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PHENOMENOLOGY FROM LATTICE QCD

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PHENOMENOLOGY FROM LATTICE QCD

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1. INTRODUCTION

In this series of lectures I will discuss the theoretical aspects, describe the numerical techniques and report the most recent results in the computation of hadronic matrix elements using lattice QCD.

Lattice QCD offers in fact among all other non-perturbative approaches, the unique possibility, of computing with the same method many different matrix elements which are of interest in the phenomenology of the Standard Model. Among the other quantities I will report the most recent results obtained for the pion and proton structure functions and electromagnetic form factors, the nucleon σ -term, the proton decay amplitude, the meson decay constants and the weak hamiltonian matrix elements.

Particular attention will be devoted to the discussion of the systematic errors actually present in the lattice approach, especially of those coming from the "quenched" approximation and from effects due to the finiteness of the lattice spacing a . In Section 2 the general framework of lattice QCD and the problems connected with the lattice regularization of the fermionic action will be described. In Sect.3 some basic notions on the calculations of operator matrix elements on the lattice will be introduced. In Section 4 and 5 I will discuss in length the pion and nucleon structure functions from the lattice as a prototype of these calculations. In Section 6 the problem of the σ -term and its connection to the lattice results in the quenched approximation will be presented. Section 7 will be devoted to the weak decays of the hadrons. Section 8 will summarize the present situation and the perspectives in this field.

2. THE LATTICE REGULARIZATION

The lattice is simply an ultraviolet regularization of the theory. The trascription of the QCD action on the lattice is discussed in the lectures by M.Creutz at this school, together with the numerical methods currently used to generate the gluon field configurations and to compute the quark propagators. We assume that the reader has already some basic notions on this regularization and technique and on the continuum limit of the theory. In the following we will analyze in length only the peculiar problems related to the regularization of the Dirac action on the lattice.

When we introduce a regularization in the process of renormalizing a theory some of the symmetries present in the original (bare) action are in general lost. It is precisely for this reason that

anomalies are present in quantum field theories. In most of the versions of lattice QCD we break the Lorentz invariance and, in presence of fermions, we get in troubles with the chiral symmetry of the theory. In this section the problem of the chiral symmetry for a wide class of lattice actions is discussed in detail.

2.1. Basic definitions and Ward identities

A quite general form of the fermion action on the lattice is:

$$S_\psi = \sum_{x,f} \left\{ \frac{1}{2a} \sum_{\mu} [\bar{\psi}_f(x) (\tau - \gamma_\mu) U_\mu(x) \psi(x + \hat{\mu}) + \bar{\psi}_f(x+\mu) (\tau + \gamma_\mu) U_\mu^\dagger(x) \psi(x)] + \right. \\ \left. \bar{\psi}_f(x) [M_0 + \frac{\tau}{a}] \psi_f(x) \right\} = \sum_f \bar{\psi}_f \Delta_f(U) \psi_f \quad (2.1)$$

The terms proportional to the Dirac matrices γ^μ are the naive lattice transcription of the covariant derivative $\bar{\psi} \gamma^\mu D_\mu \psi$ and an “irrelevant” operator, proportional to the parameter τ , has been introduced to avoid the fermion species doubling (Wilson term¹). The Wilson term acts as a mass term and breaks chiral symmetry explicitly, even in the limit of the vanishing quark mass M_0 . As a consequence, the Ward identities for the vector and axial vector currents contain anomalous pieces, and tree-level chiral properties of composite operators are spoiled by the interaction between quarks and gluons. The Ward identities for the vector and axial vector currents can be easily derived by a chiral rotation of the fermion fields in the action of Eq.(2.1):

$$\psi \rightarrow \psi + i(\alpha_V^a + \gamma_5 \alpha_A^a) \frac{\lambda^a}{2} \psi \quad (2.2)$$

$$\bar{\psi} \rightarrow \bar{\psi} - i\bar{\psi}(\alpha_A^a - \gamma_5 \alpha_V^a) \frac{\lambda^a}{2}$$

One obtains²:

$$\langle \alpha | \nabla_\mu V_\mu^a(x) | \beta \rangle = - \langle \alpha | \bar{\psi}(x) | \left[\frac{\lambda^a}{2}, M_0 \right] \psi(x) | \beta \rangle + \\ \langle \alpha | X_V^a(x) | \beta \rangle \quad (2.3)$$

$$\langle \alpha | \nabla_\mu A_\mu^a(x) | \beta \rangle = \langle \alpha | \bar{\psi}(x) | \left\{ \frac{\lambda^a}{2}, M_0 \right\} \gamma_5 \psi(x) | \beta \rangle + \\ \langle \alpha | X_A^a(x) | \beta \rangle$$

where

$$\nabla_\mu f(x) = f(x) - f(x - \hat{\mu}) \quad (2.4)$$

FIGURE 1

The operator X_A^a is of order a (Fig.1a). One-loop corrections, because of the bad ultra-violet behaviour of X_A^a ($\sim p^2$) give a linearly divergent contribution as $a \rightarrow 0$ (Fig.1b)

and

$$V_\mu^a(x) = \frac{1}{2} \left[\bar{\psi}(x) U_\mu(x) \gamma_\mu \frac{\lambda^a}{2} \psi(x + \hat{\mu}) + \text{h.c.} \right] \quad (2.5)$$

$$A_\mu^a(x) = \frac{1}{2} \left[\bar{\psi}(x) U_\mu(x) \gamma_\mu \gamma_5 \frac{\lambda^a}{2} \psi(x + \hat{\mu}) + \text{h.c.} \right]$$

The Ward identities in Eq.(2.3) look very similar to the corresponding identities in the continuum limit but for the last term on the right-hand side. In the free theory, this term is of order $O(a)$, a being the lattice spacing, and disappears as $a \rightarrow 0$. Unfortunately, in the real case, the interaction promotes these anomalous pieces, so that not only they do not vanish as $a \rightarrow 0$, but become linearly divergent ($\sim 1/a$), as illustrated in Fig.1. For the vector current this is not a problem, since one can show that X_V^a is itself the divergence of a current²:

$$X_V^a = \nabla_\mu^a J_\mu^a \quad (2.6)$$

X_V^a can be moved on the left-hand side of the Ward identity. The current $\hat{V}_\mu^a = V_\mu^a - J_\mu^a$ now obeys the same Ward identity of the continuum theory. This happens because the Wilson term respects vector symmetries (e.g., baryon number), which is not the case for the axial current. In the latter case, however, one can show that the matrix elements of X_A^a between on-shell states α and β can be written as^{2,3}:

$$\begin{aligned} \langle \alpha | X_A^a | \beta \rangle = & - \langle \alpha | \bar{\psi} \left\{ \frac{\lambda^a}{2}, \bar{M}(M_0) \right\} \gamma_5 \psi | \beta \rangle \\ & - (Z_A - 1) \langle \alpha | \nabla_\mu A_\mu^a | \beta \rangle + O(a) \end{aligned} \quad (2.7)$$

The relation (2.7) is true at all orders in the coupling constant and is the more general one which is compatible with the symmetries of the action. $O(a)$ indicates matrix elements of operators of

dimension larger than four, which vanish as $a \rightarrow 0$. One can also show that the coefficient $\bar{M}(M_0)$ is linearly divergent in $1/a$ and that $Z_A = Z_A(g_0)$ is a function only of the bare lattice coupling constant g_0 (and r)^{3,4}. Let us rewrite the last of equations (2.3) as follows

$$Z_A < \alpha | \nabla_\mu A_\mu^a | \beta > = < \alpha | \left\{ M_0 - \bar{M}(M_0), \frac{\lambda^a}{2} \right\} \gamma_5 \psi | \beta > + < \alpha | \bar{X}_A^a | \beta > \quad (2.8)$$

where

$$\bar{X}_A^a = X_A^a + \psi \left\{ \frac{\lambda^a}{2}, \bar{M} \right\} \gamma_5 \psi + (Z_A - 1) \nabla_\mu A_\mu^a \quad (2.9)$$

has matrix elements between on-shell states which vanish in the continuum limit [$< \alpha | \bar{X}_A^a | \beta > \sim 0(a)$]. From Eq.(2.8) we see that we have recovered the usual continuum Ward identity, provided that we identify the good axial current A_μ^a and the bare quark mass m with:

$$\hat{A}_\mu^a = Z_A A_\mu^a \quad m = M_0 - \bar{M}(M_0) \quad (2.10)$$

We also note that Eq.(2.8) is not sufficient alone to fix Z_A and m separately, but only the ratio $\frac{m}{Z_A}$. We need at least another Ward identity to disentangle the two constants. The simplest Ward identity we can think of involves two- and three-point correlation functions of vector and axial vector currents:

$$\begin{aligned} < \nabla_\mu \hat{A}_\mu^a(x) \hat{A}_\nu^b(y) \hat{V}_\rho^c(0) > = < \bar{\psi}(x) \left\{ \frac{\lambda^a}{2}, m \right\} \gamma_5 \psi(x) \hat{A}_\nu^b(y) \hat{V}_\rho^c(0) > \\ & + i f^{abd} \frac{Z_A}{Z_V} \delta(x-y) < \hat{V}_\nu^d(y) \hat{V}_\rho^c(0) > + i f^{acd} \frac{Z_V}{Z_A} \delta(x) < \hat{A}_\rho^b(y) \hat{A}_\rho^d(0) > + \\ & < \bar{X}_A^a(x) \hat{A}_\nu^b(y) \hat{V}_\rho^c(0) > + \dots \end{aligned} \quad (2.11)$$

The dots indicate possible Schwinger terms. Z_V is the renormalization constant analogous to Z_A for the vector current V_ρ , [Eq.(2.8)], ($\hat{V}_\rho = Z_V V_\rho$), and m and Z_A have been defined before. The last term on the right-hand side of Eq.(2.11) is non-zero even in the continuum limit. The matrix elements of $\bar{X}_A^a$ vanish between on-shell states. $\bar{X}_A^a$ can, however, still give rise to contact terms [i.e., in Eq.(2.11) terms proportional to $\delta(x-y)$ or $\delta(x)$] in more complicated correlation functions because of extra divergences present when more fields are at the same point. These terms transform the anomalous Ward identity in Eq.(2.11) into the corresponding identity of the continuum:

$$\begin{aligned}
\langle \nabla_\mu \hat{A}_\mu^a(x) \hat{A}_V^b(y) \hat{V}_\rho^c(0) \rangle &= \langle \bar{\psi}(x) \left\{ \frac{\lambda^a}{2}, m \right\} \gamma_5 \psi(x) \hat{A}_V^b(y) \hat{V}_\rho^c(0) \rangle \\
&+ i f^{abd} \delta(x-y) \langle \hat{V}_V^d(y) \hat{V}_\rho^c(0) \rangle + i f^{acd} \delta(x) \langle \hat{A}_V^b(y) \hat{A}_\rho^d(0) \rangle + \dots
\end{aligned} \tag{2.12}$$

as it can be proved in perturbation theory³. This implies the following relation:

$$\begin{aligned}
\sum_x \langle [\nabla_\mu A_\mu^a(x) - \bar{\psi}(x) \left\{ \frac{\lambda^a}{2}, \frac{m}{Z_A} \right\} \gamma_5 \psi(x)] A_V^b(y) V_\rho^c(0) \rangle &= \\
i f^{abd} \left(\frac{Z_V}{Z_A^2} \right) \langle V_V^d(y) V_\rho^c(0) \rangle + i f^{acd} \frac{1}{Z_V} \langle A_V^b(y) A_\rho^d(0) \rangle
\end{aligned} \tag{2.13}$$

Equations (2.8) and (2.13) allow us to separate Z_A and m in order to find the axial current and the quark mass of the continuum theory.

The strategy based on continuum Ward identities as a mean of identifying the "good" parameters and renormalized operators (m and $\hat{A}_\mu^a$ this case) is quite general, as it will also be clear from the discussion of the renormalization of four-fermion operators given in the next sub-section. Current algebra relations of the type discussed above have been verified by numerical simulation on the lattice in Ref.[5].

2.2 Renormalization of Four-Fermion Operators of the Weak Hamiltonian

The problem of the construction of renormalized, four-fermion operators is quite intriguing because of the presence of severe power divergences and of the bad chiral behaviour of the bare operators on the lattice, induced by the Wilson term.

It is simple to start by giving a specific example. Let us consider the (8,1) operator:

$$O_\alpha = (\bar{\psi}_L \frac{\lambda_\alpha}{2} \gamma_\mu \psi_L) (\bar{\psi}_R \gamma^\mu \psi_R) \tag{2.14}$$

where $\psi_{L,R} = \frac{(1 \pm \gamma_5)}{2} \psi$, and λ_α is one of the Gell-Mann matrices acting in flavour space. Even at zero order in the bare strong coupling constant and in the chiral limit, the operator in Eq.(2.14) can mix with an operator of dimension three with a power-divergent coefficient through the diagram in Fig.(2):

$$\delta O_\alpha = \frac{1}{a^3} \left(\bar{\psi}_L \frac{\lambda_\alpha}{2} \psi_R + \bar{\psi}_R \frac{\lambda_\alpha}{2} \psi_L \right) \tag{2.15}$$

δO_α is a (3,3) operator. The factor $1/a^3$ is there because of dimensional reasons. Usually, for massless quarks, this mixing is not possible because the original operator cannot flip the helicity; on the lattice the mixing is given by the Wilson term which acts as a mass term when inserted into the loop of the diagram in Fig.(2). This example shows that a bare operator, which one would naively expect to be an (8,1) operator, is really a mixture of (8,1) and (3,3) operators. A definite chiral

behaviour and the removal of lattice artifact divergences can be obtained at the same time, using the Ward identities along a line similar to the one followed in the previous subsection for the axial current^{3,6}.

First, one subtracts from the naive operator a combination of operators with equal or lower dimensions and all possible naive chiralities, in such a way as to obtain an operator with the correct chiral properties with respect to the "good" axial and vector charges. Generally, at this point not all divergences have been eliminated, but one is left with an overall multiplicative renormalization which is determined by requiring the lattice operators to be normalized equally to the continuum ones.

FIGURE 2

The insertion of the Wilson term in the loop of this diagram mixes the four-fermion operator with a two-fermion operator by flipping the helicity of the quarks.

In the chiral limit, this procedure is straightforward. Let us consider a particular operator O_α which transforms, under naive axial transformation, according to some representation R:

$$\frac{\delta O_\alpha}{\delta \epsilon^a} = i R_{\alpha\beta}^a O_\beta \quad (2.16)$$

R^a are the axial generators of the particular representation to which O belongs: α and β are flavour labels appropriate to R. The operator which truly transforms according to R is

$$\tilde{O}_\alpha = O_\alpha + \sum_{n,\sigma} d_{\alpha\sigma}^{(n)} O_\sigma^{(n)} \quad (2.17)$$

with $O^{(n)}$ transforming according to $R^{(n)} \neq R$. The coefficient $d^{(n)}$ are restricted by the condition that they must respect the conserved vector symmetry.

In the chiral limit, the integrated axial current Ward identity for $\tilde{O}$ reads:

$$\begin{aligned} \sum_x \nabla_\mu \langle \alpha | T(\hat{A}_\mu^a(x) \tilde{O}(0)) | \beta \rangle = \\ \sum_x \langle \alpha | T(\bar{X}^a(x) \tilde{O}(0)) | \beta \rangle + i \langle \alpha | \frac{\delta \tilde{O}}{\delta \epsilon^a} | \beta \rangle \end{aligned} \quad (2.18)$$

where

$$\frac{\delta \tilde{O}}{\delta \epsilon} = i \left(R O + \sum_n d^n R^n O^{(n)} \right) \quad (2.19)$$

and we have omitted flavour indices. For the appropriate choice of the coefficients $d^{(n)}$, the contact terms arising in the correlation function of $\bar{X}$ with $\tilde{O}$ will correct the complicated naive chiral variation, Eq.(2.19), in such a way as to reproduce the desired transformation property. The condition for this is:

$$\sum_x \langle \alpha | T(\bar{X}^a(x) \tilde{O}_\alpha(0)) | \beta \rangle + i \langle \alpha | \frac{\delta \tilde{O}_\alpha}{\delta \epsilon^a} | \beta \rangle = - R_{\alpha\beta}^a \langle \alpha | \tilde{O}_\beta | \beta \rangle \quad (2.20)$$

equations (2.20) are a set of linear, inhomogeneous equation in $d^{(n)}$ which has a unique solution. The $\tilde{O}$ thus constructed is not yet finite, to $a \rightarrow 0$. Equations (2.18) and (2.20) show that the only freedom left is to take a linear superposition of operators transforming in the same way under chiral rotations. We thus define

$$\hat{O}(\mu) = Z_{\text{LATT}} \tilde{O} \quad (2.21)$$

Z_{LATT} being a matrix which mixes equivalent representations only, and it is such that:

$$\langle \alpha | Z_{\text{LATT}} \tilde{O} | \beta \rangle = \text{finite} \quad (2.22)$$

This condition is consistent with the left-hand side of Eq.(2.18). In fact, if both $\hat{A}_\mu$ and $Z_{\text{LATT}} \tilde{O}$ have finite matrix elements and, possible divergent terms in their product are localized operators whose contribution to the left-hand side of Eq.(2.18) vanishes upon taking the four-divergence and summing over x .

To complete the programme, we must specify more precisely the normalization condition (2.22). In the chiral limit, this requires the introduction of a subtraction point μ , so that:

$$Z_{\text{LATT}} = Z_{\text{LATT}}(\mu a, g_0) \quad (2.23)$$

$$\hat{O}(\mu) = Z_{\text{LATT}} \tilde{O}$$

$Z_{\text{LATT}}(\mu a, g_0)$ is completely determined, including the finite terms, by the requirement that $\hat{O}(\mu)$, for some value $\mu \ll a^{-1}$, obeys the same normalization condition as the continuum operator $O(\mu)$.

By construction, the operator $\tilde{O}$ obeys the continuum Ward identity

$$\int dx \langle \alpha | \nabla_\mu \hat{A}_\mu^a(x) Z_{\text{LATT}} \tilde{O} | \beta \rangle = - R^a \langle \alpha | Z_{\text{LATT}} \tilde{O} | \beta \rangle \quad (2.24)$$

where all the matrix elements in Eq.(2.24) are finite. Equation (2.24) is equivalent to the statement that the operator $\tilde{O}$ obeys soft pion theorems. In fact, upon integration on x , only the pion pole can give a contribution to the correlation function on the left-hand side of Eq.(2.24), as illustrated in Fig.(3). So we obtain

$$f_\pi \langle \alpha + \pi | \tilde{O} | \beta \rangle = R^a \langle \alpha | \tilde{O} | \beta \rangle = \langle \alpha | [Q_5^a, \tilde{O}] | \beta \rangle \quad (2.25)$$

FIGURE 3

Pion pole contribution to the matrix element of the renormalized operator.

i.e., the usual soft pion theorem of the continuum theory. This also implies that we can compute the physical $k \rightarrow \pi\pi$ amplitude from the $k \rightarrow \pi$ matrix element alone. At this point, we may turn the argument around. We can forget the Ward identity in Eq.(2.24) and determine the coefficients $d^{(n)}$ in the expansion (2.17) simply by requiring that the matrix elements of $\tilde{O}$ on the lattice obey the standard soft pion theorems. The latter conditions determine the operator apart from the overall normalization matrix Z_{LATT} .

To complete the analysis, we have to accomplish two further steps (see also Sect.4 and 7):

- i) identify in the various cases the operators that we need to subtract from the bare operators which are relevant to the weak Hamiltonian;
- ii) give a practical set of rules to determine the coefficients $d^{(n)}$ from the matrix elements on pseudoscalar meson states of the previous operators.

A final remark is necessary. The use of perturbation theory to fix coefficient $d^{(n)}$ which are finite (i.e., the coefficients of operators of dimension six) is completely justified. On the other hand, the use of perturbation theory to fix power-divergent coefficients leads in general to wrong results.

This happens because, for dimensional reasons, the coefficients of these lower dimension operators diverges as inverse powers of the lattice spacing (i.e. a^{-N}). Any non-perturbative spurious term given by these operators will then remain even for $g_0 \rightarrow 0$ (in fact $1/a e^{-1/2\beta_0 g_0^2} \rightarrow \text{const. as } g_0 \rightarrow 0$).

The mixing problem discussed above is present for all composite operators, as for example the

two quark operators $\bar{\psi} \gamma^\mu D^{\nu 1} \dots D^{\nu 2} \psi$, which are relevant in the computation of the structure functions. The mixing with lower dimensional operators poses the same problems as in the case of the four fermion operators and non perturbative subtractions are needed also in this case^{7,8}. Few explicit examples will be discussed in the section on the structure functions.

3. BASIC NOTIONS ON MONTECARLO TECHNIQUES AND ON THE CALCULATION OF HADRONIC MATRIX ELEMENTS

3.1 Two point functions, hadron masses and meson decay constants

We start by defining operators carrying the same quantum number of the particles under study. For example, in the case of the pion and of the rho, we may use

$$\begin{aligned} \pi^+(x) &= \bar{u}^A(x) \gamma_5 d^A(x) \\ \rho_\mu^+(x) &= \bar{u}^A(x) \gamma_\mu d^A(x) \end{aligned} \quad x \equiv (x, t) \quad (3.1)$$

From the action in Eq.(2.1), it is straightforward to compute the correlation function for these operators. In the pion case:

$$G(x, t) = \langle \pi(x, t) \pi^+(0, 0) \rangle = \frac{\int d[U] \prod_f [\det \Delta_f(U)] \text{tr} [S^u(x, 0) \gamma_5 S^d(0, x) \gamma_5] e^{-S_G(U)}}{\int d[U] \prod_f [\det \Delta_f(U)] e^{-S_G(U)}} \quad (3.2)$$

$S_G(U)$ is the gluon action; Δ_f has been defined in Eq.(2.1). $S^{u,d}(x, 0)$ is the up, down quark propagator between 0 and x:

$$\sum_z \Delta_{u,d}(x, z) S_{u,d}(z, 0) = \delta_{x,0} \quad (3.3)$$

The right-hand side of Eq.(3.2) is represented by the diagram in Fig.4. In a Monte Carlo simulation the integral over the gluon fields U_μ is replaced by the sum over gluon field configurations generated by some numerical algorithm (Metropolis, Langevin, ...^{9,10}) on a finite, generally hypercubic lattice. $\Delta(x, 0)$ is inverted [see Eq.(3.3)] by some numerical technique, such as the Gauss-Seidel method, for example. All the numerical results discussed in the following have been obtained in the so-called "quenched" approximation*.

* For further information on the validity of this approximation in which one neglects the effects of the fermion determinant, see, for example [10]. I will discuss the effects of the quenching in the section on the nucleon σ -term.

FIGURE 4

Typical diagram for the pion propagator.

We may fix the momentum by summing $G(\mathbf{x},t)$ over the space components:

$$G(\mathbf{q},t) = \sum_{\mathbf{x}} G(\mathbf{x},t) e^{i \mathbf{q} \cdot \mathbf{x}} \quad (3.4)$$

$G(\mathbf{q},t)$ will propagate all possible intermediate states of momentum $\mathbf{q}$ carrying the same quantum numbers of the pion.

At zero momentum, considering only one-particle intermediate states, one has:

$$G(t) = \sum_{\mathbf{x}} G(\mathbf{x},t) = \sum_n \frac{Z_n}{2m_n} e^{-m_n t} \quad (3.5)$$

$m = m_\pi, m_\pi^*, \dots$ For large time distances the correlation function is dominated by the pole corresponding to the lowest-lying state, the pion, since heavier states are exponentially suppressed as $\exp[-(m_n - m_\pi)t]$:

$$G(t) \xrightarrow{t \rightarrow \infty} \frac{Z_\pi}{2m_\pi} e^{-m_\pi t} \quad (3.6)$$

On the other side, if one uses eq.(3.6) when t is too small, one has a systematic overestimate of the mass of the lowest-lying state. The logarithm of $G(t)$, as we observe it in a Monte Carlo simulation, is plotted in Fig.5 as a function of t . At large t , $\ln[G(t)]$ goes like a straight line because only one particle is propagating [Eq.(3.6)]. The slope of the straight line corresponds to the mass of the particle in lattice units ($m_\pi a$ in this case), and the intercept in $t=0$ allows the determination of Z_π .

Z_π is connected to the matrix element of the operator which we have used as interpolating field:

$$Z_\pi = | \langle 0 | \bar{u}^A \gamma_5 d^A | \pi^+ \rangle |^2 \quad (3.7)$$

A check of consistency of the results can be obtained by using some other operator with the same quantum numbers: different operators should lead to the same value of m_π . A possible alternative for the pion at zero momentum, is the fourth component of the axial current

$$A_0^+(x) = \bar{u}A(x) \gamma^0 \gamma^5 dA(x) \quad (3.8)$$

We may use the operators in Eqs.(3.1) and (3.8) to compute the pseudoscalar meson decay constants, which are of particular interest in the physics of D and B mesons.

FIGURE 5
 $\ln[G(t)]$ plotted versus t .

In fact, let us consider the following correlation functions:

$$\sum_{\mathbf{x}} \langle 0 | A_0(\mathbf{x}, t) \pi^+(0, 0) | 0 \rangle \xrightarrow{t \rightarrow \infty} \frac{\langle 0 | A_0 | \pi \rangle \langle \pi | \pi^+ | 0 \rangle}{2m_\pi} e^{-m_\pi t} \quad (3.9)$$

$$\sum_{\mathbf{x}} \langle 0 | \pi(\mathbf{x}, t) \pi^+(0, 0) | 0 \rangle \xrightarrow{t \rightarrow \infty} \frac{\langle 0 | \pi | \pi \rangle \langle \pi | \pi^+ | 0 \rangle}{2m_\pi} e^{-m_\pi t} \quad (3.10)$$

$$\frac{\langle A_0 \pi^+ \rangle}{\langle \pi \pi^+ \rangle} \xrightarrow{t \rightarrow \infty} \text{const} = \frac{f_\pi m_\pi}{Z_A \langle 0 | \pi | \pi \rangle} \quad (3.11)$$

By fitting the propagator in Eq.(3.6), using Eq.(3.11) and taking Z_A computed on the lattice with the Ward identities (see the previous section), we can thus obtain $f_\pi(f_k, \dots)$.

In Table 1 the value of the pseudoscalar meson decay constants obtained on the lattice^{11,12,13} are compared with predictions from QCD sum rules^{14,15} and (when known) with their experimental

values.

The systematic error quoted for f_π in the Table originates from the calibration of the lattice spacing with different physical masses (e.g., the ρ or proton mass). A detailed discussion can be found in Ref.11. This error is absent if we compute the other pseudoscalar meson decay constants f_{PS} from the ratio f_{PS}/f_π . This method is equivalent to fixing the lattice spacing using f_π itself. The further advantage is that the poorly-known renormalization constant of the axial current^{2,3}, Z_A , cancels in the ratio. It is interesting to notice the good agreement of f_K with its experimental value. $f_{D,DS}$ agrees well with the results from QCD sum rules. The B-meson decay constant in the Table has been obtained from f_D by extrapolating with the non-relativistic formula:

$$f\sqrt{M} = \text{const.} \quad (3.12)$$

Meson	f_{lattice}	$f_{\text{QCD-sum rules}}$	$f_{\text{experiment}}$
π	[140±20±20(syst)] [11] MeV		131 MeV
K	(158±13) MeV [11] (173±83) MeV [12]		164 MeV
D	(180±30) MeV [11] (215±60) MeV [12] (136±35) MeV [13]	(172±15) MeV [14]	< 290 MeV
D_S	(218±30) MeV [11] ~ 251 MeV [12] (220±33) MeV [13]	~ 220 MeV [14]	-
B_d	~ 120 MeV* [11] (90±23) MeV* [13]	(187±24) MeV [14]	-
B_s	~ 150 MeV* [11] (160±30) MeV* [13]		-

TABLE 1

Pseudoscalar decay constants from lattice calculations (second column), QCD sum rules (third column) and their experimental value (fourth column). The results denoted by a * have been obtained from f_{D,D_S} using Eq.(3.12).

where M is the heavy meson mass. Equation (3.12) is expected to work well (up to logarithmic corrections) for really heavy quarks but it is not clear that it can already be used for the charm quark. The value of f_B in the Table is sensibly lower than the result from QCD sum rules.

3.2 Three point functions

We now consider the more complicated case of three-point functions involving two sources of pseudoscalar particles (e.g. $\bar{\psi}\gamma_5\psi$) and a local operator whose matrix elements we want to compute. Three-point correlations are needed for the calculation of the pion and proton structure function and form factors, of the kaon B-parameter, i.e. the matrix element $\langle K^0 | (\bar{s}_L \gamma^\mu d_L) | \bar{K}^0 \rangle$, and of the K- π matrix elements of the weak Hamiltonian.

They have the general form:

$$C(t_1, t_2) = \sum_{\mathbf{x}_1, \mathbf{x}_2} \langle P_5(\mathbf{x}_1, t_1) O(0,0) P_5(\mathbf{x}_2, t_2) \rangle e^{i \mathbf{q}_1 \cdot \mathbf{x}_1 + \mathbf{q}_2 \cdot \mathbf{x}_2} \quad (3.13)$$

where $P_5 = \bar{\psi} \gamma_5 \psi$ and $O \sim (\bar{\psi}\Gamma\psi)$ ($\bar{\psi}\Gamma\psi$) or $\bar{\psi}\Gamma\psi$, Γ being one of the Dirac matrices.

For $\Delta S = 2$ ($\Delta T = 3/2$) transitions only the "eight" diagram in Fig.6 contributes. As shown in figure all the quark lines stem from the origin and we can use directly the quark propagators $S_{\alpha\beta}^{AB}(x,0)$ (A,B = colour, α,β = spin) used for hadron spectroscopy. The calculation of the three-point correlation for $\Delta T = 1/2$ operators is considerably more difficult because of the presence of the so called "eye" graphs of Fig.7. The same problem is present for quark bilinear operators ($\sim(\bar{\psi}\Gamma\psi)$), which enters in the form factors and structure functions as shown in Fig.8.

FIGURE 6

FIGURE 7

FIGURE 8

The difficulty arises because the last two cases involve propagators from any point x_1 to any point x_2 on the lattice, besides the propagator from the origin to any point x (typically the number of points is of the order of $10^4 - 10^5$). The method usually adopted consists of computing a new set of quark propagators with one of the two pseudoscalar densities inserted at a fixed time (say t_2) and a fixed momentum (q_2)^{16,17}:

$$S_1(x,0) = \sum_{\substack{x_2 \\ \text{at } t_2 \text{ fixed}}} S(x, x_2) \gamma_5 S(x_2, 0) e^{i q_2 \cdot x_2} \quad (3.14)$$

$S_1(x,0)$ satisfies an equation similar to eq.(3.3):

$$\sum_z \Delta(x,z) S_1(z,0) = \gamma_5 S(x,0) e^{i q \cdot x} \delta_{x,t_2} \quad (3.15)$$

The computation of S_1 takes about the same computer time as that of $S(x,0)$. When t_2 is large enough, the t_2 dependence is given by the meson propagator and we do not need the correlations at all t_2 's. We cannot change either q_2 but we can freely vary q_1 and t_1 . In terms of S_1 the diagrams in Fig.7 and Fig.8 reduce to two-point correlation functions as shown in Figs.9 and 10.

All the propagators again stem from the origin where the operator is located. For large time distances ($t_1 = t_K \rightarrow -\infty$; $t_2 = t_\pi \rightarrow +\infty$) the correlation in eq.(3.13) is dominated by the matrix element of the operator sandwiched between the lightest pseudoscalar mesons:

$$G(t_K, t_\pi) \rightarrow \frac{Z_K^{1/2} Z_\pi^{1/2}}{(2E_K)(2E_\pi)} \langle \pi(p) | O | K(k) \rangle \exp(-E_K |t_K| - E_\pi t_\pi) \quad (3.16)$$

In eq.(3.16) we have chosen $q_1 = k$ and $q_2 = p$ and $Z_{K,\pi}$ and the meson energies $E_{K,\pi}$ can be found by studying the two-point correlation function in eq.(3.6). Eq.(3.16) is then used for the determination of $\langle \pi | O | K \rangle$.

To compute $K \rightarrow 2\pi$ matrix elements we consider the four-point correlation function:

$$G(t_1, t_2, t_3) = \sum_{x_1 x_2 x_3} \langle P_5(x_1, t_1) O(0,0) P_5(x_2, t_2) P_5(x_3, t_3) \rangle \quad (3.17)$$

For simplicity we have put to zero all spatial momenta. All the formulae will refer to $m_K^2 = m_\pi^2 = m^2$ and $Z_{K,\pi} = Z_5$ in the following. The calculation of the four point correlation in eq.(3.17) requires the use of at least one S_1 (for "eight" diagrams) or two S_1 's (for "eye" diagrams) as explained in the two examples reported in Fig.11 and Fig.12 but it does not present any new problem.

At large time distances ($t_1 = t_K \rightarrow -\infty$; $t_{2,3} = t_{\pi 1, \pi 2} \rightarrow +\infty$ and $t_{\pi 1} \neq t_{\pi 2}$) one has:

$$G(t_K, t_{\pi 1}, t_{\pi 2}) = \frac{Z_5^{3/2}}{(2m)^3} \langle \pi_1 \pi_2 | O | K \rangle \exp[-m(|t_K| + t_{\pi 1} + t_{\pi 2})]$$

FIGURE 9

FIGURE 10

FIGURE 11

FIGURE 12

Thus from the knowledge of Z_5 and m one is able to compute the $K \rightarrow 2\pi$ matrix elements.

3.3 Nucleon Correlation Functions and Operator Matrix Elements

The nucleon case is slightly more complicated because of the spinor structure. In this case, the matrix elements are obtained by computing the Euclidean three-point correlation function

$$C(t_x, t_y) = \int d^3x d^3y \langle 0 | T [J_\gamma(0) O(y) \bar{J}_\gamma(x)] | 0 \rangle e^{i p \cdot x} e^{i q \cdot y} \quad (3.18)$$

where J is an interpolating operator for the nucleon (which, for definiteness, for the rest of this section we assume to be a proton), γ and γ' are spinor labels, t_x and t_y are the time components of x and y and we integrate only over the space coordinates. Inserting a complete set of states between each pair of operators in eq.(3.18) and taking $0 < t_y < t_x$, with t_y and $t_x - t_y$ sufficiently large so that only the lightest state contributes significantly, we have

$$C(t_x, t_y) = - \frac{e^{-E' t_y} e^{-E(t_x - t_y)}}{4EE'} \sum_{s, s'} \langle 0 | J_\gamma(0) | p, s \rangle \langle p, s | O(0) | p', s' \rangle \langle p', s' | \bar{J}_\gamma(0) | 0 \rangle \quad (3.19)$$

where $|p, s\rangle$ represents a proton states with four-momentum p and a third component of spin s . p and p' are given by

$$p = (E, \mathbf{p}) \quad E = \sqrt{\mathbf{p}^2 + m^2}$$

and

$$p' = (E', \mathbf{p} + \mathbf{q}) \quad E' = \sqrt{(\mathbf{p} + \mathbf{q})^2 + m^2} \quad (3.20)$$

where m is the mass of the proton. The required matrix element $\langle p, s | O(0) | p', s' \rangle$ is one of the factors on the right-hand side of eq.(3.19), while the remaining factors can be obtained by computing two-point correlation functions, as will be explained below.

For the proton we may use the following interpolating operator:

$$J_\gamma = \epsilon^{ijk} (u^i C \gamma^5 d^j) u_\gamma^k \quad (3.21)$$

where i, j, k are colour labels and C is the charge conjugation matrix. This isospin- $\frac{1}{2}$ interpolating operator has been shown in several lattice studies of hadron spectroscopy to have a significant overlap with the proton. Using Lorentz- and parity invariance, we can write

$$\langle 0 | J_\alpha(0) | p, s \rangle = \sqrt{Z_p} N_\alpha^{(s)}(p) \quad (3.22)$$

where Z_p is a scalar quantity and $N_\alpha^{(s)}(p)$ is the proton's spinor, which satisfies the Dirac equation. Z_p and m can be obtained from the two-point Euclidean correlation function

$$K(t_x) = \int d^3x \langle 0 | T [J_\beta(0) \bar{J}_\alpha(x)] | 0 \rangle e^{i p \cdot x} \quad (3.23)$$

For $p = 0$, inserting a complete set of states between the interpolating operators J and taking t_x to be sufficiently large so that the single proton state is the dominant one, we have:

$$K(t_x) = Z_p \frac{(1+\gamma^0)}{2} \beta_\alpha e^{-m t_x} \quad (3.24)$$

We have neglected in (3.24) the contributions from heavier states, in particular that from the opposite parity baryon, $N^*(\frac{1}{2}^-)$, which is equal to

$$K(t_x) = Z_{N^*} \frac{(1-\gamma^0)}{2} \beta_\alpha e^{-m' t_x}$$

One can further suppress its contribution by an appropriate choice of spinor indices (e.g., $\alpha = \beta = 1$ or $\alpha = \beta = 2$). Thus from the behaviour of $K(t_x)$ with t_x , both m and Z_p can be determined and from Eq.(3.19) we can extract the matrix elements $\langle p, s | O | p', s' \rangle$.

The evaluation of $K(t_x)$ and $C(t_x, t_y)$ requires the computation of the diagrams in Fig.13 and Fig.14 respectively.

FIGURE 13

FIGURE 14

$$\Sigma_u(y, 0; t_x, p) = \sum_{\underline{x}} e^{ip \cdot \underline{x}}$$

$$\Sigma_d(y, 0; t_x, p) = \sum_{\underline{x}} e^{ip \cdot \underline{x}}$$

FIGURE 15

In these figures, the propagators are quark propagators in a gluon background field. The crosses denote the insertion of the interpolating operator (x) and the shaded circles that of the operator whose matrix element we wish to compute*.

For the diagram in Fig.13 it is sufficient to have the set of quark propagators from an arbitrary lattice point to the origin $S(x,0)$. For simplicity we keep the up and down quark masses equal and hence the propagators $S(x,0)$ are identical for both flavours. In order to compute the diagrams in Fig.14 we need the sets of generalized propagators Σ_d and Σ_u (in analogy with the pion case) defined by (see Fig.15)¹⁸

$$\begin{aligned} [\Sigma_d(0,y; t_x, p)]_{\rho\sigma}^{XY} = & -\epsilon^{ABC} \epsilon^{A'XC'} (C\gamma^5)_{\alpha\beta} (C\gamma^5)_{\alpha'\rho} \\ & \sum_x \left\{ e^{i p \cdot x} S_{\beta\sigma}^{BY}(x,y) [S_{\alpha\alpha'}^{AA'}(x,0) S_{\gamma\gamma'}^{CC'}(x,0) - S_{\alpha\gamma'}^{AC'}(x,0) S_{\gamma\alpha'}^{CA'}(x,0)] \right\} \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} [\Sigma_u(0,y; t_x, p)]_{\rho\sigma}^{XY} = & -\epsilon^{ABC} (C\gamma^5)_{\alpha\beta} \sum_x e^{i p \cdot x} S_{\beta\beta'}^{BB'}(x,0) \\ & \left\{ \epsilon^{XB'C'} (C\gamma^5)_{\rho\beta'} [S_{\alpha\sigma}^{AY}(x,y) S_{\gamma\gamma'}^{CC'}(x,0) - S_{\alpha\gamma'}^{AC'}(x,0) S_{\gamma\sigma}^{CY'}(x,y)] \right. \\ & \left. + \epsilon^{A'B'X} (C\gamma^5)_{\alpha\beta'} \delta_{\gamma\rho'} [S_{\alpha\alpha'}^{AA'}(x,0) S_{\gamma\sigma}^{CY}(x,y) - S_{\alpha\sigma}^{AY}(x,y) S_{\gamma\alpha'}^{CA'}(x,0)] \right\} \end{aligned} \quad (3.26)$$

The upper case Latin superscripts and the lower case Greek subscripts (both primed and unprimed) are colour and spinor labels respectively. The interpolating operators for the proton were chosen to be $\bar{J}_\gamma(0)$ and $J_{\gamma'}(x)$ [with J defined in Eq.(3.21)], so that the spinor indices γ and γ' are external labels. There is a different Σ_d and Σ_u for each pair (γ, γ') .

To illustrate the significance of the Σ_u 's and Σ_d 's, consider the matrix element $\langle p | \bar{\psi} \Gamma \psi | p \rangle$ where ψ can be an up or down quark and Γ is one of the 16 matrices of Dirac theory. The three-point correlation function eq.(3.18) corresponding to the operator $\bar{\psi} \Gamma \psi$, computed in the quenched approximation, is the weighted average over all configurations of

$$\sum_y \text{tr} [\Sigma_\psi(0,y; t_x, p) \Gamma S(y,0)] e^{i q \cdot y} \quad (3.27)$$

where $\psi = u$ or d . Thus the set $\{\Sigma_\psi(0,y; t_x, p), \psi = u \text{ or } d\}$, together with the set of quark propagators $S(y,0)$, is sufficient to compute the matrix elements of a large class of local operators

* I do not consider here the diagrams relevant for hyperon decays but only those which are relevant for form factors and structure functions.

(including operators containing covariant derivatives).

The Σ_d 's and Σ_u 's satisfy equations which can be solved using the same numerical techniques as those used to generate the usual quark propagators $S(x,0)$. For example, Σ_d satisfies

$$\sum_y \Sigma_d(0,y; t_x, p)_{\rho\sigma}^{XY} \Delta_{\alpha\tau}^{YZ}(y,z) = -e^{i p \cdot z} \in^{AZC} \in^{A'XC'} (C\gamma_5)_{\alpha\tau} (C\gamma_5)_{\alpha'\rho} \delta_{t_x, t_z} \cdot$$

$$[S_{\alpha\alpha'}^{AA'}(z,0) S_{\gamma\gamma'}^{CC'}(z,0) - S_{\alpha\gamma}^{AC'}(z,0) S_{\gamma\alpha'}^{CA'}(z,0)] \quad (3.28)$$

where Δ is as before the kernel of the fermionic action.

Each set $\{\Sigma_u(0,y; t_x, p)\}$ or $\{\Sigma_d(0,y; t_x, p)\}$ depends on t_x , on p and on the choice of spinor indices, γ and γ' , in the interpolating operators $J_\gamma, \bar{J}_{\gamma'}$, (this dependence on γ and γ' is implicit throughout the above discussion). Therefore, we have to choose each of these parameters in a way which will optimized the amount of information obtainable.

In particular the choice of the spinor indices γ and γ' depends on the physics one wishes to study. To sum over the two cases $\gamma = \gamma' = 1$ and $\gamma = \gamma' = 2$, enables us to compute the low moments of the unpolarized deep inelastic structure functions, the electric form factor of the proton at several values of momentum transfer and the sigma term. In order to compute other interesting quantities, such as the matrix elements of the axial vector current or pseudoscalar density, the magnetic form factor of the nucleon or the recently-measured lowest moment of the polarized structure function g_1 ¹⁹, other choices of γ and γ' must be made ($\gamma = \gamma' = 1$ minus $\gamma = \gamma' = 2$).

4. THE PION ELECTROMAGNETIC FORM FACTOR AND STRUCTURE FUNCTION

Structure functions have played an important role in our understanding of the parton model and of quantum chromodynamics. They are measure in deep inelastic experiments by probing the hadron with a photon (W or Z) as shown in Figs.16 and 17. In the parton model, the photon interacts directly with the quarks inside the hadron and measures the probability $q(x, Q^2)$ of finding a parton with a fraction x of the hadron momentum and a momentum transfer $-q^2 = Q^2$ (Fig.16). Scaling violations allow us to measure the gluon distribution $G(x, Q^2)$ as shown in Fig.17.

In principle it is possible to compute directly the structure functions on the lattice through the diagram in Fig.18. This, however, requires that the distance between the two currents satisfies the condition:

$$l/m_h \gg z \gg a \quad (4.1)$$

in order to be in the inelastic region ($l/m_h \gg z$) and to avoid lattice artefacts (present when $z/a \sim 1$).

To satisfy Eq.(4.1), one needs an inverse lattice spacing of several GeV, and hence a very big lattice in order to contain the hadronic system under study. Such a calculation is not practicable at present. However, using the operator product expansion and the renormalization group one can express the T-product of two currents as a combination of local operator (Fig.19). The problem is thus reduced to the computation of matrix elements of local operators between hadronic states.

FIGURE 16

The structure function $F(x, Q^2)$ is proportional to the probability of finding a quark with a fraction x of the hadron momentum at $-q^2 = Q^2$, $q(x, Q^2)$.

FIGURE 17

Scaling violations allow us to measure the gluon density $G(x, Q^2)$.

FIGURE 18

Forward Compton scattering which allow us to measure the structure functions.

FIGURE 19

Operator product expansion of the diagram in Fig.18.

In the non-singlet case (valence quarks only) and keeping only the contribution of lowest twist operators, the moments of the structure function can be written as:

$$\int_0^1 dx x^{N-2} F(x, Q^2) = \int_0^1 dx x^{N-1} q(x, Q^2) = C_N(Q^2/\mu^2, g(\mu)) A_N(\mu) \quad (4.2)$$

where $A_N(\mu)$ are obtained from the matrix elements of the operators:

$$O_N^{\mu_1 \dots \mu_N} = i^N \bar{\psi} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} \psi - \text{traces} \quad (4.3)$$

(with suitable flavour indices and symmetrized over Lorentz indices) using:

$$\langle h(p) | O_N^{\mu_1 \dots \mu_N} | h(p) \rangle = p^{\mu_1} p^{\mu_2} \dots p^{\mu_N} A_N \quad (4.4)$$

In Eq.(4.2), μ is the renormalization scale of the operator. Since the left-hand side is independent of μ , the relation in Eq.(4.2) gives the renormalization group equation to which the coefficients C_N must obey. Provided $\mu \gg \Lambda_{\text{QCD}}$, the coefficient functions can be computed in perturbation theory. The non-perturbative physics is contained in the operator matrix elements [Eq.(4.4)]. We can compute them on the lattice.

4.1. Renormalization of the Relevant Operators

Following the discussion of Sect.2 we now discuss the renormalization properties of the operators which are relevant for the calculation of the electromagnetic form factor and for the structure functions.

To determine the electromagnetic form factor of the pion, we have to compute the matrix elements of the vector current. We can consider for example the local vector current:

$$V_\mu^{\text{loc}}(x) = \bar{q}(x) \gamma_\mu q(x) \quad (4.5)$$

As discussed in Section 2, the matrix elements of the conserved current $\hat{V}_\mu$ and V_μ^{loc} between on-shell states are related through the equation

$$\langle \alpha | \hat{V}_\mu | \beta \rangle = Z_V^{\text{loc}}(g_0) \langle \alpha | V_\mu^{\text{loc}} | \beta \rangle \quad (4.6)$$

The renormalization constant Z_V^{loc} is a finite function of the bare lattice coupling g_0 and can be computed in perturbation theory. On the lattice we can measure these constants non-perturbatively by comparing the matrix elements of V_μ^{loc} with those of $\hat{V}_\mu$.

We can compute for example Z_V^{loc} from the proton matrix elements of the vector currents:

$$Z_V^{\text{loc}} = \frac{\langle p | \hat{V}_\mu(0) | p \rangle}{\langle p | V_\mu^{\text{loc}}(0) | p \rangle} \quad (4.7)$$

In ref.18 it was found

$$Z_V^{\text{loc}} = 0.74 \pm 0.02 \quad (4.8)$$

For sufficiently small lattice spacing Z_V^{loc} is independent of the states between which the conserved and local vector currents are sandwiched and can be calculated in perturbation theory. The result reported in (4.8) and is similar to those obtained with the currents sandwiched between pion states, $Z_V^{\text{loc}} \sim 0.71^8$. They differ, however, from the one-loop perturbative result $(0.83)^{20,21,22}$ and from the value obtained with the currents sandwiched between the rho meson and the vacuum $(0.57)^5$. These discrepancies are certainly due to the presence of terms of order a (a = lattice spacing) which

arise in the renormalization of the local current and provide an indication of some of the systematic uncertainties these calculations at finite lattice spacing.

In order to calculate $\langle x \rangle$, the average momentum carried by the valence quarks in the pion, we need to compute the matrix element

$$\langle \pi(p) | O_2^{\mu\nu}(0) | \pi(p) \rangle = p^\mu p^\nu A_2(p) \quad (4.9)$$

where $|\pi(p)\rangle$ is a pion state with momentum p and

$$O_2^{\mu\nu}(0) = i^2 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \psi - \text{traces} \quad (4.10)$$

(with suitable flavour indices and symmetrized over the Lorentz indices). I now discuss the two cases $\mu = \nu$ and $\mu \neq \nu$ in turn.

The three linearly-independent operators with $\mu = \nu$ transform like one of the three-dimensional irreducible representation of the hypercubic group (where the character table for this group can be found in ref.23). The subtraction of the trace removes the singlet component (which is proportional to $(\bar{\psi}\psi/a)$) and hence eliminates the power divergence. Such subtractions have, in previous cases, led to significant numerical cancellations and hence to large statistical errors⁷.

The subtraction of power divergences can be avoided by choosing μ to be different from ν . The six operators $O_2^{\mu\nu}$ with $\mu \neq \nu$ transform like the six-dimensional irreducible representation of the hypercubic group, and for this reason there is no mixing with lower-dimensional operators. In addition, there is no numerical subtraction of divergent terms in the calculation itself, and hence the choice $\mu \neq \nu$ is likely to give results with a smaller statistical error. The price to be paid for choosing to have $\mu \neq \nu$ is that the pion must be given a non-zero component of momentum in at least one spatial direction, otherwise the matrix element in eq.(4.9) vanishes. Finally, as was also the case for the vector current, a finite multiplicative renormalization is needed to relate the lattice operator to the corresponding continuum operator defined in a given renormalization scheme. For the $\overline{MS}$ scheme, this renormalization is given by

$$\langle \pi(p) | O_2^{\mu\nu}(0) | \pi(p) \rangle_{\overline{MS}} = \langle \pi(p) | O_2^{\mu\nu} | \pi(p) \rangle_{\text{latt}} (1 + \alpha_S C_F \delta/4\pi) \quad (4.11)$$

where $\delta = -2.5$ for $\mu = \nu$ and $\delta = -1.2$ for $\mu \neq \nu$ ²⁴. At $\beta \sim 6.0$ ($g_0=1$) the perturbative corrections are negligible and the matrix elements in the two schemes can be taken to be equal.

The average fraction of momentum carried by gluons in a pion can be evaluated from the matrix element

$$\langle \pi(p) | F^{\mu\rho}(0) F^{\nu\rho}(0) - \text{traces} | \pi(p) \rangle \quad (4.12)$$

where $F_{\mu\nu}$ is the field-strength tensor. To avoid subtractions of power divergences, we may take $\mu \neq \nu$.

We define the lattice field-strength tensor to be (see also ref.23)

$$F^{\mu\nu}(x) = \frac{1}{4a^2} \sum_{4 \text{ plaquettes}} \frac{1}{2i} (U - U^\dagger) \quad (4.13)$$

where the sum extends over the four plaquettes in the $\mu\nu$ plane which have the point x at one corner. The matrices U are the products of the four link variables for each plaquette, taken in an anticlockwise sense. With this definition of $F^{\mu\nu}$, no power divergences are encountered when evaluating the matrix element of eq.(4.12). This is no longer true if other definitions of $F^{\mu\nu}$ are used. The essential point is that $F^{\mu\nu}$, as defined in eq.(4.13), transforms like the direct sum of two three-dimensional irreducible representations of the hypercubic group ($E + B$ and $E - B$, each of which transform irreducibly). If one does not symmetrize over the four plaquettes, then this is not the case. Suppose, for example, that we define $F^{\mu\nu}$ using just one of the plaquettes (this is an equally good definition at the tree level). At the quantum level, the matrix element in eq.(4.12) will diverge as inverse powers of the lattice spacing. In fact, $F^{\mu\nu}$ defined by means of a single plaquette transforms like a 24-dimensional reducible representation of the hypercubic group and $\{F^{\mu\rho} F^{\nu\rho}, \mu \neq \nu\}$ like a 192-dimensional reducible representation. This 192-dimensional representation contains the singlet, so that $F^{\mu\rho} F^{\nu\rho}$ can mix with the identity, the mixing coefficients containing power divergences. By using the definition (4.13) we automatically avoid this problem.

The evaluation of the second moment of the momentum distribution of the valence partons requires the calculation of the matrix element

$$\langle \pi(p) | O_3^{\mu\nu\rho}(0) | \pi(p) \rangle = p^\mu p^\nu p^\rho A_3(p) \quad (4.14)$$

where

$$O_3^{\mu\nu\rho}(0) = i^3 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \overleftrightarrow{D}^\rho \psi - \text{traces} \quad (4.15)$$

(with suitable flavour indices and symmetrized over the Lorentz indices).

The transformation properties of $O_3^{\mu\nu\rho}$ under the hypercubic group have been discussed in detail in refs.[7]. Taking $\mu = \nu = \rho$, we find that $O_3^{\mu\mu\mu}$ transforms like the four-dimensional vector representation $((\frac{1}{2}, \frac{1}{2}))$ in the notation of ref.23), and hence can mix with lower dimensional operators such as $\bar{\psi} \gamma^\mu \psi$. One can therefore not study the matrix elements of $O_3^{\mu\mu\mu}$, but rather of $O^{411} - O^{433}$ (which transforms like the eight-dimensional irreducible representation). To have a non-vanishing matrix element we shall give the pion a component of momentum in the 1 direction. The subtraction in $O^{411} - O^{433}$ involves a subtraction of power divergences, and hence we expect

our results for $\langle x^2 \rangle$ to have larger statistical errors than for $\langle x \rangle$.

The operator O^{412} , which transforms like the $(\frac{1}{2}, \frac{1}{2})$ four-dimensional irreducible representation (in the notation of ref.23) is also free of power divergences. In this case, however, we must give the pion non-zero components of momentum in the 1 and 2 directions, increasing the systematic errors which grow as ap_l increases [the granularity of the lattice becomes apparent once ap_l is $O(1)$; see also below]. Also in this case a renormalization factor is needed to relate the lattice to the continuum result:

$$\langle \pi(p) | O_3^{\mu\nu\rho}(0) | \pi(p) \rangle_{\overline{MS}} = \langle \pi(p) | O_3^{\mu\nu\rho} | \pi(p) \rangle_{\text{latt}} (1 + \alpha_S C_F \delta'/4\pi) \quad (4.16)$$

For the quantities which are obtained from matrix elements of the form $\langle \pi | O | \pi \rangle$ (e.g., $\langle x \rangle$, $\langle x^2 \rangle$ and the electromagnetic form factor), the optimal use of computing resources requires us to have one of the two pion states at the same momentum in all cases. In view of the above discussion, for one of the two pion states we may choose $p_1 \neq 0$ and $p_2 = p_3 = 0$. This means that for the calculation of $\langle x^2 \rangle$ we have to use the operator $O_3^{411} - O_3^{433}$, while all the other matrix elements do not require any subtractions.

4.2 The electromagnetic form factor of the pion

For the computation of the electromagnetic form factor of the pion we start with the three-point correlation function $C(t_x, t_y)$ of eq.(3.13) generalized to nonzero momentum transfers

$$C(t_x, t_y) = \sum_{x,y} e^{ip \cdot x} e^{iq \cdot y} \langle 0 | T [J_\pi^+(0) V_\mu(y) J_\pi(x)] | 0 \rangle \quad (4.17)$$

where V_μ is the electromagnetic current (e.g., the conserved or local current defined in Sect.2) and J_π is the pion source (e.g. P_5). For sufficiently large values of t_y and $t_x - t_y$, eq.(4.17) reduces to

$$C(t_x, t_y) = \frac{e^{-E(t_x-t_y)} e^{-E't_y}}{4EE'} |\langle \pi | J_\pi(0) | 0 \rangle|^2 \cdot Z_V^{-1} \cdot (p+p')_\mu F(q^2) \quad (4.18)$$

where $E = \sqrt{m_\pi^2 + p^2}$, $E' = \sqrt{m_\pi^2 + (p+q)^2}$ and p' is the four-vector $(E', p+q)$. $F(q^2)$ is the electromagnetic form factor and Z_V is the renormalization constant for the electromagnetic current being used.

The results for $F(q^2)$ with $q^2 \neq 0$, using the local vector current $\bar{\psi} \gamma^\mu \psi$ with $\mu = 1$ or 4 are presented in the following.

In Fig.20 the form pion factor calculated is on 15 gauge field configurations on a $20 \times 10^2 \times 40$ lattice at $\beta = 6/g_0^2 = 6$ ($a^{-1} \sim 2$ GeV) is reported for a pion with a mass of ~ 1 GeV ($m_\pi a \sim 0.52$). For reference, the expression (suggested by the vector dominance model) $1/(1+(-q^2)/m_\rho^2)$, where m_ρ is the mass of the vector meson at this value of the pion mass, is also plotted. The results are

encouraging, the form factor falls with q^2 and is "measured" reasonably accurately for at least four of the five momentum transfers.

FIGURE 20

The pion's form factor at $m_\pi a=0.52$ at $\beta=6$ on a $20 \times 10^2 \times 40$ lattice. The curve is the result suggested by the vector dominance model $1/(1+(-q^2)/m_\rho^2)$. The circles and triangles signify that the Lorentz index was chosen to be 1 or 4 respectively. Few points, obtained by using the conserved vector current $\hat{V}_\mu$, are given in the figures as squares.

Of course we are really interested in the form factor of a real pion, and not the one which has a mass of about 1 GeV. We must, therefore, extrapolate to the chiral limit.

In Fig.21 the results, extrapolated to the physical pion mass, are plotted versus $\sqrt{-q^2}$ using $a^{-1} = 2.2$ GeV. We also plot the results from the NA7²⁵ and Cornell²⁶ experiments*. The results (in particular for the two most reliable points $p'=0$ and $p'=-p$) are quite satisfactory.

FIGURE 21

4.3. The deep inelastic structure function of the pion

We define $\langle x \rangle_{\text{latt}}$ and $\langle x^2 \rangle_{\text{latt}}$ in terms of the three-point correlation functions (3.13) as follows

$$C^{(2)}(t_x, t_y) = \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{p} \cdot \mathbf{x}} \langle 0 | T [J_\pi^+(0, 0) O_2^{41}(\mathbf{y}, t_y) J_\pi(\mathbf{x}, t_x)] | 0 \rangle \rightarrow$$

$$| \langle \pi | J_\pi(0) | 0 \rangle |^2 \frac{P(\mathbf{x})_{\text{latt}}}{2E} e^{-Et_x} \quad (4.19)$$

* The Cornell experiments determine the pion's form factor indirectly, by studying the electroproduction processes $ep \rightarrow e\pi^+n$ and $en \rightarrow e\pi^-p$.

$$C^{(3)}(t_x, t_y) = \sum_{x,y} e^{ip \cdot x} \langle 0 | T [J_\pi^+(0,0) \tilde{O}_3(y, t_y) J_\pi(x, t_x)] | 0 \rangle \rightarrow$$

$$| \langle \pi | J_\pi(0) | 0 \rangle |^2 \frac{p^2 \langle x^2 \rangle_{\text{latt}}}{E} e^{-Et_x} \quad (4.20)$$

where $\tilde{O}_3 = O_3^{411} - \frac{1}{2} (O_3^{422} + O_3^{433})$, $p \equiv (p, 0, 0)$ and $E = \sqrt{m_\pi^2 + p^2}$.

Let $C^{(2)}(t_x, t_y)$, $C^{(3)}(t_x, t_y)$ and $C^{(\hat{V})}(t_x, t_y)$ be the three-point correlation functions corresponding to the operators O_2^{41} , $\tilde{O}_3$ and the conserved vector current $\hat{V}^1$ (with Lorentz index $\mu = 1$), respectively. Then

$$\frac{C^{(2)}(t_x, t_y)}{C^{(\hat{V})}(t_x, t_y)} = 2 E \langle x \rangle_{\text{latt}}, \quad t_x - t_y \text{ large, } t_y \text{ large}, \quad (4.21)$$

$$\frac{C^{(3)}(t_x, t_y)}{C^{(\hat{V})}(t_x, t_y)} = 4pE \langle x^2 \rangle_{\text{latt}}, \quad t_x - t_y \text{ large, } t_y \text{ large}, \quad (4.22)$$

Using (4.21) and (4.22) at $\beta = 6$, on a $20 \times 10^2 \times 40$ lattice it was found⁸:

$$\begin{aligned} \langle x \rangle_{\text{latt}} &= 0.60 \pm 0.08 \\ \langle x^2 \rangle_{\text{latt}} &= 0.23 \pm 0.06 \end{aligned} \quad (4.23)$$

Most of the error in eqs.(4.23) is due to the linear extrapolation to the chiral limit. Using perturbation theory to rewrite these results in continuum schemes for which the coefficient function is close to 1 (such as the $\overline{\text{MS}}$ scheme), and at a renormalization scale $\mu \sim 7 \text{ GeV}$. From eqs.(4.23) one finds:

$$\begin{aligned} \langle x \rangle_{\text{cont}} &= 0.46 \pm 0.07 \\ \langle x^2 \rangle_{\text{cont}} &= 0.18 \pm 0.05 \end{aligned} \quad (4.24)$$

The above result for $\langle x \rangle$ well agrees with the experimental results:

$$\begin{aligned} \langle x \rangle_{\text{exp}} &= 0.4 \pm 0.1, & \text{ref. [27],} \\ \langle x \rangle_{\text{exp}} &= 0.35 \pm 0.05, & \text{ref. [28],} \\ \langle x \rangle_{\text{exp}} &= 0.4 - 0.5, & \text{ref. [29],} \\ \langle x \rangle_{\text{exp}} &= \sim 0.34, & \text{ref. [30],} \end{aligned} \quad (4.25)$$

and $\langle x^2 \rangle_{\text{cont}}$ is well compatible with the experimental observation that the distribution function $V_\pi(x) \sim (1-x)^\alpha$, as $x \rightarrow 1$ with the exponent α being close to 1.

5. THE NUCLEON STRUCTURE FUNCTION AND ELECTROMAGNETIC FORM FACTOR

5.1. Deep inelastic structure functions

We now come to the computation of the first two moments of the non-singlet structure function of the nucleons (for unpolarized scattering) by using O_2^{41} and $\tilde{O}_3 = O_3^{411} - \frac{1}{2}(O^{422} + O^{433})$ as for the pion case.

The insertion of the operators O_2 and $\tilde{O}_3$ in Eq.(3.18) gives

$$C_{SF}^{(2)}(t_x, t_y) = \sum_{\gamma=\gamma=1}^2 \sum_{x,y} \langle 0 | T [J_\gamma(x) O_2(y) \bar{J}_\gamma(0)] | 0 \rangle e^{i p \cdot x} = - \frac{2 \langle x \rangle_{\text{latt}} (E+m)}{E} p Z_p e^{-E t_x} \quad (5.1)$$

and

$$C_{SF}^{(3)}(t_x, t_y) = \sum_{\gamma=\gamma=1}^2 \sum_{x,y} \langle 0 | T [\bar{J}_\gamma(0) O_3(y) J_\gamma(x)] | 0 \rangle e^{i p \cdot x} = - \frac{4 \langle x^2 \rangle_{\text{latt}} (E+m)}{E} p^2 Z_p e^{-E t_x} \quad (5.2)$$

where $p \equiv (p, 0, 0)$ $E = \sqrt{m^2 + p^2}$ and Z_p is defined in Eq.(3.22). The subscript "latt" reminds us that the quantities are computed with the corresponding operators renormalized in the lattice renormalization scheme, and at a renormalization scale equal to the lattice spacing. Using perturbation theory, we are then able to reduce $\langle x \rangle$ and $\langle x^2 \rangle$ in standard continuum renormalization schemes (such as the $\overline{MS}$ scheme).

By using (5.1) and (5.2) one can determine the first two moments of the protons' structure function. In ref.18 they found for the two up quarks:

$$\begin{aligned} \langle x \rangle_{\text{LATT}} &\sim 0.45 \\ \langle x^2 \rangle_{\text{LATT}} &\sim 0.13 \end{aligned} \quad (5.3)$$

and for the down quark:

$$\begin{aligned} \langle x \rangle_{\text{LATT}} &\sim 0.22 \\ \langle x^2 \rangle_{\text{LATT}} &\sim 0.08 \end{aligned} \quad (5.4)$$

The perturbative correction relating the operator $O_2^{\mu\nu}$ renormalized in the lattice renormalization scale to that renormalized in the $\overline{MS}$ scheme is negligibly small. Thus the values given in (5.3) and (5.4) can be taken as being $\langle x \rangle_u$ and $\langle x \rangle_d$, the average fraction of momentum carried by the up and down valence quarks, as defined from the operator $O_2^{\mu\nu}$ in the $\overline{MS}$ scheme at a renormalization scale of about 2 GeV. Within the errors, $\langle x \rangle_u$ is twice $\langle x \rangle_d$.

$$\frac{\langle x \rangle_u}{\langle x \rangle_d} = 2.0 \pm 0.25$$

The values reported above are larger than those measured experimentally,

$$\langle x \rangle_v = \langle x \rangle_u + \langle x \rangle_d \sim 0.4 \text{ at } \mu \sim 2 \text{ GeV} \quad (5.5)$$

Part of the discrepancy (but less than about 0.1 in $\langle x \rangle_v$) can be attributed to the fact that there are no sea quarks in the quenched approximation. The effect of the quenching alone seems however unable to explain the discrepancy of the lattice calculation and the experimental result. This is probably due to effects of order a , as it was the case of the vector current discussed before.

Using the tadpole dominance of lattice perturbation theory one may estimate that the values of $\langle x^2 \rangle$ in the $\overline{\text{MS}}$ scheme are about 13% smaller than those in the lattice renormalization scheme. Given the large statistical errors in our results for $\langle x^2 \rangle$, I do not comment on them further, other than to note that they are reasonable.

5.2. The Electric Form Factor of the Proton

For the electric proton form factor the correlation functions which we compute are of the form:

$$C_{FF}^\mu(t_x, t_y) = \sum_{\gamma=\gamma=1}^2 \sum_{\mathbf{x}, \mathbf{y}} \langle 0 | T [J_\gamma(\mathbf{x}) \hat{V}^\mu(\mathbf{y}) \bar{J}_\gamma(0)] | 0 \rangle e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{q} \cdot \mathbf{y}} \quad (5.6)$$

where $\hat{V}^\mu(\mathbf{y})$ is the electromagnetic conserved current. In general, the matrix element depends on the two form-factors F_1 and F_2 , defined by (in the notation of ref.31)

$$\langle p, s | \hat{V}^\mu(0) | p', s' \rangle = \bar{u}^{(s)}(p) [\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} k F_2(q^2)] u^{(s')}(p') \quad (5.7)$$

with the normalization chosen so that $F_2(0)=1$ and $F_1(0)=1(0)$ for the proton (neutron) and $q \equiv p' - p$. In actual calculations the non-relativistic approximation $E = m + p^2/2m$ holds to very good precision, one sums over $\gamma = \gamma = 1$ and 2 and the spatial momentum is always in the x -direction (ref.18). Then the correlation function is proportional to the electric form factor, defined by

$$G_E = F_1 + \frac{kq^2}{4m^2} F_2 \quad (5.8)$$

In fact it is easy to show that:

$$C_{FF}^4(t_x, t_y) = \frac{Z_p}{2EE'} e^{-E't_y} e^{-E(t_x - t_y)} 4m^2 G_E(q^2) \quad (5.9)$$

and

$$C_{FF}^1(t_x, t_y) = \frac{-i Z_0}{2EE'} e^{-E't_y - E'(t_x - t_y) 2m(p_x + p_x')} G_E(q^2) \quad (5.10)$$

The factor of i in (5.10) comes from the continuation to Euclidean space. G_E , evaluated at the five values of momentum transfer corresponding to the initial state proton using the conserved vector current is reported in Fig.(22) against q/m_p .

The solid curve is the prediction obtained using the vector meson dominance model

$$G_E(q^2) = \frac{1}{1 + \frac{(-q^2)}{m_\rho^2}} \quad (5.11)$$

The results have the correct trend of the experimental results³². The points lie a little below the vector dominance prediction, and if we replace m_ρ by λm_ρ in (5.11), then the best fit gives

$$\lambda = 0.85 \pm 0.05 \quad (5.12)$$

to be compared to $\lambda \sim 0.65$ which corresponds to the best fit to the results of Ref.32.

FIGURE 22

6. THE σ -TERM OF THE NUCLEON

The Hamiltonian part which breaks the SU(3) flavour symmetry can be written as:

$$H_{SB} = -\frac{(m_s - m)}{\sqrt{3}} (\bar{q} \lambda_8 q) \quad (6.1)$$

with $H_{TOT} = H_{SYM} + H_{SB}$.

H_{SYM} is the Hamiltonian which is symmetric in the s, u and d quarks and gives a common mass to all the hadrons belonging to the same multiplet, i.e. $\langle B | H_{SYM} | B \rangle = M$, $|B\rangle$ being the state of a given baryon at rest. m_s is the mass of the strange quark and $m = m_u = m_d$ is the common mass of the up and down quarks.

The baryon matrix elements of $(\bar{q} \lambda_8 q)$ can be parametrized, as usual, in terms of the F and D couplings:

$$\langle B' | \bar{q} \lambda_8 q | B \rangle = F \text{tr}(B'^+ [\lambda_8, B]) + D \text{tr}(B'^+ \{\lambda_8, B\}) \quad (6.2)$$

B is the matrix which represents the baryon multiplet:

$$B \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \Sigma^0 + \sqrt{\frac{1}{6}} \Lambda^0 & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}} \Sigma^0 + \sqrt{\frac{1}{6}} \Lambda^0 & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}} \Lambda^0 \end{pmatrix} \quad (6.3)$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From eq.(6.2) and (6.3) one finds:

$$M_{\Xi} - M_p = (m_s - m) (2F) = 0.379 \text{ GeV (exp)}$$

$$M_{\Xi} + M_p - 2M_{\Sigma} = (m_s - m) (2D) = -0.129 \text{ GeV (exp)} \quad (6.4)$$

$$-\frac{3}{2} (M_{\Sigma} - M_{\Lambda}) = (m_s - m) (2D) = -0.116 \text{ GeV (exp)}$$

and $(D/F) \sim -0.32$.

The above values are related the matrix elements of the scalar densities in the hadrons; in the case of the proton:

$$\langle P | \bar{u}u | P \rangle = A$$

$$\langle P | \bar{d}d | P \rangle = B \quad (6.5)$$

$$\langle P | \bar{s}s | P \rangle = C$$

It is easy to verify that:

$$A - C = 2F$$

$$A - 2B + C = 2D \quad (6.6)$$

We cannot determine A, B and C with only two relations. However, given the ratio D/F, if we assume that the contribution of the strange quarks is negligible, C=0, the above relations fix completely A and B. In particular we can compute the so-called σ -term:

$$\sigma = m \langle P | \bar{u}u + \bar{d}d | P \rangle =$$

$$\frac{3}{2} \left(\frac{m}{m_s - m} \right) \left[M_\Xi - M_p + \frac{1}{2} (M_\Sigma - M_\Lambda) \right] \quad (6.7)$$

Using the PCAC relation $\frac{m}{m_s - m} = \frac{M_\pi^2}{M_K^2 - M_\pi^2}$ and the experimental meson and baryon masses one then finds:

$$\sigma \sim 26 \text{ MeV} \quad (6.8)$$

On the other hand σ is also related to the $I=0$ π -Nucleon scattering amplitude through PCAC reduction of the initial and final pions in the soft limit. The value obtained from the scattering amplitude is³³

$$\sigma \sim 55 - 60 \text{ MeV} \quad (6.9)$$

about a factor of two larger than the value obtained by neglecting the strange quark contribution to the proton mass.

The matrix elements of the scalar density in the proton can be computed on the lattice using the same technique used for the structure function and the electromagnetic form factor³⁴.

One can also use the Ward identity:

$$m (A + B) = m \frac{\partial M_p}{\partial m} \quad M_p = \text{proton mass} \quad (6.10)$$

which is valid in the quenched case.

FIGURE 23

The sigma term, defined in eq.(6.7) and multiplied by M_N/M_π^2 (M_N being the nucleon mass) as a function of M_π/M_Q , as found in ref.34, black circles, and ref.35, white circles, by computing the scalar density matrix element. The black squares (ref.18) and white squares (ref.36) have been obtained by measuring the σ -term from eq.(6.10).

A priori we expect that the results from the lattice, in the quenched approximation, will be about the same as those obtained by assuming $C=0$, since we have imposed, by "fiat", that there are no strange quarks in the proton. A collection of results from several groups is reported in Fig.(23)^{18,34,35,36}. All the values have been obtained by the explicit calculation of matrix element of the scalar density or by measuring the dependence of the mass of the proton on the quark mass. Possibly with the exception of ref.35, all the groups find a value of σ which is even lower than the expected value, $\sigma = 26$ MeV.

This is a typical case where the "quenched" approximation does not work well and a full calculation is needed. It should also be noticed that there could be a relationship between the low value of σ found in the quenched case and the observation that, in the same approximation, the proton to ρ mass ratio on the lattice turns out to be systematically larger than its experimental value (see Fig.(24)). Without excluding other systematic effects (finite size, extrapolation to the chiral limit and so on), this is not surprising, given that σ is related to the slope of the proton mass versus the quark mass. This fact, together with the unsatisfactory results for the proton structure function, and the recent experimental measurement of the proton polarized structure function by EMC, seems to indicate that the quenched approximation is worse of what it was thought before, at least for baryon physics.

FIGURE 24

M_N/M_p as a function of M_π/M_q . The shaded area indicates the results of numerical simulations in the quenched approximation. The cross corresponds to the experimental value. For details see ref.10.

7. WEAK HAMILTONIAN MATRIX ELEMENTS

7.1 Renormalization of the $\Delta S=2$ and $\Delta I=\frac{1}{2}$ lattice operators

In the following the general theory of the renormalization of lattice quark composite operators discussed in Sect.2 is applied to the operators which are relevant for kaon systems, i.e., the $\Delta S=2$ operator:

$$O^{\Delta S=2} = (\bar{s}_L \gamma^\mu (1 - \gamma_5) d) (\bar{s} \gamma_\mu (1 - \gamma_5) d) \quad (7.1)$$

and the two (8,1) operators responsible for $\Delta T = \frac{1}{2}$ transitions:

$$O^- = [(\bar{s}_L \gamma^\mu d_L) (\bar{u}_L \gamma_\mu u_L) - (\bar{s}_L \gamma^\mu u_L) (\bar{u}_L \gamma_\mu d_L)] - [u \leftrightarrow c]$$

$$O^+ = \frac{1}{5} [(\bar{s}_L \gamma^\mu d_L) (\bar{u}_L \gamma_\mu u_L) + (\bar{s}_L \gamma^\mu u_L) (\bar{u}_L \gamma_\mu d_L) +$$

$$2 (\bar{s}_L \gamma^\mu d_L) (\bar{d}_L \gamma_\mu d_L) + 2 (\bar{s}_L \gamma^\mu d_L) (\bar{s}_L \gamma_\mu s_L)] -$$

$$[(\bar{s}_L \gamma^\mu d_L) (\bar{c}_L \gamma_\mu c_L) + (\bar{s}_L \gamma^\mu c_L) (\bar{c}_L \gamma_\mu d_L)] \quad (7.2)$$

The operator in Eq.(7.1) cannot mix with lower dimension operators since there are no two-quark operators with $\Delta S = 2$ (or $\Delta T=3/2$). As a consequence the renormalized operator $O^{\Delta S=2}$ has the form:

$$O^{\Delta S=2}(\mu) = Z_{\text{LATT}}^{\Delta S=2}(\mu a, g_0) [O^{\Delta S=2} + \sum_i C_i(\bar{s} \Gamma_i d) (\bar{s} \Gamma_i d)] \quad (7.3)$$

where the coefficients $C_i = C_i(g_0)$ are finite, depend only on g_0 (the lattice bare QCD coupling constant) and can be safely computed in perturbation theory. They are different from zero because of the Wilson term. The corresponding operators $O_i = (\bar{s} \Gamma_i d) (\bar{s} \Gamma_i d)$ (Γ_i are Dirac matrices) respect vector symmetries and CPS, are parity conserving but belong to chiral representations different from the (27,1).

Contrary to the previous case the two operators in Eqs.(7.2) can also mix with operators of dimension five and three or, more precisely, with $(\bar{s} \sigma_{\mu\nu} F_{\mu\nu} d)$, $(\bar{s} \sigma_{\mu\nu} \tilde{F}_{\mu\nu} d)$, $(\bar{s} d)$ and $(\bar{s} \gamma_5 d)$. On dimensional grounds, the coefficients of the operators of dimension less than six are expected to diverge as powers of a^{-1} . However, it is simple to show that the coefficient of $(\bar{s} \sigma_{\mu\nu} \tilde{F}_{\mu\nu} d)$ is finite because of CPS symmetry and that the coefficient of $(\bar{s} \sigma_{\mu\nu} F_{\mu\nu} d)$ can be made finite by the GIM cancellation when we allow the charm quark to propagate in the loop in diagrams like the one shown in Fig.25 (this implies that the condition $m_{ca} \ll 1$ must be satisfied). On the other hand the coefficients of the scalar and pseudoscalar densities are always divergent.

FIGURE 25

In summary the renormalized (8,1) operator $O^\pm$ have the form:

$$O^\pm(\mu) = Z_{\text{LATT}}^\pm (\mu a, g_0) [O^\pm + \delta_6 O^\pm + \delta_5 O^\pm + \frac{(m_c - m_u)}{a^2} C_S^\pm(\bar{s} d) + \frac{(m_c - m_u)(m_s - m_d)}{a} C_P^\pm(\bar{s} \gamma_5 d)] \quad (7.4)$$

where $\delta_6 O^\pm$ and $\delta_5 O^\pm$ are the corrections due to the operators of dimension six (as for the $\Delta S = 2$ case) and five whose coefficients are finite and can be computed perturbatively*. The coefficients of the pseudoscalar and scalar densities in Eq.(7.4), being power divergent, cannot be computed in perturbation theory. The reason is that any non-perturbative contribution $\sim e^{-(1/2\beta_0 g_0^2)}$ to power divergent coefficients will not disappear even when $g \rightarrow 0$, i.e., in the continuum limit.

The overall renormalization constants $Z_{\text{LATT}}^{\Delta S=2}$ and $Z_{\text{LATT}}^\pm$ are completely determined, including the finite terms, by the requirement that the renormalized operators $O(\mu)$ obey, for some value of $\mu \ll a$, the same renormalization conditions on quark states as the continuum operators of the effective Hamiltonian (cfr. Sect.2).

A few comments are necessary at this point.

- (i) As shown by the explicit one-loop calculations of Refs.37-39 and by the general theoretical analysis of Ref.6, the off-diagonal operators in Eq.(7.3) as well as the operators $\delta_6 O^\pm$ in Eq.(7.4) are all parity even. This result is true also non-perturbatively and is a consequence of an exact symmetry as discussed in Ref.40.
- (ii) The parity violating off-diagonal operators of dimension five and three, i.e., $(\bar{s} \sigma_{\mu\nu} F_{\mu\nu} d)$ and $(\bar{s} \gamma_5 d)$ have coefficients which vanish for $m_s = m_d$.
- (iii) Non-perturbative methods to fix the coefficients of operators of dimension three, $C_{S,P}^\pm$ are necessary.

For practical reasons, we have to rely on perturbation theory to fix the coefficients of the

* The explicit expressions for $\delta_6 O^\pm$ and $\delta_5 O^\pm$ as well as the computation of the mixing coefficients can be found in Refs.37, 38, 39.

operators of dimension six and five. As it will appear from the discussion of the lattice calculation of the kaon B-parameter, a certain systematic error is entailed in the procedure of constructing the renormalized operators using lowest-order perturbation theory and neglecting terms of order a (as for the vector current).

At lowest order in chiral symmetry breaking, soft pion theorems imply the following relations between matrix elements:

$$\begin{aligned}\langle 0 | O^\pm(\mu) | K^0 \rangle &= 1(m_K^2 - m_\pi^2) \delta^\pm \\ \langle \pi^+(p) | O^\pm(\mu) | K^+(k) \rangle &= -\frac{m_K^2}{f} \delta^\pm + \gamma^\pm(p \cdot k)\end{aligned}\quad (7.5)$$

$$\langle \pi^- \pi^+ | O^\pm(\mu) | K^0 \rangle = \frac{1}{f} (m_K^2 - m_\pi^2) \gamma^\pm$$

(all the $K \rightarrow \pi\pi$ matrix elements being taken for vanishing four-moment transfer of the operator).

On the other hand, for small masses and momenta we also have:

$$\begin{aligned}\langle 0 | O^\pm + \delta_6 O^\pm + \delta_5 O^\pm | K^0 \rangle &= 1 \delta_1^\pm \\ \langle \pi^+(p) | O^\pm + \delta_6 O^\pm + \delta_5 O^\pm | K^+(k) \rangle &= \delta_2^\pm + \gamma_2^\pm(p \cdot k)\end{aligned}\quad (7.6)$$

where $O^\pm + \delta_6 O^\pm + \delta_5 O^\pm$ are the bare lattice operators in Eqs.(7.2) added with the corrections due to the operators of dimension six ($\delta_6 O^\pm$) and five ($\delta_5 O^\pm$) whose coefficients are taken from perturbation theory.

Similarly:

$$\begin{aligned}\langle 0 | \bar{s} \gamma_5 d | K^0 \rangle &= 1 \delta_p \\ \langle \pi^+(p) | \bar{s} d | K^+(k) \rangle &= \delta_s + \gamma_s(p \cdot k)\end{aligned}\quad (7.7)$$

Then by a suitable linear combination of $O^\pm + \delta_6 O^\pm + \delta_5 O^\pm$ with $(\bar{s}d)$ and $(\bar{s}\gamma_5 d)$ we can enforce the relations in (7.5). This implies:

$$\gamma^\pm = Z_{\text{LATT}}^\pm (\mu a, g_0) \left[\gamma_2^\pm - \frac{\delta_2^\pm \gamma_s}{\delta_s} \right] + O(m_K^2) \quad (7.8)$$

In Eq.(7.8) the $O(m_K^2)$ terms depend on $\delta^\pm$ but are of higher order in the chiral expansion.

The method explained above requires the computation of the $\langle \pi | O | K \rangle$ matrix elements at different values of the transferred momentum $q^2 = (p-k)^2$ in order to disentangle δ from γ in Eqs.(7.6) and (7.7). Equation (7.8) thus allows the determination of the $\Delta T = \frac{1}{2} K \rightarrow 2\pi$ amplitude

at lowest order in chiral perturbation theory:

$$\frac{A(K_S \rightarrow \pi^+ \pi^-)}{m_K} = 2 G_F \sin \theta_c \cos \theta_c \frac{m_K^2 - m_\pi^2}{m_K f} [C^-(\mu) \gamma + C^+(\mu) \gamma^*] \quad (7.9)$$

The procedure described above cannot be used for the "penguin" operators relevant for CP violation (i.e., ϵ'/ϵ). In fact, short of putting the top quark on the lattice (this would correspond to $m_t a \ll 1$), the "penguin" operators need two non-perturbative subtractions ($\bar{s} \sigma_{\mu\nu} F_{\mu\nu} d$ and $\bar{s} d$) for the parity-conserving part. On the other hand, they still require only one subtraction in the parity-violating sector. So in principle one can use the $K \rightarrow 0$ amplitude to fix the coefficient of $\bar{s} \gamma_5 d$ (e.g., by imposing the $\langle 0 | O^\pm | K^0 \rangle = 0$) and compute directly the amplitude $K \rightarrow 2\pi$ of the renormalized operator.

A very nice alternative procedure has been recently proposed in Ref.40. The idea is to evaluate the form factor $K \rightarrow 2\pi$ at the point $p_K^2 = p_\pi^2 = m_K^2 = m_\pi^2 = m^2$. In this limit, because of CPS symmetry, the coefficient of $\bar{s} \gamma_5 d$ (as well as the coefficient of $\bar{s} \sigma_{\mu\nu} F_{\mu\nu} d$) vanishes and the Cabibbo/Gell-Mann theorem is evaded because the operator carries some energy. Moreover all the counter-terms of dimension six, being parity conserving, cannot contribute to the matrix element. Thus we have:

$$\langle \pi^+ \pi^- | O^\pm(\mu) | K^0 \rangle = \langle \pi^+ \pi^- | Z_{\text{LATT}}^\pm O^\pm | K^0 \rangle = i \frac{2m^2}{f} \gamma^\pm \quad (7.10)$$

With this method $\gamma^\pm$ is directly obtained from the lattice operator (times the factor $Z_{\text{LATT}}^\pm$) without any further subtraction.

7.2 The Kaon B-Parameter and the $\Delta I=3/2$ Amplitude

The simplest (and first studied^{11,40,41,42,43}) weak transition amplitude concerns the matrix element of the $\Delta S=2$ four-fermion operator:

$$\langle K^0 | O_{(\mu)}^{\Delta S=2} | \bar{K}^0 \rangle = \frac{8}{3} f_K^2 m_K^2 B_{K^0 \bar{K}^0} \quad (7.11)$$

which is usually expressed in terms of the B-parameter appearing on the right-hand side of Eq.(7.11). To compute $B_{K^0 \bar{K}^0}$ we extract, at several values of the pseudoscalar meson mass, m , the matrix elements $\langle K^0 | O | \bar{K}^0 \rangle$ and $\langle 0 | O | \bar{K}^0 \bar{K}^0 \rangle$ from the three-point correlation functions [cf. Eq.(3.13)].

Lowest-order chiral perturbation theory predicts:

$$\langle 0 | O_{(\mu)}^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle = \alpha + \beta m^2 + \gamma p_{K^0} \cdot p_{K^0} + O(m^4) = \alpha + (\beta + \gamma) m^2 + O(m^4)$$

$$\langle K^0 | O_{(\mu)}^{\Delta S=2} | \bar{K}^0 \rangle = \alpha + (\beta + \gamma) m^2 + O(m^4) \quad (7.12)$$

when the mesons are at rest.

For the truly renormalized operator $O_{\text{REN}}^{\Delta S=2}$, α and β should vanish. However, since the subtraction has been computed only at lowest order in g_0^2 and terms of order a still play a role, the operator, renormalized in perturbation theory, has α and β different from zero. So we fit the matrix elements of $O^{\Delta S=2}$ to the expressions in Eq.(7.14) and take as our best estimate of the matrix element the coefficient γ . In so doing we are certainly left with a systematic error. In fact the residual unknown subtraction will not only cancel α and β but also modify γ ($\gamma_0 \rightarrow \gamma_{\text{REN}}$ only as $a \rightarrow 0$). As discussed before the measurement of the renormalization constant of the local vector on the lattice, which can be evaluated quite precisely in a non-perturbative way, strongly suggests that the main systematic effects in the renormalization of the operators are not due to an inadequate accuracy of perturbation theory but to effects of order $\Lambda_{\text{QCD}} a$, which are certainly present at finite lattice spacing¹⁸.

Actually, to evaluate the B-parameter, it is convenient to fit the matrix elements of the operator as a function of the squared matrix element of the axial current. The coefficient of the fit is the B-parameter (up to an overall renormalization constant $8/3Z_A^2$). In Fig.26, the ratio $\langle 0 | O^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle / Z_5$ ($Z_5 = | \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle |^2$) and $\langle K^0 | O^{\Delta S=2} | K^0 \rangle / Z_5$ are reported as a function of $8/3(\langle 0 | A_0 | \bar{K}^0 \rangle)^2 / Z_5 = 8/3 f_K^2 m_K^2 / (Z_A^2 Z_5)$ at $\beta = 6.2$ on a $16^3 \times 48$ lattice (15 configurations). These combinations of matrix elements are particularly convenient for reducing the errors since they are directly computed by taking ratios of correlation functions.

FIGURE 26

$\langle O \rangle / Z_5$ as a function of $8/3[(\langle 0 | A_0 | \bar{K}^0 \rangle)^2 / Z_5 = 8/3 f_K^2 m_K^2 / (Z_A^2 Z_5)]$ at $\beta = 6.2$ on the $16^3 \times 48$ lattice. The points correspond to the matrix elements $\langle 0 | O^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle$, i.e., $p_K \cdot p_K = -m_K^2$ on the left-hand side, and $\langle K^0 | O^{\Delta S=2} | K^0 \rangle$, i.e., $p_K \cdot p_K = +m_K^2$ on the right-hand side. Six different combinations of quark masses have been used.

The value of the B-parameter obtained from a linear and a quadratic fit to the points in Fig.26 and the analogue at $\beta = 6$ are reported in Table 2. We take as our best estimate for the B-parameter the results of the quadratic fits, given the fact that terms of order m^4 are still important in the range of masses considered. To combine the results at $\beta = 6$ and $\beta = 6.2$ we must take into account that the B-parameter is a function of the scale, and use the renormalization group evolution of the corresponding operator.

B-parameter		
Linear fit	Quadratic fit	
0.81 ± 0.16	0.75 ± 0.20	$\beta = 6$ 30 configurations
0.88 ± 0.20	0.60 ± 0.20	$\beta = 6.2$ 15 configurations

TABLE 2

Results for the B-parameter obtained from a linear and a quadratic fit to the matrix elements of $O_{\Delta S=2}$.

Combining the results of the Table, one obtains:

$$B_{K^0 \bar{K}^0}(\mu) = 0.65 \pm 15 \quad a^{-1} \sim 3 \text{ GeV} \quad (7.13)$$

which in terms of the renormalization group invariant B-parameter corresponds to:

$$B_{K^0 \bar{K}^0} = [\alpha_s(\mu)]^{-6/25} B_{K^0 \bar{K}^0}(\mu) = 0.87 \pm 0.20 \quad (7.14)$$

for $\Lambda_{QCD} = 200 \text{ MeV}$. Using $SU(3) \times SU(3)$ chiral perturbation theory⁴⁴ the value of $B(\mu)$ can be translated into the weak non-leptonic $\Delta I = 3/2$ amplitude $K^+ \rightarrow \pi^+ \pi^0$:

$$\frac{\langle \pi^+ \pi^0 | H_W | K^+ \rangle}{m_K} = G_F \frac{\sin \theta_c \cos \theta_c}{2} \frac{3}{4} \frac{m_K^2 - m_\pi^2}{m_K} C_4(\mu) \times \left[\frac{\langle K^0 | \bar{Q}(\mu) | \bar{K}^0 \rangle}{f_K m_K^2} \right]$$

One finds¹¹:

$$A_{3/2} = \frac{\langle \pi^+ \pi^0 | H_W | K^+ \rangle}{m_K} = [6 \pm 2 \pm 1 \text{ (syst)}] 10^{-8} \quad (7.15)$$

to be compared with the experimental value $A_{3/2}^{\text{exp}} = 3.7 \times 10^{-8}$.

With the techniques exposed in Section 3, using four-point correlation functions, it has also been possible to compute directly the $\Delta I = 3/2$ matrix element $\langle \pi^+ \pi^0 | H_W | K^+ \rangle$. Chiral perturbation theory gives the following relations when all mesons are at rest and $m_K^2 = m_\pi^2 = m^2$:

$$\langle \pi^+ | H_W | K^+ \rangle = g_{3/2} m^2$$

$$\langle \pi^+ \pi^- | H_W | K^0 \rangle = \frac{21}{1} g_{3/2} m^2$$

$$\langle \pi^- | H_W | \pi^- K^0 \rangle = -\frac{61}{1} g_{3/2} m^2 \quad (7.16)$$

An explicit lattice calculation¹¹ of the last two matrix elements has shown that the relation:

$$\frac{\langle \pi^- | H_W | K^0 \pi^- \rangle}{\langle \pi^+ \pi^- | H_W | K^0 \rangle} \approx -3$$

is satisfied with a good approximation in the mass range $m \approx 0.6-1.0$ GeV. The values of the coupling $g_{3/2} a^2$, a being the lattice spacing, from the $K-\pi$ and $K-\pi\pi$ transitions of eqs.(7.16) at $\beta = 6$ and 6.2 are reported in Table 3. From the Table we see that soft-pion theorems and lowest-order chiral relations are satisfied at the 30% level.

The results for $-1/3 \langle K^0 \pi^- | O_4 | \pi^- \rangle$ as a function of m^2 at $\beta = 6$ and $\beta = 6.2$ are reported in Figs.27a and 27b respectively. Only a linear fit in m^2 has been done. With the present accuracy, we cannot exclude corrections of order m^4 or effects of order a such that the matrix element does not really vanish as $m \rightarrow 0$. However, the expected chiral behaviour seems to be well reproduced by the numerical results.

FIGURE 27a and b

The quantity $g_{3/2} m^2 / f$, defined in Eqs.(7.16) and derived from the matrix element $-1/3 \langle K^0 \pi^- | O_4 | \pi^- \rangle$ is reported at $\beta = 6.2$ and $\beta = 6$ as a function of the pseudoscalar meson mass $m_K^2 = m_\pi^2 = m^2$.

Given the off-diagonal subtractions are only needed for K- π amplitudes, and that meson masses are never very small, the K- π and K- $\pi\pi$ comparison from Table 3 is an excellent test of our computation and of the approach to the chiral limit. Soft pion relations are indeed a very powerful way to control the renormalization of the relevant operators on the lattice. A similar test should also be performed in the more complicated case of $\Delta I = \frac{1}{2}$ transition amplitudes.

$g_{3/2} a^2$		
$K^+-\pi^+$	$K^0-\pi^+\pi^-$	
$(2.0 \pm 0.8) 10^{-2}$	$(1.3 \pm 0.4) 10^{-2}$	$\beta = 6$ 15 configurations
$(5.9 \pm 1.2) 10^{-3}$	$(4.4 \pm 1.2) 10^{-3}$	$\beta = 6.2$ 15 configurations

TABLE 3

The coupling $g_{3/2}$ defined in Eqs.(7.16) as computed from the K- π and K- $\pi\pi$ matrix elements. Notice the good agreement between the two values both at $\beta = 6$ and $\beta = 6.2$.

7.3 The $\Delta I = \frac{1}{2}$ amplitude

To lowest order in masses and momenta, chiral perturbation theory gives the expressions for the $K_S \rightarrow \pi^+\pi^-$ amplitude

$$\frac{A(K_S \rightarrow \pi^+ \pi^-)}{m_K} = 2 G_F \sin \theta_c \cos \theta_c \frac{m_K^2 - m_\pi^2}{m_K f_\pi} [C^-(\mu) \gamma + C^+(\mu) \gamma^*] \quad (7.17)$$

where $\gamma^\pm$ are defined according to:

$$\langle \pi^+ \pi^- | O^\pm(\mu) | K^0 \rangle = \frac{1}{f_\pi} (m_K^2 - m_\pi^2) \gamma^\pm \quad (7.18)$$

In the same approximation, $\gamma^\pm$ appear in the following K- π and K- $\pi\pi$ matrix elements:

$$\langle \pi^+ (0) | O^\pm(\mu) | K^+(q) \rangle = \delta^\pm m^2 + \gamma^\pm E(q) m \quad (7.19)$$

$$\langle 0 | O^\pm(\mu) | K^+(0) \pi^-(0) \rangle = \delta^\pm m^2 - \gamma^\pm m^2 \quad (7.20)$$

$$\langle \pi^+ (0) \pi^-(0) | O^\pm(\mu) | K^0(0) \rangle = \frac{2i}{f_\pi} m^2 \gamma^\pm \quad (7.21)$$

$$\langle \pi^+ (0) | O^\pm(\mu) | K^0(0) \pi^+ (0) \rangle = -\frac{2i}{f_\pi} m^2 \gamma^\pm \quad (7.22)$$

where we have taken K and π degenerate in mass ($m_K^2 = m_\pi^2 = m^2$), q is the spatial momentum, ($E = \sqrt{m^2 + q^2}$, and $f_\pi \approx 131$ MeV).

The calculation on the lattice of the matrix elements on the left-hand side of Eqs.(7.19) to (7.22) allows a determination of $\gamma^\pm$, i.e., of the physical amplitude, to lowest order in chiral perturbation theory.

As discussed in Refs.3 and 6 (see also Section 7.1), in the SU(3) symmetric limit, the renormalized operators $O^\pm(\mu)$ are related to the lattice operator $O^\pm$ as follows:

$$O^\pm(\mu) = Z_{\text{LATT}}^\pm (\mu a, g_0) [O_{\text{pert}}^\pm + c_9(\bar{s} d) + c_P(\bar{s} \gamma_5 d)] \quad (7.23)$$

where $O_{\text{pert}}^\pm = O^\pm + \delta_6 O^\pm + \delta_5 O^\pm$ is the naive lattice operator plus the appropriate operators of dimensions six and five.

To fix $C_S^\pm$ one can impose:

$$\langle \pi^+ (0) | O^\pm(\mu) | K^+(0) \rangle = 0 \quad (7.24)$$

which corresponds to $\delta^\pm = -\gamma^\pm^*$ (eqs. 7.19 - 7.20).

The smoothness assumption on the behaviour of the matrix elements as a function of m^2 made in Eqs.(7.20) - (7.22) may fail in the presence of octet scalar particles comparatively light with respect to the pseudoscalar mass m . Exchanges of scalar particles as shown for example in the diagram of Figs.28 gives non-smooth contributions to the matrix elements of Eqs.(7.20) - (7.22), enhanced by the scalar propagator when $M_S^2 = 4m^2$. The matrix element in Eq.(7.19) is relatively unaffected, because the momentum transfer is space-like. For quark masses at which current numerical studies are done, the scalar octet mass M_S is close to $2m^{**}$.

FIGURE 28

Feynman diagram relative to the octet scalar contribution K- $\pi\pi$ transitions. The symbol $\otimes$ indicates the insertion of the weak Hamiltonian.

* As discussed in Ref.6, $\delta^\pm$ do not appear in the physical decay amplitude and in fact are arbitrary. Different value of $\delta^\pm$ correspond to different and equivalent subtraction prescriptions needed to define $O^\pm(\mu)$.

** A scalar mass even smaller than $2m$ is found by high statistics preliminary results from the APE collaboration.

In such a situation the only theoretically safe way to extract $\gamma^\pm$ is from the matrix element of Eq.(7.19). By imposing the subtraction condition Eq.(7.24), one has:

$$\gamma^\pm = \frac{\langle \pi^\pm(o) | O^\pm(\mu) | K^\pm(q) \rangle}{m[E(q) - m]} \quad (7.25)$$

Because of the subtraction involved, the signal in the numerator of Eq.(7.25) is small and affected by large statistical fluctuations. The direct calculation of the $K\text{-}\pi\pi$ amplitude needs no subtraction, is subject to smaller statistical fluctuations, but may be contaminated by spurious octet scalar contributions unless $M_S^2 \gg 4m^2$.

$K\text{-}\pi$ and $K\text{-}\pi\pi$ matrix elements are computed from three- and four-point correlations of the operators and pseudoscalar sources. In the following we will denote the contribution to $\gamma^\pm$ from the eye and from the other (eight-shaped) diagrams, by $\gamma_{eye}^\pm$ and $\gamma_{eight}^\pm$ respectively.

We discuss first $K\text{-}\pi$ matrix elements⁴⁵.

The values of $\gamma_{eye}^\pm$ obtained using Eq.(7.25) are reported in Table 4 and plotted in Figs.29a and b versus $(ma)^2$. Given the large errors, to extrapolate to the chiral limit $\gamma_{eye}^\pm$ was either fitted to a constant, taking the weighted average of the three points (I) or extrapolated linearly discarding the last point (II). The results are also given in Table 4 together with the eight-diagram contribution $\gamma_{eight}^\pm$.

Using Eq.(7.17) with the experimental values for m_K , m_π and f_π and using $a^{-1} = 1.8$ GeV we find the amplitude for $K_S \rightarrow \pi^+\pi^-$ reported in Table 6 where we have used $c^{(-)}(\mu) = -1.7$ and $c^{(+)}(\mu) = 0.75$ at $\mu \sim 1.8$ GeV. In the Table, R is the following ratio:

$$R = \frac{\langle \pi^+ \pi^- | H_W | K_S \rangle}{\langle \pi^+ \pi^0 | H_W | K^+ \rangle} \quad \text{exp-21.2} \quad (7.26)$$

The results for $A(K_S \rightarrow \pi^+\pi^-)$ are still compatible with zero, given that the statistical error is $\sim 100\%$. The result for $\gamma_{eye}^\pm$ is definitely worse than the one for $\gamma_{eye}^\mp$. The trend of the data is, however, encouraging. In particular they favour a positive sign for R, in agreement with the fact that $B(K_S \rightarrow \pi^+\pi^-) > 2B(K_S \rightarrow \pi^0\pi^0)$. More statistics are clearly needed.

If we use the matrix element in Eq.(7.20) to extract $\gamma_{eye}^\pm$ we obtain results with smaller statistical errors, corresponding to a large signal, but R comes out to be negative. These results have been presented and discussed in length in Ref.46. The anomalous sign may be again due to the presence of a nearby scalar pole which dominates the amplitude in this range of masses.

A scalar pole may affect also the $K\text{-}\pi\pi$ calculation, where subtractions are not needed because of CPS symmetry⁴⁰. If this is the case this method is reliable only at low pseudoscalar masses. The values of $2m^2\gamma_{eye}^\pm/f(m)$, reported in Table 5, have been obtained from the average of the matrix elements, Eqs.(7.21) and (7.22). $f(m)$ is the pseudoscalar meson axial coupling as a function of the quark mass. In the Table, also $2m^2\gamma_4/f(m)$, which refers to the $\Delta I = 3/2$ operator is given:

$$O_4 = [(\bar{s}_L \gamma_\mu d_L) (\bar{u}_L \gamma^\mu u_L) + (\bar{s}_L \gamma_\mu u_L) (\bar{u}_L \gamma^\mu d_L)] - (\bar{s}_L \gamma_\mu d_L) (\bar{d}_L \gamma^\mu d_L) \quad (7.27)$$

FIGURES 29a and 29b

γ_{eye}^- (a) and γ_{eye}^+ (b) versus m^2a^2 from the K- π matrix elements at $q \neq 0$, see Eq.(7.25) of the text.

Assuming a constant behaviour in m^2 of all γ^s , we obtain the physical amplitudes reported in Table 6.

Within errors the $K \rightarrow \pi\pi$ results are quite compatible with those obtained from the $q \neq 0$ $K \rightarrow \pi$ matrix elements, and do not show any non-smooth variation with the pseudoscalar meson mass m . Within the present data, we are unable to decide whether any contribution of the scalar particle is hidden in the errors, or is really small.

m a	$\gamma_{eye}(q \neq 0) \times 10^3 \times a^2$	$\gamma_{eye}^+(q \neq 0) \times 10^3 \times a^2$	$\gamma_{eight} \times 10^3 \times a^2$	$\gamma_{eight}^+ \times 10^3 \times a^2$
0.523	(-3 ± 9)	(-2 ± 11)	—	—
0.436	(-8 ± 10)	(0 ± 13)		
0.327	(-24 ± 17)	$(+11 \pm 16)$		
chiral limit	$(-8 \pm 9)(I)$ $(-20 \pm 12)(II)$	$(+2 \pm 12)(I)$ $(+5 \pm 19)(II)$	-2.9 ± 1.5	3.9 ± 1.5

TABLE 4

γ_{eight}^+ extrapolated to the chiral limit from $K-\pi$ matrix elements.

$\frac{2m^2}{f(m)} \gamma_{eye} \times 10^4 \times a^3$	$\frac{2m^2}{f(m)} \gamma_{eye}^+ \times 10^4 \times a^3$	$\frac{2m^2}{f(m)} \gamma_{eight} \times 10^4 \times a^3$	$\frac{2m^2}{f(m)} \gamma_{eight}^+ \times 10^4 \times a^3$	$\frac{2m^2}{f(m)} \gamma_4 \times 10^4 \times a^3$
-8.0 ± 5.4 -5.5 ± 3.7	5.4 ± 2.3 3.7 ± 1.7	-0.33 ± 0.28 -0.14 ± 0.24	0.51 ± 0.07 0.38 ± 0.06	0.59 ± 0.13 0.37 ± 0.11

TABLE 5

γ_{eight}^+ from $K-\pi\pi$ matrix elements at two different values of the pseudoscalar meson mass.

β	SOURCE	$A(K_S \rightarrow \pi^+ \pi^-)/m_K$	$A(K^+ \rightarrow \pi^+ \pi^0)/m_K$	R
6.0	K- $\pi(q \neq 0)$	$(1.2 \pm 1.3) 10^{-6}$ (I) $(2.5 \pm 1.8) 10^{-6}$ (II)	$(16.0 \pm 6.0) 10^{-8}$	8 ± 24 (I) 16 ± 40 (II)
6.2	K- $\pi\pi$	$(2.1 \pm 1.3) 10^{-6}$	$(5.9 \pm 1.4) 10^{-8}$	35 ± 31
Experiment		$0.78 \cdot 10^{-6}$	$3.8 \cdot 10^{-8}$	21.2

TABLE 6

Results from Ref.45 obtained at two different values of the strong coupling constant $\beta = 6/g_0^2$, $\beta = 6$ and $\beta = 6.2$.

In conclusion, the method based on the calculation of K- π matrix elements with a non-vanishing space-momentum, q , supports the observed enhancement of the $\Delta I = \frac{1}{2}$ amplitude, although within very large statistical fluctuations. The eye-diagrams are the source of the enhancement, an indirect confirmation of the scheme of Ref.47 since penguin diagrams are indeed generated by eye-diagrams. The vacuum saturation approximation gives $\gamma_{eye} \gamma_{eye}^* = -\frac{1}{2}$, while we find a large ratio which seems to be just the continuation of the short-distance enhancement.

The K- $\pi\pi$ method gives results in agreement with the previous ones, but its validity is subject to the demonstration that the scalar pole has little influence in the present quark mass range. Finally, the K- π timelike region method gives results incompatible with the previous ones, which may be due to the effect of the scalar pole and/or imperfect subtraction.

7.4 Electropenguin contribution to ϵ'/ϵ .

The electropenguin contribution to the CP-violating parameter ϵ' induced by the electropenguin diagrams has been widely discussed in recent literature⁴⁸. After integration of t and c quark virtual loops, the electropenguin contribution to the $I=2$, $K \rightarrow \pi\pi$ amplitude, A_2 , is determined by the matrix elements of the following four-fermion operators:

$$O_{LR} = \frac{1}{3} \{ (\bar{s}_L \gamma_\mu d_L) (\bar{u}_R \gamma^\mu u_R - \bar{d}_R \gamma^\mu d_R) + (\bar{s}_L \gamma_\mu u_L) (\bar{u}_R \gamma^\mu d_R) \} \quad (7.28)$$

$$(O_D)_{LR} = \frac{1}{3} \{ (\bar{s}_L \gamma_\mu t^A u_L) (\bar{u}_R \gamma^\mu t^A u_R - \bar{d}_R \gamma^\mu t^A d_R) + (\bar{s}_L \gamma_\mu t^A u_L) (\bar{u}_R \gamma^\mu t^A d_R) \} \quad (7.29)$$

The operators in Eqs.(7.28) and (7.29) have pure $\Delta I=3/2$. From the point of view of flavour SU(3), they are a superposition of 27-plet and decuplet, unlike the fully left-handed operator O_4 given in eq.(7.27) which is a pure 27.

In a recent paper⁴⁹, the $K-\pi$ and $K-\pi\pi$ matrix elements of the operators in Eqs.(7.28) and (7.29) have been computed on the lattice. The results of Ref.49 can be summarized as follows.

- (i) In the chiral limit, the $K-\pi$ matrix elements of O_{LR} and $(O_C)_{LR}$ are both very close to their vacuum saturation value, the latter being also computed on the lattice. More precisely they find:

$$B_{LR}(a^{-1} \simeq 2.2 \text{ GeV}) = 1.0 \pm 0.1$$

$$(B_C)_{LR}(a^{-1} \simeq 2.2 \text{ GeV}) = 0.95 \pm 0.1 \quad (7.30)$$

- (ii) Chiral relations are obeyed within about 50% accuracy, similarly to what is found in Ref.11 for O_4 .
- (iii) For both O_{LR} and $(O_C)_{LR}$, the matrix elements $\langle \pi^+ \bar{K}^0 | O | \pi^- \rangle$, which allow for a scalar octet intermediate state, are between 5 to 10 times larger than the $\langle \pi^+ \pi^- | O | K^0 \rangle$ and $\langle \pi^- | O | \pi^+ K^0 \rangle$ matrix elements, which do not. This is strongly suggestive of an important role of the scalar octet also in the $K \rightarrow \pi\pi$, $\Delta I = \frac{1}{2}$ amplitude. It suggests that a reliable calculation of $\Delta I = \frac{1}{2}$ CP-conserving and CP-violating $K \rightarrow \pi\pi$ amplitudes requires quark masses much closer to the chiral limit.

The results on the electropenguin contribution to ϵ'/ϵ can be stated in two different ways:

- a) from a full-lattice calculation of $(\text{Im } A_2/A_2)_{\text{EMP}}$ one finds:

$$\left| \frac{\epsilon'}{\epsilon} \right|_{\text{EMP}} = (0.74 \pm 0.27) 10^{-3} \left(\frac{A}{1.05} \right)^2 \left(\frac{\lambda}{0.221} \right)^4 \left(\frac{\rho}{0.6} \right) \sin\phi \quad (7.31)$$

where the error is purely statistical; A , λ , ρ and ϕ are the parameters of the C-K-M matrix in the Wolfenstein parametrization⁵⁰; alternatively:

$$\epsilon' = -1 e^{i(\delta_2 - \delta_0)} \frac{\omega}{\sqrt{2}} \frac{\text{Im } A_0}{A_0} (1 + \Omega_{\text{EMP}} - \Omega_{\eta,\eta'}) \quad (7.32)$$

with $\Omega_{\eta,\eta'}$, arising from $\pi-\eta$ and $\pi-\eta'$ mixing. One gets⁴⁹:

$$\Omega_{\text{EMP}} \simeq 0.17/B_6 \quad (7.33)$$

where B_6 is the B-parameter of the penguin operator O_6 . The above value for Ω_{EMP} is of the same order of the estimated value of $\Omega_{\eta,\eta'}$ but with the opposite sign.

This result is quite different from what was obtained using the $1/N_c$ expansion. It agrees with previous lattice determinations of the B-parameter of left-right operators⁵¹. All previous lattice calculations found a B-parameter close to unity both with Wilson⁴⁰ and staggered fermions⁵¹. In

Ref.51 they also estimated the ordinary penguin B-parameter with the result $B_6 = 0.5 \pm 0.2$. According to Eq.(7.33), and assuming $B_6 = 0.5$, this would imply that the electropenguin effects can be as large as 35%.

8. CONCLUSION AND PERSPECTIVES

The quality of the results in the computation of hadronic matrix elements on the lattice is already comparable to other theoretical approaches. The same method is used in all the cases (structure functions, form factors, σ -term, weak decays, ...) and relies only on calculations from first principles which use the renormalization group and the Wilson operator expansion, as basic tools.

The error on the evaluation of the matrix elements ranges between 20 and 100%. The main problem is to reduce the systematic errors in the final results. Two facts seem to be at the moment the major sources of uncertainties: the effects of order a and the quenched approximation (finite size effects seem reasonably small for all the cases). For the effects of order a , one could think to improve the situation by using the Symanzik proposal, which easily reduce them to order $g^2 a$. On the contrary it is rather difficult to do at the moment a complete, unquenched calculation on large lattices and at realistic quark masses.

My conclusion is that, however, in spite of the difficulties, the lattice approach is already one of the leading techniques and it will become a fundamental (if not unique) method to predict precisely transition amplitudes which are relevant in the phenomenology of the hadrons.

REFERENCES

- 1) K.G.Wilson in "New Phenomena in Subnuclear Physics", A.Zichichi ed, Plenum Press, New York (1977).
- 2) L.H.Karsten and J.Smit, Nucl.Phys. B183 (1981) 103.
- 3) M.Bohicchio, L.Maiani, G.Martinelli, G.C.Rossi and M.Testa, Nucl.Phys. B262 (1985) 331.
- 4) G.Curci, Phys.Lett. B167 (1986) 425.
- 5) L.Maiani and G.Martinelli, Phys.Lett. 168B (1986) 265.
- 6) L.Maiani, G.Martinelli, G.C.Rossi and M.Testa, Phys.Lett. 176B (1986) 445; Nucl.Phys. B289 (1987) 505.
- 7) G.Martinelli and C.T.Sachrajda, Phys.Lett. B190 (1987) 151.
- 8) G.Martinelli and C.T.Sachrajda, Phys.Lett. B196 (1987) 184; Nucl.Phys. B306 (1988) 865.
- 9) M.Creutz, at this school.
- 10) A.Ukawa, at this school.
- 11) M.B.Gavela, L.Maiani, G.Martinelli, S.Petrarca, F.Rapuano and C.T.Sachrajda, Nucl.Phys. B306 (1988) 677; M.B.Gavela, L.Maiani, O.Pène and S.Petrarca, Phys.Lett. B206 (1988) 113.

- 12) T.A.De Grand and R.D.Loft, Preprint COLO-HEP-165 (nov. 1987) to appear in Phys.Rev.D.
- 13) C.Bernard et al. presented by C.Bernard at the Int. Symp. "Theorie des Champs sur Reseau", Seillac (1987), Nucl.Phys. B4 (1988) 483; IUHET-141 (1988).
- 14) S.Narison, CERN-TH 4768 (1987).
- 15) C.A.Dominguez and N.Paver, Phys.Lett. B197 (1987) 423.
- 16) For the exponentiation method see:
F.Fucito, G.Parisi, S.Petrarca, Phys.Lett. 115B (1982) 148; G.Martinelli, G.Parisi, R.Petronzio, F.Rapuano 116B (1982) 432; C.Bernard et al., Phys.Rev.Lett. 49 (1982) 1076; S.Gottlieb, P.B.Mackenzie, H.B.Thacker, D.Weintgarten, Nucl.Phys. B263 (1986) 704.
- 17) For the source method see:
C.Bernard in "Gauge Theories on a lattice", 1984 eds. Zachos et al., Nat.Tech. Information Service, Springfield, VA (1984); R.Gupta et al., Phys.Lett. 164B (1985) 347.
- 18) G.Martinelli and C.T.Sachrajda, CERN-TH.5042 (May 1988).
- 19) J.Ashman et al. (EMC Collaboration), Phys.Lett. B206 (1988) 364.
- 20) B.Meyer and C.Smith, Phys.Lett. 123B (1983) 62.
- 21) G.Martinelli and Y.C.Zhang, Phys.Lett. 123B (1983) 77.
- 22) R.Groot, J.Hoek and J.Smit, Nucl.Phys. B237 (1984) 111.
- 23) J.Mandula, G.Zweig and J.Govaerts, Nucl.Phys. B228 (1983) 9.
- 24) G.Martinelli and Zhang Yi Cheng, Phys.Lett. B125 (1983) 7.
- 25) S.R.Amendolia et al., Nucl.Phys. B277 (1986) 168.
- 26) C.J.Bebek et al., Phys.Rev. D17 (1987) 1693 and refs. therein.
- 27) C.Newman et al., Phys.Rev.Lett. 42 (1979) 951.
- 28) J.Badier et al., Proc. EPS Conference on HEP, Geneva (1979) p.751.
- 29) R.Barate et al., Phys.Rev.Lett. 43 (1979) 1541.
- 30) B.Betev et al., Z.Phys. C28 (1985) 15.
- 31) J.D.Bjorken and S.D.Drell, "Relativistic Quantum Mechanics" (Mc Graw Hill, 1965)
- 32) G.Höler et al., Nucl.Phys. B114 (1976) 505 and references therein.
- 33) G.Höler, F.Kaiser, R.Koch and E.Pietarinen, "Handbook of pion-nucleon scattering" (Fachs-information-Zentrum, Karlsruhe, 1979) p.427 and references therein.
- 34) L.Maiani, G.Martinelli, M.L.Paciello, B.Taglienti, Nucl.Phys. B293 (1987) 420.
- 35) S.Güsken et al., CERN-TH 5027/88 (1988).
- 36) Bacilieri et al., (Ape Collaboration), presented by F.Rapuano at the XXIII HEP Conference, Munich, August 1988.
- 37) G.Martinelli, Phys.Lett. 141B (1984) 395.

- 38) C.Bernard, A.Soni and T.Draper, Phys.Rev. D36 (1987) 363.
- 39) G.Curci, E.Franco, L.Maiani and G.Martinelli, Phys.Lett. 202B (1988) 363.
- 40) C.Bernard, T.Draper, G.Hockney and A.Soni, Talk given by C.Bernard at the "Int. Symposium: Field Theory on the Lattice", Seillac (France), Nucl.Phys. B (Proc. Sup.) 4 (1988) 483.
- 41) N.Cabibbo, G.Martinelli and R.Petronzio, Nucl.Phys. B244 (1984) 381.
- 42) R.C.Brower, M.B.Gavela, R.Gupta and G.Maturana, Phys.Rev.Lett. 53 (1984) 1318.
- 43) L.Maiani and G.Martinelli, Phys.Lett. B181 (1986) 344.
- 44) J.F.Donoghue, E.Golowich and B.R.Holstein, Phys.Lett. B199 (1986) 442.
- 45) M.B.Gavela, L.Maiani, G.Martinelli, O.Pène and S.Petrarca, Phys.Lett. 211B (1988) 139.
- 46) M.B.Gavela, L.Maiani, G.Martinelli, O.Pène and S.Petrarca, Talk given by L.Maiani at the "Int. Symposium: Field Theory on the Lattice", Seillac (France), Nucl.Phys. B (Proc. Sup.) 4 (1988) 466.
- 47) M.A.Shifman, a.I.Vainshtein and V.J.Zakharov, Nucl.Phys. B120 (1977) 346; Sov.Phys. Jept. 45 (1977) 670.
- 48) J.Bijens and M.B.Wise, Phys.Lett B137 (1984) 245; J.F.Donoghue, E.Golowich, B.R.Holstein and J.Trampetic, Phys.Lett. B179 (1986) 361 and B188 (1987) 511; A.Buras and J.M.Gérard, Phys.Lett B192 (1987) 156; S.Sharpe, Phys.Lett. B194 (1987) 551.
- 49) E.Franco, L.Maiani, G.Martinelli and A.Morelli, CERN-TH 5101 (1988).
- 50) L.Wolfenstein, Phys.Rev.Lett. 51 (1983) 1945.
- 51) R.Gupta, G.Guralnik, W.Kilcup, A.Patel and S.R.Sharpe, Phys.Lett. B192 (1987) 149.