{G}$ operators. First, the positivity condition arising from the norms of the states $\tilde{G}_{-m+\frac{1}{2},+}$ leads to the strongest bound when $m = 1$ and it is the same as that obtained above by reducing to $N = 2$ (eq.(3.30)).

By analogy with the $N = 2$ line of arguments, we looked at the positivity condition on the norms of states

$$|hws; m^2/2 \rangle \equiv \tilde{G}_{-m+1/2,+} \dots \tilde{G}_{-3/2,+} \tilde{G}_{-1/2,+} |hws \rangle \quad (3.31)$$

for $m = 1, 2, 3, \dots$. We have

$$\begin{aligned} \langle hws; 2 | hws; 2 \rangle &= \langle hws; 1/2 | hws; 1/2 \rangle (k\tilde{h} - H_{\frac{1}{2}}) \\ H_{\frac{1}{2}} &\equiv H_{\frac{1}{2}} - \tilde{k}^- q^+ - \tilde{k}^+ q^-, \end{aligned} \quad (3.32)$$

where we have defined

$$\begin{aligned} q^+ &= \tilde{k}^+ - 2\ell^+ - 1 \\ q^- &= \tilde{k}^- - 2\ell^- - 1. \end{aligned} \quad (3.33)$$

When these quantities are negative, the state $|hws; 1/2 \rangle$ is zero, and these calculations do not give anything new. So we are only interested in $q^+, q^- \geq 0$. The new bound obtained here, is indeed better than that from $g_{3/2} \geq 0$ (eq.(3.25)), but still does not improve eq.(3.30).

The norm of $|hws; 9/2 \rangle$ gives after a long computation

$$\begin{aligned} k^2 \langle hws; 9/2 | hws; 9/2 \rangle / \langle hws; 1/2 | hws; 1/2 \rangle &= \\ (k\tilde{h} - H_{\frac{1}{2}})(k\tilde{h} - H_{\frac{1}{2}} + \tilde{k}^- q^+ + \tilde{k}^+ q^- + 2\tilde{k}^+ \tilde{k}^- - \tilde{k}^+ - \tilde{k}^-) \\ + 2q^+ q^- + \tilde{k}^- q^+ + \tilde{k}^+ q^-. \end{aligned} \quad (3.34)$$

From this expression it is clear that we do not get a stronger bound than those mentioned above. Only if $q^+ = q^- = 0$ are the bounds the same.

We can thus conclude that we found no new information beyond that from reducing to $N = 2$.

4 Representations for $k^- = 1$ ($3 \leq c < 6$).

From the previous analysis, it is clear that $k^- = 1$ is a special case. We found already that all representations are "massless" and satisfy eq.(2.8). Now we will prove that all the allowed values for ℓ^+ indeed exist, and we will explicitly construct them. We use the realization given in [6] for $k^- = 1$. Let us recall that this is built from bosonic (J^a) and fermionic (ψ^a) fields associated with $SU(2) \otimes U(1)$. The bosonic fields of $SU(2)$ form a KM algebra whose level (n) determines k^+ ($k^+ = n + 1$).

We construct hws starting from a vacuum annihilated by the positive modes of ψ^a and J^a . The requirement that $|hws \rangle$ is annihilated by Q_n (which is now just ψ_n) for $n \geq 1/2$ implies that there are no fermionic creation operators on the vacuum to build $|hws \rangle$. Then the annihilation by positive modes of A^{+i} and U implies the annihilation by positive modes of J^a . So the vacuum is characterized by the eigenvalues of their zero modes. The eigenvalue of $i\sqrt{n+2}J^0$ is u , and can be an arbitrary real number. The spin under the $SU(2)$ from which we started becomes also

ℓ^+ , the spin of the $SU(2)^+$ subalgebra of $\mathcal{A}_\gamma$. A uhwr of the $\widehat{SU(2)}$ KM algebra must be characterized by ℓ^+ in the range

$$\ell^+ = 0, \frac{1}{2}, 1, \dots, \frac{n}{2}. \quad (4.1)$$

From this realization we find then

$$\begin{aligned} h &= \frac{1}{k}(u^2 + \ell^+(\ell^+ + 1)), \\ \ell^- &= 0. \end{aligned} \quad (4.2)$$

This agrees with the results before and shows that there are indeed representations for all the allowed values of ℓ^+ .

To prove that no other uhwr exist, one can also argue in the following way. If one considers again the realization given in [6], one can convert the expressions for Q^a , U and A^{+i} such that ψ^a and J^a are functions of them. This allows to define the other generators of the algebra in terms of the previously mentioned ones (giving expressions which we call L' , G' , A'^-), e.g.

$$L' = \frac{1}{k^+ + 1} \left(UU + (A^{+i} - \frac{1}{k^+ + 1} \alpha_{ab}^{+i} Q^a Q^b)^2 - \partial Q^a Q_a \right). \quad (4.3)$$

These operators satisfy the algebra (but only if $k^- = 1$!), with $c' = \frac{6k^+}{k^+ + 1}$. The difference $L - L'$ then also satisfies the Virasoro algebra with $c = 0$. Therefore the eigenvalue of $L_0 - L'_0$ should be zero. This implies that this difference is trivially realized, and therefore any uhwr of the algebra with $k^- = 1$ is equal to the one that we have constructed.

5 Representations for $k^- = 2$.

We start from the realization using $SU(3)$ [10], which we have written explicitly in Appendix B. From the form of Q we can derive immediately that $|hws\rangle$ should have no negative modes of ψ^m , ψ_m , ψ^θ or ψ_θ . Then the fact that $A_n^{-+}|hws\rangle$ should be zero for $n \geq 0$ implies that there can also be no negative modes of $\psi^\pm$. Finally, the positive (but not necessarily the zero) modes of A^{--} should annihilate $|hws\rangle$, which implies that there could be only one $\psi_{\pm, \frac{-1}{2}}$ mode. That fixes the fermionic content completely, and the form of A^{-3} gives then already the value of ℓ_3^- . In conclusion, the possible $|hws\rangle$ are of the following form

$$\begin{aligned} |\Omega_0\rangle &= \text{bosonic } |0\rangle, & \ell_3^- &= 0 \\ |\Omega_+\rangle &= \text{bosonic } \psi_{+, \frac{-1}{2}} |0\rangle, & \ell_3^- &= \frac{1}{2} \\ |\Omega_-\rangle &= \text{bosonic } \psi_{-, \frac{-1}{2}} |0\rangle, & \ell_3^- &= \frac{1}{2}. \end{aligned} \quad (5.1)$$

5.1 $\ell^- = 0$

Here we use $|\Omega_0\rangle = \text{bosonic } |0\rangle$, so that the fermionic operators do not contribute. It follows from other requirements (e.g. $G_r|\Omega_0\rangle = 0$ for $r > 0$) that the positive modes of all V operators should annihilate the hws. Therefore the hws belongs to the lowest level of a uhwr of the $\widehat{SU(3)}$ KM algebra. The requirement that it should be annihilated by the zero modes of A^{+3} finally requires that it is annihilated by the zero mode of $V^m + V_m$. The picture is then the following: one starts with a uhwr of $\widehat{SU(3)}$ and decomposes the states of the lowest level (vacuum) in $SU(2) \otimes U(1)$ representations. Each of these $SU(2) \otimes U(1)$ representations gives a uhwr of our conformal superalgebra with $k^- = 2$, $\ell^- = 0$. The $SU(2)$ spin determines ℓ^+ , while the eigenvalue of $V^m - V_m$ (the $U(1)$, the other component in the weight diagram) determines u . To determine h we remark that the bosonic part of the energy momentum tensor is simply the Sugawara construction for an $\widehat{SU(3)}$ KM at level n . We have thus for a certain irreducible $SU(3)$ -representation (Λ)

$$h(\Lambda) = \frac{1}{2k}C(\Lambda), \quad (5.2)$$

where $C(\Lambda)$ is the quadratic Casimir of (Λ) (we follow the notations of [15]). If we describe (Λ) by Dynkin-indices (a_1, a_2) , which simply indicate the minimal number of quarks (a_1) and anti-quarks (a_2) in the representation, then $C(\Lambda)$ is given by ([15] (5.15) and (5.11))

$$h(a_1, a_2) = \frac{1}{3k}[a_1^2 + a_2^2 + a_1a_2 + 3(a_1 + a_2)]. \quad (5.3)$$

From the representation theory of KM algebras it follows that for a certain n (level of the $\widehat{SU(3)}$ algebra), the representations are restricted to a_1, a_2 non-negative integers (that is always the case with Dynkin indices) such that $0 \leq a_1 + a_2 \leq n$.

Having obtained $h(a_1, a_2)$ we must reduce the representation to $SU(2) \otimes U(1)$ to find ℓ^+ and u .

For the irreducible representation (a_1, a_2) , the weight vectors form a lattice of equilateral triangles, bounded by a hexagon where 3 sides are a_1 units long and the other three are a_2 units long: thus the top horizontal side has a_1 units and the bottom horizontal side a_2 units (see Figure 2).

At the position of the highest weight we have

$$\ell_{hw}^+ = \frac{1}{2}(a_1 + a_2). \quad (5.4)$$

Every time we move one line up from there or one line down from there, ℓ^+ is decreased by $1/2$. Since we may move a_2 steps up and a_1 steps down, this gives the list of ℓ^+ -values.

These are then further distinguished by u . In the usual notation

$$u = -\frac{\sqrt{3}}{2}Y. \quad (5.5)$$

So that in this particular realization, u is quantized in steps of $\sqrt{3}/2$. For the highest weight state, $Y_{hw} = \frac{1}{3}(a_1 - a_2)$. Thus we get the following list of (ℓ^+, u) -values corresponding to *the boundary* of the weight diagram:

$$\begin{aligned} \ell^+ &= \frac{1}{2}(a_1 + a_2) - |m|/2, \\ u &= -\frac{1}{2\sqrt{3}}(a_1 - a_2 + 3m), \\ m &= -a_1, -a_1 + 1, \dots, +a_2. \end{aligned} \quad (5.6)$$

That is not all, however. In general weights are degenerate. Only the ones on the border, which we have described, are not. Inside the weight-diagram lies a sub-diagram, very similar with $a'_1 = a_1 - 1, a'_2 = a_2 - 1$, (if both $a_1 \neq 0$ and $a_2 \neq 0$). The corresponding border, defined as above, has multiplicity 2. So we get a new list obtained from the previous one by replacing (a_1, a_2) by (a'_1, a'_2) . And so on and so forth until $a'_1 = 0$ or $a'_2 = 0$. One can now summarize for $\ell^- = 0$:

h, ℓ^+, u are given in terms of four integers: a_1, a_2, p, m with the following restrictions among them:

a_1, a_2 non-negative such that $a_1 + a_2 \leq n$

p non-negative $\leq \min(a_1, a_2)$

$m = -a_1 + p, -a_1 + p + 1, \dots, a_2 - p$.

Then

$$\begin{aligned} h &= \frac{1}{3k}[a_1^2 + a_2^2 + a_1 a_2 + 3(a_1 + a_2)] \\ \ell^+ &= \frac{1}{2}(a_1 + a_2) - p - |m|/2 \\ u &= -\frac{1}{2\sqrt{3}}(a_1 - a_2 + 3m) \\ k\tilde{h} &= \frac{1}{4}(a_1 + a_2)^2 + (a_1 + a_2) + \frac{m}{2}(a_2 - a_1 - \frac{3}{2}m) \\ &= (\ell^+ + p + \frac{1}{2}|m|)^2 + 2\ell^+ + 2p + |m| + \frac{m}{2}(a_2 - a_1 - \frac{3}{2}m) \end{aligned} \quad (5.7)$$

The bound which we obtained before for this case was

$$k\tilde{h} \geq (\ell^+)^2 + 2\ell^+, \quad (5.8)$$

and we had derived that it should be saturated when $2\ell^+ = \tilde{k}^+ = n$, which one can verify here. However, we see that it is even saturated for any $SU(3)$ hws ($p = m = 0$), also when $2\ell^+ = a_1 + a_2 < n$. We do not expect that the list eq.(5.7) exhausts the possibilities. In fact it is clear from these equations that, when $\ell^\pm$ is fixed, $\tilde{h}$ is not yet determined in the case when massive representations can exist.

5.2 $\ell^- = 1/2$

We use here $|\Omega_+ \rangle = \text{bosonic } \psi_{+,-1/2}|0 \rangle$. Using $|\Omega_- \rangle$ is equivalent with $a_1 \leftrightarrow a_2, u \rightarrow -u$. We get new constraints this time from $(\tilde{G}_a)_{+1/2}$ which should vanish on $|\Omega_+ \rangle$, since now $\psi_{1/2}^+$ does not vanish. The term with 3 ψ 's will always give zero here. So we are left with conditions from

$$\begin{aligned}(\tilde{G}_-)_{1/2} &= \frac{1}{\sqrt{2}} V_{+,0} \psi_{1/2}^+, \\(\tilde{G}_{+K})_{1/2} &= -\frac{i}{\sqrt{2}} V_0^- \psi_{1/2}^+.\end{aligned}\tag{5.9}$$

Thus we require that both $(V_+)_0$ and $(V_-)_0$ should vanish. This restricts us to the weights on the *top border* of the weight diagram of (Λ) . So, e.g., the highest weight is in general *not* allowed. Similarly for $|\Omega_- \rangle$ we would get the *bottom border* of the weight diagram.

The other new feature is to look for the ψ -contributions to L and U :

$$\begin{aligned}(L_F)_n &= \sum_r (r + 1/2) : [\psi_{i,r} \psi_{n-r}^i + \psi_r^i \psi_{i,n-r}] : \\(L_F)_0 &= -\psi_{+,-1/2} \psi_{1/2}^+ = \frac{1}{2} N_{+,1/2}\end{aligned}\tag{5.10}$$

on $|\Omega_+ \rangle$ where $N_{+,1/2}$ is the number operator. Similarly on $|\Omega_+ \rangle$

$$iU_F = -\sqrt{3} : \psi_{1/2}^+ \psi_{+,-1/2} : = -\frac{\sqrt{3}}{2} N_{+,1/2}\tag{5.11}$$

So we can summarize for $\ell^- = 1/2$:

In the above notation we only have the possibilities $p = 0, m = a_2$ giving

$$\begin{aligned}h &= \frac{1}{3k} [a_1^2 + a_2^2 + a_1 a_2 + 3(a_1 + a_2)] + \frac{1}{2} \\ \ell^+ &= \frac{1}{2} a_1 \\ u &= -\frac{1}{2\sqrt{3}} [a_1 + 2a_2 + 3] \\ k\tilde{h} &= \frac{1}{4} a_1^2 + \frac{1}{2} a_1 + \frac{k}{2} - \frac{3}{4} \\ &= (\ell^+)^2 + \ell^+ + \frac{k^+}{2} + \frac{1}{4}\end{aligned}\tag{5.12}$$

The latter is indeed the value for massless representations $H_{1/2}$ (see eq.(2.7)) as it should be since $2\ell^- = \tilde{k}^-$. It follows that for $\ell^- = 1/2$, our construction exhausts the possibilities for $k^- = 2$.

In figure 3 we have illustrated the results for the case $k^- = 2, k^+ = 6$. The line from figure 1 has become a parabola by expressing it in terms of $\tilde{h}$ instead of $\hat{h}$. The figures clearly show the main results. If $2\ell^\pm = \tilde{k}^\pm$ there is only one dot, corresponding to a massless representation, while for other cases there are massless as well as massive representations.

5.3 Arbitrary values of $k^- > 2$

The foregoing analysis for $SU(3)$ and $k^- = 2$ may be generalized using any realization for $\mathcal{A}_\gamma$ in [10], which is based on spaces of the form, a Wolf space $\otimes SU(2) \otimes U(1)$. The parameter $\tilde{k}^-$ becomes equal to the quaternionic dimension of the Wolf space.

Our aim will be to provide an existence proof for all possible massless representations, that is, for all fixed values of k^+ , k^- we want to find realizations for all the possible values of ℓ^+ , ℓ^- such that $\tilde{h}$ is given by the boundary value eq.(2.7).

To this end it is sufficient to restrict attention to the Wolf spaces

$$W = \frac{SU(N)}{SU(N-2) \otimes SU(2) \otimes U(1)} \quad (5.13)$$

so that we deal with realizations on the cosets

$$W \otimes SU(2) \otimes U(1) = SU(N)/SU(N-2) \quad (5.14)$$

which appear as stages in the construction of the quaternionic structure that defines the $N = 4$ supersymmetries. For details of the root system we refer to Appendix C and for more details on the realizations we refer to [11] and [10].

We introduce a bosonic KM algebra of $SU(N)$ with level $n = \tilde{k}^+$ and currents V_i , V^i written in complex notation as well as a set of free fermionic fields, ψ_i , ψ^i uncorrelated with the V_i 's. We shall in fact never use the fermions ψ_i , ψ^i with index $i = \hat{i}$ belonging to the subgroup $SU(N-2)$. We further introduce the fermionic $SU(N-2)$ construction

$$V_i^F = -2F_{i+\alpha}^{+\beta} \psi_{+\beta} \psi^{+\alpha} - 2F_{i-\alpha}^{-\beta} \psi_{-\beta} \psi^{-\alpha}. \quad (5.15)$$

where $\psi_{+\alpha}$, $\psi_{-\alpha}$ ($\alpha = 1, \dots, N-2$) and their complex conjugates with upper indices are the Wolf space fermions (cf. App. C). (The factor -2 originates in peculiarities of the complex notation employed in [10].) Finally we introduce the modified $SU(N-2)$ generators

$$\hat{V}_i = V_i + V_i^F \quad (5.16)$$

The level of the KM algebra with currents $\hat{V}$ is $n+2$ since V^F is built using two complex sets of fermions in the defining representation of $SU(N-2)$.

The realization of the energy-momentum tensor is now based on the supersymmetric coset space GKO construction [14] [11]

$$L = (L_F + L_{SU(N)}) - L_{SU(N-2)} \quad (5.17)$$

where L_F is a free Fermi piece built from the Wolf space fermions $\psi_{\pm\alpha}$, $\psi^{\pm\alpha}$ and the fermions ψ_θ , ψ^θ and the associated ψ_m , ψ^m in the CSA. These last

fermions in fact are used to represent the Q_a operators in $\mathcal{A}_\gamma$. Their contribution will disappear in $\tilde{L}$. $L_{SU(N)}$ is the Sugawara construction based on V and $L_{SU(N-2)}$ is the Sugawara construction based on the modified $\hat{V}$. The two terms in eq.(5.17) both transform $\hat{V}_i$ as a conformal dimension 1 operator, so $L_{SU(N-2)}$ commutes with L as is required in the GKO construction. One easily checks that c for eq.(5.17) is given by eq.(1.1).

For the remaining operators of $\mathcal{A}_\gamma$ it is simpler to think in terms of the algebra $\tilde{\mathcal{A}}_\gamma$ eq.(1.5). One finds [11] up to normalizations that $\tilde{A}^{\pm\pm}$ is simply given by the V_θ , V^θ . $\tilde{A}^{\pm\pm}$ is given by the fermionic construction eq.(5.15) with i replaced by θ , corresponding to an $SU(2)$ KM current built from $N-2$ complex Fermi doublets, the Wolf space fermions. Thus indeed $\tilde{k}^- = N-2$ is the quaternionic dimension of the Wolf space.

Similarly the $\tilde{G}$ operators are given by expressions generalizing eq.(B.5) in a simple way [11].

The realization of U is described in Appendix C, and finally the Q_a 's are given by ψ^θ , ψ_θ and the associated ψ_m , ψ^m in the CSA, all suitably normalized, as mentioned above.

To show that all parameters of massless representations can be constructed consider the states (cf. App. C for details of notation)

$$|\Omega(a, m)\rangle = \psi_{(+\alpha)_{i_1}, \frac{-1}{2}} \cdots \psi_{(+\alpha)_{i_m}, \frac{-1}{2}} |(a\ 0 \dots 0)\rangle \quad (5.18)$$

As in the $SU(3)$ case this is always a highest weight state of $\mathcal{A}_\gamma$. The state $|(a\ 0 \dots 0)\rangle$ denotes the hws of the $SU(N)$ KM algebra corresponding to the representation of $SU(N)$ with Dynkin labels $(a\ 0 \dots 0)$. This direction is orthogonal to the $SU(N-2)$ roots and is thus left invariant by $SU(N-2)$.

From the above account of the realizations it follows that

$$\ell^+ = \frac{1}{2} \sum_i a_i = \frac{1}{2} a, \quad \ell^- = \frac{1}{2} m \quad (5.19)$$

Note that the maximum value of a is $n = \tilde{k}^+$ and for m the maximal number of fermionic $\frac{-1}{2}$ modes which we can have for a hws is $N-2 = \tilde{k}^-$. Both these numbers are thus in agreement with eq.(2.3).

The value of h is determined by the following contributions :

- the fermionic modes give $h_F = \frac{m}{2} = \ell^-$.
- the bosonic representation of $SU(N)$ gives with eq.(5.2) (cf. eq.(5.10) of [15])

$$h_{SU(N)} = \frac{1}{2k} \left(\frac{N-1}{N} a^2 + (N-1)a \right) = \frac{k^-}{k} \left(\frac{2}{k^-+1} (\ell^+)^2 + \ell^+ \right) \quad (5.20)$$

- Concerning the Sugawara construction for $L_{SU(N-2)}$, using the operators $\hat{V}$ (eq.(5.15)), there is no contribution from the bosonic V 's on

the state eq.(5.18) since the representation of $SU(N)$ has been chosen such that the hws is a singlet under $SU(N-2)$. But the fermions sit in the m index antisymmetric representation of $SU(N-2)$ (Dynkin label $(0 \dots 010 \dots 0)$ with 1 in the m -th position). The contribution is thus

$$h_{SU(N-2)} = \frac{N-1}{2k(N-2)} m(N-2-m) = \frac{k^-}{k\tilde{k}^-} \ell^- (\tilde{k}^- - 2\ell^-). \quad (5.21)$$

Finally

$$\begin{aligned} kh &= k(h_F + h_{SU(N)} - h_{SU(N-2)}) \\ &= 2k^- \left(\frac{(\ell^+)^2}{k^- + 1} + \frac{(\ell^-)^2}{\tilde{k}^-} \right) + k^- \ell^+ + k^+ \ell^- \end{aligned} \quad (5.22)$$

The value of u has also a bosonic and a fermionic contribution. These may be read off from Appendix C with the result

$$\begin{aligned} u &= - \left(\sqrt{\frac{N-2}{N}} \frac{a}{2} + \sqrt{\frac{N}{N-2}} \frac{m}{2} \right) \\ u^2 &= \frac{\tilde{k}^-}{k^- + 1} (\ell^+)^2 + 2\ell^+ \ell^- + \frac{k^- + 1}{\tilde{k}^-} (\ell^-)^2 \end{aligned} \quad (5.23)$$

It is now trivial to verify that

$$k\tilde{h} = kh - u^2 \quad (5.24)$$

satisfies the bound eq.(2.7), proving that we have indeed constructed massless representations for all the values of the parameters that were allowed.

6 Representations with Ramond moding.

Up to here we restricted ourselves to the Neveu-Schwarz moding of the operators. But having obtained these, the representations in all other modings follow from the automorphisms constructed in [12], except for the case $k^+ = k^-$ where a twisted algebra exists, which cannot be obtained from the others. We will not treat these twisted algebras here. But we will show explicitly how to obtain a Ramond representation from a Neveu-Schwarz one.

If we have a Neveu-Schwarz representation then consider the state

$$(A_{-1}^{++})^{k^+ - 2\ell^+} |hws\rangle. \quad (6.1)$$

It is a hws for the Ramond algebra which can be obtained from a spectral flow with parameters $\rho = -1$, $\eta = 0$ in [12]. The definition of hws includes then, in addition to eq.(1.3),

$$(G_+)_0 |hws\rangle = (G_{+\kappa})_0 |hws\rangle = 0. \quad (6.2)$$

The characteristics of this Ramond representation can then be derived from expressions in [12] :

$$\begin{aligned}
(L_0)^R &= (L_0)^{NS} - (iA_0^{+3})^{NS} + \frac{k^+}{4} \\
(iA_0^{+3})^R &= (iA_0^{+3})^{NS} - \frac{k^+}{2} \\
(iA_0^{-3})^R &= (iA_0^{-3})^{NS} \\
(iU_0)^R &= (iU_0)^{NS}
\end{aligned} \tag{6.3}$$

One can repeat all the foregoing with $- \leftrightarrow +$, $\eta \leftrightarrow \rho$, so that one obtains the following equivalence of representations :

$$\begin{aligned}
(h, \ell^+, \ell^-, u)_{NS} &\equiv (h - \ell^+ + \frac{k^+}{4}, \frac{k^+}{2} - \ell^+, \ell^-, u)_R \\
&\equiv (h - \ell^- + \frac{k^-}{4}, \ell^+, \frac{k^-}{2} - \ell^-, u)_R.
\end{aligned} \tag{6.4}$$

From the limiting values for Neveu-Schwarz representations given above one then trivially finds those for Ramond. The lowest value of $\tilde{h}_R$ for given k^+ , k^- is

$$\tilde{h}_R = \frac{1 + k^+ k^-}{4(k^+ + k^-)}. \tag{6.5}$$

7 Conclusions

We have discovered properties of uhwr of the $N = 4$ superconformal algebras $\mathcal{A}_\gamma$ [6] and $\tilde{\mathcal{A}}_\gamma$ [8]. The representations of these are related in a simple way described in the introduction. In the representations of the Lie superalgebras $\mathcal{A}_\gamma$ occurs an extra quantum number u related to a free boson field. This can be projected out together with the 4 free fermions as described in [8]. Then the relevant quantum numbers are $\ell^\pm$, the isospins related to the two $\widehat{SU}(2)$ subalgebras, and $\tilde{h}$, the eigenvalue of the $\tilde{L}_0$, related to h by eq.(1.7). Our result is that the unitary representations are characterized by (for Neveu-Schwarz moding)

$$\begin{aligned}
\ell^\pm &= 0, \frac{1}{2}, 1, \dots, \frac{k^\pm - 1}{2} = \frac{\tilde{k}^\pm}{2} \\
k\tilde{h} &\geq (\ell^+ - \ell^-)^2 + k^-\ell^+ + k^+\ell^-
\end{aligned} \tag{7.1}$$

If the bound is reached we call the representation massless, otherwise massive. This is motivated by the following result:

$$k\tilde{h} = (\ell^+ - \ell^-)^2 + k^-\ell^+ + k^+\ell^- \Leftrightarrow \tilde{G}_{-\frac{1}{2},+}|hws\rangle = 0. \tag{7.2}$$

Further we have the important fact that

$$2\ell^+ = \tilde{k}^+ \text{ or } 2\ell^- = \tilde{k}^- \Rightarrow \tilde{G}_{-\frac{1}{2},+}|hws\rangle = 0. \quad (7.3)$$

The above rules imply that for $k^\pm = 1$ all representations are massless.

We have given an explicit construction of all massless representations. We also have given explicit constructions of some massive representations (in some detail for $k^- = 2$). In these realizations we find quantized values for $\tilde{h}$ above the lower bound (see figure 3). We believe that this is due to the compact nature of the space we considered ($SU(3)$), and that more general realizations should exist where $\tilde{h}$ can have all the values on the lines above

$$g_{1/2} \equiv 2(\tilde{h} - \frac{1}{k}H_{\frac{1}{2}}) = 0 \quad (7.4)$$

(as indicated in fig. 1).

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A Conventions about the N=4 algebra

For the $\mathcal{A}_\gamma$ algebra we use the commutators as given in [6]. In OPE they read (those with the Virasoro algebra $L(z)$ are as usual)

$$\begin{aligned} G_a(z)G_b(w) &= \frac{2}{3}c\delta_{ab}(z-w)^{-3} + (z-w)^{-2}2M_{ab}(w) \\ &\quad + (z-w)^{-1}(2L(w)\delta_{ab} + \partial M_{ab}(w)) + \dots \\ M_{ab} &= -\frac{4}{k}[k^-\alpha_{ab}^{+i}A_i^+ + k^+\alpha_{ab}^{-i}A_i^-] \\ A^{\pm i}(z)G_a(w) &= \alpha_a^{\pm i}{}^b(G_b(w)(z-w)^{-1} \mp \frac{2}{k}k^\pm Q_b(w)(z-w)^{-2}) + \dots \\ A^{\pm i}(z)A^{\pm j}(w) &= \varepsilon^{ijk}A_k^\pm(w)(z-w)^{-1} - \frac{k^\pm}{2}\delta^{ij}(z-w)^{-2} + \dots \\ Q_a(z)G_b(w) &= [2(\alpha_{ab}^{+i}A_i^+(w) - \alpha_{ab}^{-i}A_i^-(w)) + \delta_{ab}U(w)](z-w)^{-1} + \dots \\ A^{\pm i}(z)Q_a(w) &= \alpha_a^{\pm i}{}^bQ_b(w)(z-w)^{-1} \\ U(z)G_a(w) &= Q_a(z-w)^{-2} + \dots \\ Q_a(z)Q_b(w) &= -\frac{k}{2}\delta_{ab}(z-w)^{-1} + \dots \\ U(z)U(w) &= -\frac{k}{2}(z-w)^{-2} + \dots \end{aligned} \quad (A.1)$$

and the others are nonsingular. The matrices $\alpha_a^{\pm b}$ satisfy

$$\begin{aligned} [\alpha^{\pm i}, \alpha^{\pm j}] &= -\epsilon^{ijk} \alpha_k^{\pm} \\ [\alpha^{+i}, \alpha^{-j}] &= 0 \\ \{\alpha^{\pm i}, \alpha^{\pm j}\} &= -\frac{1}{2} \delta^{ij}. \end{aligned} \quad (\text{A.2})$$

We use complex notations. That means that for i one takes $\{+, -, 3\}$ with

$$\begin{aligned} \delta_{+-} &= \frac{1}{2} \\ \epsilon_{-3}^{+-} &= -2i \\ \epsilon_{\pm}^{3\pm} &= \mp i \end{aligned} \quad (\text{A.3})$$

For $a, b, \dots$ values one takes $\{+, -, +K, -K\}$ and the non zero δ_{ab} and $\alpha_{ab}^{\pm i}$ (up to symmetry) are

$$\begin{aligned} \delta_{+-} &= \frac{1}{2} & \delta_{+K-K} &= \frac{1}{2} \\ \alpha_{+-}^{\pm 3} &= -\frac{i}{4} & \alpha_{+K-K}^{\pm 3} &= \mp \frac{i}{4} \\ \alpha_{++K}^{+-} &= \frac{i}{2} & \alpha_{--K}^{++} &= -\frac{i}{2} \\ \alpha_{-+K}^{-+} &= -\frac{i}{2} & \alpha_{+-K}^{--} &= \frac{i}{2}. \end{aligned} \quad (\text{A.4})$$

The 4- index antisymmetric tensor (in the real coordinates $\epsilon^{1230} = 1$) can be defined by

$$\alpha^{\pm iab} \alpha_{\pm i}^{\pm cd} = \frac{1}{4} (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \pm \epsilon^{abcd}). \quad (\text{A.5})$$

The hermiticity relations are

$$\begin{aligned} (L_n)^\dagger &= L_{-n} \\ (G_n^+)^\dagger &= G_{-n}^- \\ (G_n^{+K})^\dagger &= G_{-n}^{-K} \\ (A_n^{\pm 3})^\dagger &= -A_{-n}^{\pm 3} \\ (A_n^{\pm +})^\dagger &= -A_{-n}^{\pm -} \\ (U_n)^\dagger &= -U_{-n} \\ (Q_n^+)^\dagger &= -Q_{-n}^- \\ (Q_n^{+K})^\dagger &= -Q_{-n}^{-K}. \end{aligned} \quad (\text{A.6})$$

For the $\tilde{\mathcal{A}}_\gamma$ algebra the nontrivial OPE are given by

$$\begin{aligned} \tilde{A}^{\pm i}(z) \tilde{G}_a(w) &= \alpha_{ab}^{\pm i} \tilde{G}^b(w) (z-w)^{-1} + \dots \\ \tilde{G}_a(z) \tilde{G}_b(w) &= \frac{4\tilde{k}^+ \tilde{k}^-}{k} \delta_{ab} (z-w)^{-3} + 2\tilde{L} \delta_{ab} (z-w)^{-1} \end{aligned}$$

$$\begin{aligned}
& -\frac{8}{k} \left(\tilde{k}^- \alpha_{ab}^{+i} \tilde{A}_i^+ + \tilde{k}^+ \alpha_{ab}^{-i} \tilde{A}_i^- \right) (z-w)^{-2} \\
& -\frac{4}{k} \partial \left(\tilde{k}^- \alpha_{ab}^{+i} \tilde{A}_i^+ + \tilde{k}^+ \alpha_{ab}^{-i} \tilde{A}_i^- \right) (z-w)^{-1} \\
& -\frac{8}{k} (\alpha^{+i} \tilde{A}_i^+ - \alpha^{-i} \tilde{A}_i^-)_{\alpha a} (\alpha^{+j} \tilde{A}_j^+ - \alpha^{-j} \tilde{A}_j^-)_{\beta b} \epsilon^{ab} (z-w)^{-1}.
\end{aligned} \tag{A.7}$$

B The realization using $\widehat{SU}(3)$

Following [10] we get a realization with $k^+ = n+1$, $k^- = 2$ if we start from the sigma model on $SU(3)$. We define 4 complex generators labeled m (in the CSA), θ (highest positive root), $+$ and $-$. Then the complex structure constants are ($k = n+3$)

$$\begin{aligned}
F_{m\theta} &= \frac{2i}{\sqrt{k}} \\
F_{m\pm} &= \frac{i}{\sqrt{k}} (1 \pm i\sqrt{3}) \\
F_{+-} &= \frac{2i}{\sqrt{k}} \\
F_{ij}^k &= (F_{ij}^k)^*
\end{aligned} \tag{B.1}$$

The second complex structure is defined by the antisymmetric matrix

$$K_{m\theta} = K_{+-} = i \tag{B.2}$$

and with upper indices given by the complex conjugate.

The fundamental fields are the bosons V^i , fermions ψ^i ³, and their complex conjugates with lower indices. Their singular OPE are

$$\begin{aligned}
\psi_i(z) \psi^j(w) &= -\frac{1}{2} \delta_i^j (z-w)^{-1} \\
V^i(z) V^j(w) &= F_{ij}^k V^k(w) (z-w)^{-1} \\
V^i(z) V_j(w) &= -\frac{4n}{k} \delta_j^i (z-w)^{-2} + \left(F_{jk}^i V^k(w) + F_{ij}^k V_k(w) \right) (z-w)^{-1}
\end{aligned} \tag{B.3}$$

Now the realization is

$$\begin{aligned}
L &= -\frac{1}{8} (V^i V_i + V_i V^i) - \partial \psi_i \psi^i - \partial \psi^i \psi_i \\
G_+ &= \frac{1}{\sqrt{2}} (V^i \psi_i + F_{ij}^k \psi_i \psi_j \psi^k) \\
G_- &= \frac{1}{\sqrt{2}} (V_i \psi^i + F_{ij}^k \psi^i \psi^j \psi_k)
\end{aligned}$$

³To avoid confusion we always call here ψ the operators Q from [10]. Note also the factors $\sqrt{2}$ difference in the normalization of G and Γ (here called Q).

$$\begin{aligned}
G_{+K} &= \frac{1}{\sqrt{2}}(V^i \psi^j K_{ij} + 2\psi^\ell K_{\ell i} F^{ij} \psi^k \psi_j) \\
G_{-K} &= \frac{1}{\sqrt{2}}(V_i \psi_j K^{ij} + 2\psi_\ell K^{\ell i} F_{ij} \psi^k \psi^j) \\
iA^{+3} &= \frac{i}{4}\sqrt{k} (V_m + V^m) + \psi_\theta \psi^\theta - \psi_m \psi^m \\
iA^{++} &= -\frac{1}{2}\sqrt{k} V^\theta - 2i\psi_m \psi^\theta \\
iA^{+-} &= +\frac{1}{2}\sqrt{k} V_\theta + 2i\psi_\theta \psi^m \\
iA^{-3} &= \psi^i \psi_i \\
iA^{-+} &= -2i(\psi_m \psi_\theta + \psi_+ \psi_-) \\
iA^{--} &= -2i(\psi^m \psi^\theta + \psi^+ \psi^-) \\
Q_+ &= i\sqrt{\frac{k}{2}} \psi_m \\
Q_- &= -i\sqrt{\frac{k}{2}} \psi^m \\
Q_{+K} &= -\sqrt{\frac{k}{2}} \psi^\theta \\
Q_{-K} &= -\sqrt{\frac{k}{2}} \psi_\theta \\
U &= -\frac{i}{4}\sqrt{k} (V_m - V^m) + i\sqrt{3}(\psi^+ \psi_+ - \psi^- \psi_-). \quad (\text{B.4})
\end{aligned}$$

The operators of the $\tilde{\mathcal{A}}_\gamma$ algebra take a simpler form

$$\begin{aligned}
\tilde{L} &= -\frac{1}{8}(V^i V_i + V_i V^i) - \partial\psi_+ \psi^+ - \partial\psi^+ \psi_+ - \partial\psi_- \psi^- - \partial\psi^- \psi_- + \frac{1}{k}UU \\
\tilde{G}_+ &= \frac{1}{\sqrt{2}}(V^+ \psi_+ + V^- \psi_-) \\
\tilde{G}_- &= \frac{1}{\sqrt{2}}(V_+ \psi^+ + V_- \psi^-) \\
\tilde{G}_{+K} &= \frac{i}{\sqrt{2}}(V^+ \psi^- - V^- \psi^+) \\
\tilde{G}_{-K} &= -\frac{i}{\sqrt{2}}(V_+ \psi_- - V_- \psi_+) \\
\tilde{A}^{+3} &= \frac{1}{4}\sqrt{k} (V_m + V^m) \\
\tilde{A}^{++} &= \frac{i}{2}\sqrt{k} V^\theta \\
\tilde{A}^{+-} &= -\frac{i}{2}\sqrt{k} V_\theta \\
\tilde{A}^{-3} &= (-i)(\psi^+ \psi_+ + \psi^- \psi_-)
\end{aligned}$$

$$\begin{aligned}
\tilde{A}^{-+} &= -2\psi_+\psi_- \\
\tilde{A}^{--} &= -2\psi^+\psi^-
\end{aligned}
\tag{B.5}$$

C The root system for the realization using $SU(N)/SU(N-2)$

Let the simple roots of $SU(N)$ be $\alpha_1, \dots, \alpha_{N-1}$. The highest root may be obtained as

$$\theta = \sum_{i=1}^{N-1} \alpha_i \tag{C.1}$$

or in the Dynkin basis [15]

$$\theta = (100 \dots 01) \tag{C.2}$$

The $SU(N-2)$ subgroup must be chosen such that its simple roots are orthogonal to θ , and are therefore $\alpha_2, \dots, \alpha_{N-2}$. The roots in the Wolf space $SU(N)/[SU(N-2) \otimes SU(2) \otimes U(1)]$ may be given as follows. They come in four sets of $N-2$ roots: $(+\alpha)_{(1)}, \dots, (+\alpha)_{(N-2)}$ and their negatives, as well as $(-\alpha)_{(1)}, \dots, (-\alpha)_{(N-2)}$ and their negatives. Then we have

$$\begin{aligned}
(-\alpha)_{(1)} &= \alpha_1 \\
(-\alpha)_{(i)} &= \alpha_1 + \sum_{j=2}^i \alpha_j, \quad i = 2, \dots, N-2 \\
(+\alpha)_{(j)} &= \theta - (-\alpha)_{(j)}, \quad j = 1, \dots, N-2
\end{aligned}
\tag{C.3}$$

This shows that these Wolf space roots are transformed into each other by the action of $SU(N-2)$ and that we have in the notation of [15] (eq.(C.3) give $(\pm\alpha)'s$ in the dual basis)

$$(\theta, (\pm\alpha)) = 1 \tag{C.4}$$

Together with $\pm\theta$ and the (non-zero) roots of the $SU(N-2)$, they make up the non-zero roots of $SU(N)$. In the notation of [10], eq.(C.4) is written in the complex basis as ($x_\theta = 1$ for $SU(N)$ and $SU(N-2)$)

$$i^m \theta_m + i_m \theta^m = -\frac{1}{n + \tilde{h}} \tag{C.5}$$

where $\tilde{h} = N-2$ is the dual Coxeter number of the subgroup, in this case $SU(N-2)$. (i^m, i_m) are the complex coordinates of a Wolf space root $(\pm\alpha)$ and $i_m = -(i^m)^*$. We see that the normalization of the scalar product in eq.(C.4) differs from eq.(C.5) by the factor indicated, and that the projection of θ provides the real part whereas the imaginary part, or direction of U , is given by

$$i^m \theta_m - i_m \theta^m \tag{C.6}$$

suitably normalized [10].

Correspondingly, in the Dynkin basis, this direction in root space is provided by

$$I_U \equiv \sqrt{\frac{N}{N-2}}(100 \cdots 0 - 1) \quad (\text{C.7})$$

which is obviously orthogonal to θ and the $SU(N-2)$ roots and has the same length squared of 2 as the roots (in notation [15]). Also

$$(I_U, (\pm\alpha)) = \mp \sqrt{\frac{N}{N-2}}. \quad (\text{C.8})$$

The fermionic part of U then becomes [10]

$$iU_F = \sqrt{\frac{N}{N-2}}(\psi^{+\alpha}\psi_{+\alpha} - \psi^{-\alpha}\psi_{-\alpha}) \quad (\text{C.9})$$

and the bosonic part acting on a state $|(a_i) \rangle$ gives [10], [15]

$$u_B = \frac{1}{2}(I_U, (a_i)) = \frac{1}{2}\sqrt{\frac{1}{N(N-2)}} \sum_{i=1}^{N-1} a_i(N-2i) \quad (\text{C.10})$$

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Figure Captions

Figure 1 : The unitary representations of the hat algebra with $\hat{c} = 6$, $k^- = 2$ and $k^+ = 6$ have their hws described by the dots on the line $g_{1/2} = 0$ and the wavy lines. The values of $\hat{q}$ above 1 can only be reached with $2\ell^+ = \tilde{k}^+$ or $2\ell^- = \tilde{k}^-$, and therefore the corresponding $\hat{h}$ values are always those which saturate the bound eq.(3.25) when $m = 1$.

Figure 2 : The well known weight diagram of a typical $SU(3)$ -representation (a_1, a_2) for $a_1 = 6, a_2 = 2$. The multiplicities of the weights increase from 1 at the border to 3 as indicated. Inside the border are sub-diagrams with $(a'_1, a'_2) = (5, 1)$ and $(4, 0)$. These three borders have $p = 0, 1, 2$ as indicated. Also the horizontal lines with $m = -6, \dots, +2$ are shown.

Figure 3 : Examples of the spectrum of $k\hat{h}$ and ℓ^+ values found for $k^- = 2$. The figure is for $k^+ = 6$. The $\ell^- = 0$ figure summarizes the possibilities from eq.(5.7) corresponding to the 21 different possible $SU(3)$ -representations in this case with $a_1 + a_2 \leq 5 = k^+ - 1$. The dashed line is the bound eq.(5.8) (or $H_{1/2}$, eq.(2.7)) which is saturated for ℓ^+ belonging to an $SU(3)$ highest weight.

Similarly the $\ell^- = 1/2$ figure summarizes eq.(5.12) for $k^+ = 6$. The dashed line is the bound $H_{1/2}$, eq.(2.7).

Fig. 1.

Fig. 2.

Fig. 3a.

Fig. 3b.