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SPACING OF ENERGY LEVELS IN TWO-BODY SYSTEMS

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Abstract

We study the spacing of energy levels in two-body systems and discuss some implications for quarkonium physics. The case of a perturbation around the harmonic oscillator is supplemented, whenever possible, by a non-perturbative study, at least in the WKB approximation. We first compare the location of a level with angular momentum $\ell+1$ to the average of the neighbouring levels with angular momentum ℓ . We also study the spacing between the various radial excitations at fixed angular momentum $\ell = 0$.

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I. Introduction

In the previous paper [1], two of us (J.M.R. and P.T.) studied the level ordering for three-body systems. In the particular case where the binding is achieved by two-body potentials close to the harmonic oscillator (h.o.), the splitting of the first excitations with positive parity depends on the sign of the quantity

$$Y(r) = \frac{d}{dr} \frac{1}{r} \frac{dV}{dr} \quad (I.1)$$

If $Y(r)$ is positive (negative) for all r , the potential is a convex (concave) function of r^2 . For two-body systems, this quantity appears in one of the theorems on the order of the energy levels : if one characterises the eigenvalues $E(n, \ell)$ by n , number of nodes of the radial wave-function, and ℓ , orbital angular momentum, then [2]

$$Y(r) \gtrless 0 \quad \forall r > 0 \Rightarrow E(n, \ell) \gtrless E(n-1, \ell+2) \quad (I.2)$$

The quantity $Y(r)$ also controls to some extent the spacing of energy levels. The harmonic oscillator (h.o.) potential has two remarkable properties :

$$E(n, \ell+1) = \frac{1}{2} [E(n, \ell) + E(n, \ell+2)] \quad (I.3)$$

$$E(n, \ell+1) = \frac{1}{2} [E(n, \ell) + E(n+1, \ell)] \quad (I.4)$$

By combination, one can also get that, for fixed angular momentum, the successive radial excitations are equally spaced.

Property (I.3) can be rephrased by saying that Regge trajectories of the h.o. potential are linear. For different potentials, in the special case $n = 0$, (I.3) is replaced by the theorem :

$$Y(r) \gtrless 0 \quad \forall r > 0 \Rightarrow E(0, \ell) \text{ concave (convex) in } \ell \quad (I.5)$$

This result, first established for perturbations around h.o. [3] and recently proved in general [4], has been used in the previous paper [1] on three-body systems. We think that a more systematic study of the level spacing in 2-body systems might be useful both for the 2-body and 3-body physics.

First, in section II, we shall be concerned with what replaces (I.4) for general potentials. We shall show that for perturbations around h.o.

$$Y(r) \geq 0 \quad \forall r > 0 \Rightarrow E(n, \ell+1) \leq \frac{1}{2} [E(n, \ell) + E(n+1, \ell)] \quad (\text{I.6})$$

For $n = 0$, the result holds outside the framework of perturbation theory. For $n > 0$, (I.6) is also established in the limit of ℓ very large.

In section III, we shall discuss the problem of level spacing for fixed ℓ , in particular $\ell = 0$, and there, we shall be surprised to see that it is not the sign of the quantity $Y(r)$ which controls the variations of the spacing with n . We have found two different sufficient conditions to guarantee that $E(n, 0)$ is concave in n . The first condition, which involves a third order differential operator, holds for perturbations around h.o. and breaks down for some large deviations from h.o. The second condition, which is just that V is concave in r , i.e. $V'' < 0$, is only established in the WKB approximation but we shall give some extra support to our belief that this condition is in fact valid in general.

II. Spacing involving different angular momenta

To establish property (I.6), we use a strategy developed by H. Grosse and one of us (A.M.) to connect various neighbouring h.o. levels by raising and lowering operators [2]. For a potential $V = \frac{r^2}{4}$, they are :

$$\begin{aligned} B_\ell^+ &= + \frac{d}{dr} - \frac{\ell+1}{r} + \frac{r}{2} \\ B_\ell^+ u_{n+1, \ell} &= \gamma_{n, \ell} u_{n, \ell+1} \\ B_\ell^- u_{n, \ell+1} &= \gamma_{n, \ell} u_{n+1, \ell} \end{aligned} \quad (\text{II.1})$$

$$\gamma_{n, \ell}^2 = 2n+2$$

and

$$\begin{aligned} C_\ell^+ &= + \frac{d}{dr} - \frac{\ell+1}{r} - \frac{r}{2} \\ C_\ell^+ u_{n, \ell} &= \beta_{n, \ell} u_{n, \ell+1} \\ C_\ell^- u_{n, \ell+1} &= \beta_{n, \ell} u_{n, \ell} \\ \beta_{n, \ell}^2 &= 2n+2\ell + 3 \end{aligned} \quad (\text{II.2})$$

Therefore, if we take a potential $V = \frac{r^2}{4} + \lambda v$, we have, to lowest order in perturbation

$$2E(n, \ell+1) - E(n, \ell) - E(n+1, \ell) = \int_0^\infty v dr \left[2u_{n, \ell+1}^2 - \left(\frac{C_\ell u_{n, \ell+1}}{\beta_{n, \ell}} \right)^2 - \left(\frac{B_\ell u_{n, \ell+1}}{\gamma_{n, \ell}} \right)^2 \right] \quad (\text{II.3})$$

After making integrations by parts, using the Schrödinger equation

$$\left[\frac{d^2}{dr^2} - \frac{(\ell+1)(\ell+2)}{r^2} + 2n + \ell + \frac{5}{2} - \frac{r^2}{4} \right] u_{n, \ell+1} = 0 \quad (\text{II.4})$$

one finds that the above quantity is proportional to $(u = u_{n, \ell+1})$

$$- (4n + \ell + 5) \int_0^\infty u^2 dr \left(v'' - \frac{v'}{r} \right) - (2\ell + 1) \int_0^\infty u^2 dr \frac{v'}{r} (r^2 - (4n + 2\ell + 5)) \quad (\text{II.5})$$

However, from the virial theorem, one has

$$\int_0^\infty u^2 r^2 dr = (4n + 2\ell + 5) \int_0^\infty u^2 dr \quad (\text{II.6})$$

and hence (II.5) can be written as

$$- (4n + \ell + 5) \int_0^\infty u^2 dr \frac{d}{dr} \left(\frac{v'}{r} \right) - (2\ell + 1) \int_0^\infty u^2 dr (r^2 - r_o^2) \left(\frac{v'(r)}{r} - \frac{v'(r_o)}{r_o} \right) \quad (\text{II.7})$$

where $r_o^2 = 4n + 2\ell + 5$. Hence, if $\frac{d}{dr} \frac{1}{r} \frac{dv}{dr}$ is positive (negative), (II.3) is negative (positive).

One question is whether this property is valid outside perturbation theory. In the case of the lowest lying levels the answer is affirmative :

$$Y(r) \geq 0 \Rightarrow E(n=0, \ell+1) \leq \frac{1}{2} [E(n=1, \ell) + E(n=0, \ell)] \quad (\text{II.8})$$

The proof uses the combination of the theorem (I.5) on the concavity in ℓ of the $n = 0$ Regge trajectories and of theorem (I.1). If for instance $Y(r) < 0$

$$E(0, \ell+1) > \frac{1}{2} [E(0, \ell) + E(0, \ell+1)] > \frac{1}{2} [E(0, \ell) + E(1, \ell)] \quad (\text{II.9})$$

Furthermore, we have indications that the property (II.8) goes beyond perturbations around $V = r^2$. For very large ℓ , the effective potential $\ell(\ell+1) r^{-2} + V(r)$ can be approximated by an harmonic oscillator potential, at least if $V(r)$ is sufficiently smooth. Then we have, for n finite, ℓ large :

$$E(n, \ell) \approx V(r_0) + \frac{r_0 V'(r_0)}{2} + (2n+1) \sqrt{\frac{1}{2} \left(V''(r_0) + \frac{3V'(r_0)}{r_0} \right)} \quad (\text{II.10})$$

where r_0 is a minimum of V_{eff} given by

$$r_0^3 V'(r_0) = 2 \ell(\ell+1) \quad (\text{II.11})$$

In addition, since

$$\frac{dE}{d\ell} = (2\ell+1) \int_0^\infty \frac{u^2}{r^2} dr \quad (\text{II.12})$$

for these sharply localized states we have

$$E(n, \ell+1) \approx E(n, \ell) + \frac{2\ell+1}{r_0^2} = E(n, \ell) + \sqrt{\frac{2V'(r_0)}{r_0}} \quad (\text{II.13})$$

and hence

$$2E(n, \ell+1) - E(n, \ell) - E(n+1, \ell) \approx 2\sqrt{\frac{2V'(r_0)}{r_0}} - 2\sqrt{\frac{1}{2} \left(V''(r_0) + \frac{3V'(r_0)}{r_0} \right)} \quad (\text{II.14})$$

which is positive if $Y(r) < 0$ and $V' > 0$. For $Y(r) > 0$, we get the opposite conclusion.

At this point, let us comment on the behaviour of $E(n, \ell)$ as a function of ℓ , at fixed n . For ℓ sufficiently large, concavity holds if $Y < 0$. Indeed,

$$\frac{d^2 E}{d\ell^2} = \frac{dr_0}{d\ell} \frac{d}{dr_0} \left(\frac{dE}{d\ell} \right) \approx \frac{dr_0}{d\ell} \frac{d}{dr_0} \left(\sqrt{\frac{2V'(r_0)}{r_0}} \right) \quad (\text{II.15})$$

so that, if ℓ is large enough and $Y < 0$, we have

$$2E(n, \ell) > E(n, \ell-1) + E(n, \ell+1) \quad (\text{II.16})$$

It is not possible, however, to claim that $E(n, \ell)$ is concave in ℓ for low values of ℓ , at fixed $n > 0$, provided that $Y(r) < 0$. This is shown in the following counterexample. Take the case $n = 1$, i.e. the first daughter trajectory. Then, after integrations by parts,

$E(1,2)+E(1,0) - 2E(1,1)$ can be shown to be proportional to

$$\int_0^{\infty} Y(r) (r^4 - 10r^2 + 21) \exp - \frac{r^2}{2} dr \quad (\text{II.17})$$

The parenthesis is negative for $\sqrt{3} < r < \sqrt{7}$. It is easy to construct a potential with $Y(r) < 0$ in this interval and $Y(r) = 0$ outside. Yet we see that $E(1,\ell)$ is not concave.

A first application of these results can be given for quarkonium physics. In that case, lattice QCD predicts [5] that the potential between (infinitely heavy) quarks is concave in r^2 , i.e. $Y(r) < 0$. So, neglecting spins, we get for the $c\bar{c}$ system.

$$2 E_{\chi_c}(\ell=1) > E_{\psi'}(\ell=0, n=1) + E_{J/\psi}(\ell=0, n=0) \quad (\text{II.18})$$

which is indeed true ($2 \times 3.52 < 3.07 + 3.66$ GeV experimentally [6], after spin averaging). If we are convinced that this applies also to $n > 0$, we get for the Y system

$$\begin{aligned} 2 E_{\chi_b} &> E_{Y'} + E_Y \\ 2 E_{\chi_b} &> E_{Y'} + E_Y \end{aligned} \quad (\text{II.19})$$

III. Spacing between levels at fixed ℓ

As mentioned at the beginning, a striking property of the h.o. is that fixed ℓ levels are equally spaced, in particular $\ell = 0$ levels. In quarkonium physics, $\ell = 0$ levels are the most directly accessible by e^+e^- experiments, and reproducing their spacing is a crucial constraint on potential models. Our goal here is to find simple sufficient conditions on the potential to ensure that the spacing between energy levels increases or decreases with the number of nodes n . We are aware that such a question cannot leave an unique answer. The knowledge of all $\ell = 0$ energy levels, indeed, does not fix uniquely the potential, even in the case where this potential is confining one has to know in addition the wave-function at the origin [7].

III.1 Perturbations around the harmonic oscillator ($\ell = 0$).

We started our investigations by looking at the vicinity of the h.o. potential, with the hope that the sign of $Y(r)$ would decide whether the spacing would increase or decrease with n . This turned out to be wrong, as shown in explicit examples with $n = 0$, $n = 1$ and $n = 2$.

A posteriori it is easy to understand why the sign of $Y(r)$ does not suffice to fix the behaviour of the spacing. There are three linearly independent perturbations of $V = r^2$ which preserve equal spacing :

$$v = A + B r^2 + C r^{-2} \quad (\text{III.1.1})$$

The first two are obvious. The third is equivalent to changing the angular momentum of all levels by some fixed amount, and, since $\frac{dE}{d\ell}$ is independent of n for the harmonic oscillator, the spacing remains constant. However, if $v = c r^{-2}$, Y is not zero.

A quantity which vanishes for any perturbation of the form (III.1.1) is

$$Z(r) = \frac{d}{dr} r^5 \frac{d}{dr} \frac{1}{r} \frac{dv}{dr} \quad (\text{III.1.2})$$

We shall proceed to prove that if $Z(r)$ is positive (negative) and $\lim_{r \rightarrow 0} r^3 v = 0$, the spacing of the $\ell = 0$ increases (decreases) with n , v being a perturbation to r^2 . If we denote for simplicity $u_{n,(\ell=0)}$ by u_n and postpone the discussion of the case of fixed $\ell \neq 0$, we have to calculate

$$\Delta_n \equiv E_{n+1} + E_{n-1} - 2 E_n = \int_0^\infty v (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr \quad (\text{III.1.3})$$

The strategy consists of performing a series of partial integrations to exhibit the quantity $Z(r)$ defined in eq. (III.1.2). Here we shall only sketch the successive steps. Details will be given in Appendix A.

We first rewrite Δ_n as

$$\Delta_n = \lim_{r \rightarrow 0} I(r) v(r) + \int_0^\infty I(r) v'(r) dr \quad (\text{III.1.4})$$

$$\text{where} \quad I(r) = \int_r^\infty (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr' \quad (\text{III.1.5})$$

However, $I(0) = 0$ since a constant v should give $\Delta_n = 0$. Hence

$$I(r) = - \int_0^r (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr' \quad (\text{III.1.6})$$

and if $r^3 v \rightarrow 0$ as $r \rightarrow 0$, we can drop the all integrated term. We repeat the operation,

introducing

$$J(r) = \int_r^\infty r' I(r') dr' \quad (\text{III.1.7})$$

then

$$\Delta_n = \lim_{r \rightarrow 0} J(r) \frac{v'(r)}{r} + \int_0^\infty J(r) \frac{d}{dr} \left(\frac{v'(r)}{r} \right) dr \quad (\text{III.1.8})$$

Again, $J(0) = 0$ since a potential $v = r^2$ should make $\Delta = 0$. Then J behaves like r^5 at the origin and if we provisionally impose $\lim_{r \rightarrow 0} r^4 \frac{dv}{dr} = 0$, the integrated term goes away. In the Appendix A, we shall give explicit examples showing that even if $Y(r)$ has a definite sign, Δ_n has not a fixed sign. The next step consists of writing

$$\Delta_n = \lim_{r \rightarrow 0} K(r) r^5 \frac{d}{dr} \frac{v'(r)}{r} + \int_0^\infty K(r) \frac{d}{dr} \left(r^5 \frac{d}{dr} \frac{v'(r)}{r} \right) dr \quad (\text{III.1.9})$$

where

$$K(r) = \int_r^\infty \frac{J(r')}{r'^5} dr' \quad (\text{III.1.10})$$

with $K(0) = 0$ since Δ_n vanishes for $v = \frac{1}{r^2}$. If we assume (again provisionally!) $\lim_{r \rightarrow 0} r^5 \frac{d^2 v}{dr^2} = 0$, the integrated term goes away and we get the desired

$$\Delta_n = \int_0^\infty K(r) Z(r) dr \quad (\text{III.1.11})$$

It is painful but straightforward to calculate $K(r)$ in terms of u_n and u'_n only, using the raising and lowering operators

$$(2n+2 - \frac{r^2}{2} + r \frac{d}{dr}) u_n = \sqrt{(2n+2)(2n+3)} u_{n+1} \quad (\text{III.1.12})$$

$$(2n+3 - \frac{r^2}{2} - r \frac{d}{dr}) u_{n+1} = \sqrt{(2n+2)(2n+3)} u_n \quad (\text{III.1.13})$$

and the radial equation

$$-u''_n + \left(\frac{r^2}{4} - E_n\right) u_n = 0 \quad (\text{III.1.14})$$

where $E_n = 2n + \frac{3}{2}$. One gets

$$\begin{aligned} 2n(2n+1)(2n+2)(2n+3) K(r) &= u_n'^2 \left(\frac{3}{4r^3} + \frac{E_n}{2r} \right) \\ &+ u_n'^2 \left(-\frac{3E_n}{4r^3} - \frac{3}{16r} + \frac{E_n^2}{2r} - \frac{E_n r}{8} \right) + u_n u_n' \left(-\frac{3}{4r^4} + \frac{E_n}{2r^2} \right) \end{aligned} \quad (\text{III.1.15})$$

The miracle, described in Appendix A is that this quadratic form in u_n and u_n' is positive everywhere. We can also get rid of the conditions $\lim_{r \rightarrow 0} r^4 \frac{dv}{dr} = 0$ and $\lim_{r \rightarrow 0} r^5 \frac{d^2v}{dr^2} = 0$. Indeed, if $Z(r)$ has a constant sign, $r^5 \frac{d}{dr} \frac{1}{r} \frac{dv}{dr}$ is monotonous and has therefore a constant sign for, say, $0 < r < r_0$, then the same is true for $\frac{1}{r} \frac{dv}{dr}$ and hence, since

$$v(2r) - v(r) = \int_r^{2r} \left(\frac{1}{r'} \frac{dv}{dr'} \right) r' dr' \quad (\text{III.1.16})$$

we have either

$$\left| \frac{1}{r} \frac{dv}{dr} \right| < \frac{|v(2r) - v(r)|}{\frac{3r^2}{2}} \quad (\text{III.1.17})$$

or

$$\left| \frac{1}{r} \frac{dv}{dr} \right| < \frac{\left| v(r) - v\left(\frac{r}{2}\right) \right|}{\frac{3r^2}{8}} \quad (\text{III.1.18})$$

depending on the sign of $\frac{dv}{dr}$ and $\frac{d}{dr} \frac{1}{r} \frac{dv}{dr}$ and hence $\lim_{r \rightarrow 0} r^4 \frac{dv}{dr} = 0$ if $\lim_{r \rightarrow 0} r^3 v = 0$. The same kind of proof applies to the second derivative.

In summary, for a perturbation $v(r)$ around r^2 , the spacing of $\ell = 0$ levels increases (decreases) with n if $\frac{d}{dr} \left[r^5 \frac{d}{dr} \left(\frac{1}{r} \frac{dv}{dr} \right) \right] \gtrless 0$ and $\lim_{r \rightarrow 0} r^3 v = 0$.

$$Z(r) \gtrless 0 \quad \forall r > 0 \text{ and } \lim_{r \rightarrow 0} r^3 v = 0 \Rightarrow \Delta \gtrless 0 \quad (\text{III.1.19})$$

III.2 Beyond perturbation around the harmonic oscillator

The above result does not hold beyond perturbation around h.o., as shown in the following examples.

First, consider the non-monotonic potential $V = \lambda(r^3 - r^2)$ which, for large λ , exhibits a well-pronounced pocket of attraction near $r_0 = 2/3$. Simple numerical investigations show that, for λ large enough, $\delta(n) \equiv E(n+1,0) - E(n,0)$ is first decreasing as long as $E(n+1,0)$ is negative and then, for larger n , increasing as a function of n .

An explicit analytical example can be given using the potential $V = r^6 - (2j+3)r^2$, with j odd, for which Turbiner [8] has shown that the first $\frac{j+1}{2}$ levels are given by solving an algebraic equation. In Appendix B, we give a "pedestrian" proof of this fact and show that these $\frac{j+1}{2}$ levels are symmetric around $E = 0$. For instance, for $j = 9$, the 5 first levels are given by $E = \pm 4 [6 (5 \pm \sqrt{11})]^{1/2}$ and $E = 0$, and the spacings are approximately 15.56, 12.68, 12.68 and 15.56, i.e. decreasing first and increasing, though again $Z(r)$ is positive.

The second class of examples consists of potentials $v(r) = A/r^2 + \omega(r)$. For ω small enough, non-perturbative treatment of the $\ell = 0$ level with $r^2 + v(r)$ is reduced to perturbation of h.o. levels with angular momentum ℓ such that $\ell(\ell+1) = A$. Explicit calculation shows that eq. (III.1.11) is replaced by

$$E_{0,\ell} + E_{2,\ell} - 2 E_{1,\ell} \propto \int_0^\infty r^{2\ell+1} (r^2 - 2\ell + 1) Z(r) \exp - \frac{r^2}{2} dr \quad (\text{III.2.1})$$

whose kernel clearly exhibits a node for $\ell > \frac{1}{2}$. One can thus choose as counterexample any potential $\omega(r)$ for which $Z(r)$ is positive but localized at short distances, say $r < \sqrt{\ell - 1/2}$.

III.3 The WKB approximation

Since the sign of $Z(r)$ does not guarantee that the spacing of the $\ell = 0$ varies monotonically, when the potential is too far away for the h.o., we try another line of attack, based on the WKB approximation. Though the errors in this approximation are difficult to control rigourously, they are usually very small for smooth potentials.

For a monotonously increasing potential, the WKB quantification condition is [9]

$$n - \frac{1}{4} = \frac{1}{\pi} \int_0^{R_T} \sqrt{E - V} \, dr \quad (\text{III.3.1})$$

where R_T is the unique turning point. This defines n as a continuous function of E . The variation of the spacing, Δ_n , as defined in eq. (III.1.3) is given approximately by

$$\Delta_n \approx \frac{d^2 E}{dn^2} = - \frac{d^2 n}{dE^2} / \left(\frac{dn}{dE} \right)^3 \quad (\text{III.3.2})$$

From (III.3.1) we have

$$\frac{dn}{dE} = \frac{1}{\pi} \int_0^{R_T} \frac{dr}{2\sqrt{E - V}} \quad (\text{III.3.3})$$

by differentiation under the integral. However we cannot repeat this a second time because we would get two infinities coming from the differentiation under the integral and with respect to the end point of integration. We have to perform a convenient integration by parts. We did this in 2 ways which lead to 2 different sufficient conditions for the decrease of the spacing

a) we can write

$$\frac{dn}{dE} = \frac{1}{2\pi} \int_0^{R_T} \frac{V' \, dr}{V' \sqrt{E - V}} = \frac{1}{\pi} \frac{\sqrt{E - V(0)}}{V'(0)} - \frac{1}{\pi} \int_0^{R_T} \sqrt{E - V} \frac{V''}{V'^2} dr \quad (\text{III.3.4})$$

and

$$\frac{d^2 n}{dE^2} = \frac{1}{2\pi} \frac{1}{V'(0) \sqrt{E - V(0)}} - \frac{1}{2\pi} \int_0^{R_T} \frac{dr}{\sqrt{E - V}} \frac{V''}{V'^2} \quad (\text{III.3.5})$$

Therefore, if WKB is valid, $\frac{d^2 n}{dE^2} > 0$ if $V''(r) < 0 \, \forall r > 0$, i.e. the spacing decreases if V is concave.

This is a very nice condition, because the potential between infinitely heavy quarks derived from QCD is precisely concave [5]. We shall indicate later the reasons why we believe that this condition is in fact truly valid, though we have not yet a full proof.

b) we can also write

$$\frac{dn}{dE} = \frac{1}{\pi} \int_0^1 \frac{dy}{\sqrt{E} \sqrt{1-y^2}} \frac{1}{\frac{d}{dr} \sqrt{\frac{V}{E}}} \quad (\text{III.3.6})$$

where $y = \sqrt{\frac{V}{E}}$, and then we get, integrating by parts

$$\frac{dn}{dE} = \frac{2\sqrt{V}}{\pi V'} \text{Arc sin } \sqrt{\frac{V}{E}} \Big|_0^{R_T} - \frac{1}{\pi} \int_0^{R_T} \text{Arc sin } \sqrt{\frac{V}{E}} \frac{V'^2 - 2 V V''}{V'^2 \sqrt{V}} dr \quad (\text{III.3.7})$$

Assume now that $V(0) = 0$ (more generally, if $V(0)$ is finite, we can replace V by $V - V(0)$) and that $V/V' \rightarrow 0$ for $r \rightarrow 0$, then

$$\frac{dn}{dE} = -\frac{1}{\pi} \int_0^{R_T} \text{Arc sin } \sqrt{\frac{V}{E}} \frac{V'^2 - 2 V V''}{V'^2 \sqrt{V}} dr \quad (\text{III.3.8})$$

and

$$\frac{d^2n}{dE^2} = \frac{1}{\pi E} \int_0^{R_T} \frac{dr}{\sqrt{E - V}} \frac{V'^2 - 2 V V''}{V'^2 \sqrt{V}} \quad (\text{III.3.9})$$

Therefore, in WKB approximation, the level spacing decreases with n if $V(0)$ is finite, $(V - V(0))/V' \rightarrow 0$ for $r \rightarrow 0$ and $V'^2 - 2 V V'' > 0 \quad \forall r > 0$. It increases if $V'^2 - 2 V V'' < 0$.

One implication is that for power potentials, $V = r^\alpha$, the level spacing decreases for $0 < \alpha < 2$ (for $0 < \alpha < 1$ use a)) and increases for $\alpha > 2$.

We have checked, by solving numerically the Schrödinger equation for pure powers $\alpha = 0.5, 1, 1.2, 1.8$ that this is indeed true up to $n = 10$. Furthermore, the WKB approximation is seen to be good to better than 1% for $n = 1$, 10% for $n = 3$, and the accuracy on the quantity $(2 E_n - E_{n-1} - E_{n+1})$ is better than 5% for $n = 2$, 30% for $n = 5$. We have therefore an extremely strong suspicion that the above statement on power potentials is completely true.

Let us now return to the sufficient condition $V'' < 0$ which guarantees, if we

believe WKB, that the level spacing decreases with increasing n , and explain why we are convinced that this condition is really sufficient.

First of all, the limiting case of a linear potential $V = r$ is such that the spacing between levels decreases. The proof is incredibly simple. For any level, the reduced wave-function is the Airy function with some change of origin, $u_n(r) = A_i(r + x_n)$. The energies are the absolute values of the zeros x_n of $A_i(x)$, and the spacing between these zeros decreases when x becomes more negative, because the effective potential becomes deeper. It is strange that such a simple argument cannot be propagated to any potential with $V'' < 0$, but so far we failed. What we did is very inelegant : consider a perturbation to the linear potential

$$V = r + \lambda v \quad (\text{III.3.10})$$

then

$$\Delta_n \equiv E_{n+1} + E_{n-1} - 2 E_n = \Delta_n(V = r) + \lambda \delta \Delta_n \quad (\text{III.3.11})$$

We know already that $\Delta_n(V = r) < 0$, and we want to prove that $\delta \Delta_n < 0$ if $v'' < 0$, so that the decrease of the spacing is still accentuated. To this end, we write

$$\begin{aligned} \delta \Delta_n &= \int_0^\infty v (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr = - \int_0^\infty v' dr \int_0^r (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr' \\ &= \int_0^\infty v'' dr \int_0^r dr' \int_0^{r'} (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr'' \end{aligned} \quad (\text{III.3.12})$$

So we have to prove that

$$F_n(r) \equiv \int_0^r dr' \int_0^{r'} (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr'' > 0 \quad (\text{III.3.13})$$

This quantity can be written in closed form in terms of the Airy function and its derivative since

$$u_n(r) = A_i(r - E_n) / A_i'(-E_n) \quad (\text{III.3.14})$$

and

$$\int_0^r dr' \int_0^{r'} u_n^2 dr'' = \frac{2}{3} \left[\frac{A_i^2(r - E_n) + (E_n - r) A_i^2(r - E_n) - 1/2 A_i(r - E_n) A_i'(r - E_n)}{A_i^2(-E_n)} - E_n \right] \quad (\text{III.3.15})$$

It has been checked numerically that $F_n(r)$ is positive for $2 \leq n \leq 9$ and $0 \leq r \leq 12$, i.e. beyond the turning point of u_9 at $r = 11.93601...$ It is clear, on the other hand, that since $A_i(x) \rightarrow 0$ for $x \rightarrow \infty$, and since $\Delta_n(V = r) = E_{n+1} + E_{n-1} - 2E_n < 0$, more precisely $\Delta_n(V=r) \approx \frac{-\pi^2}{2E_n^2}$ for n large, $F_n(r)$ is positive for large r .

For n large, we can use asymptotic expansions for F_n . We have asymptotic expansions covering 3 regions

1) $r \geq E_{n-1}$. Then

$$F_n \approx \frac{\pi^3}{E_n^{3/2}} [A_i(r - E_n)]^2 + \frac{\pi^3}{3 E_n^2} \quad (\text{III.3.16})$$

manifestly positive. It is excellent already for $n = 7$, giving an error of less than 1% for $r = E_n$.

2) $2(E_n - E_{n-1}) < r < E_{n-1}$. Then

$$F_n \approx \frac{2\pi^2}{3E_n^2} \left[\frac{3}{4t} + \frac{1}{2} - \frac{9t}{4} + t^3 + \frac{3}{8\pi^2} \frac{1 - \cos 2\pi t}{t^3} \sin \left(\frac{4}{3} (E_n - r)^{3/2} \right) \right] \quad (\text{III.3.17})$$

with $t = \left(\frac{E_n - r}{E_n} \right)^{1/2}$

Again, it can be proved that this expression is positive. For $n = 9$, the error in the whole interval is less than 15%.

3) $r < 2(E_1 - E_{n-1})$. Defining with $\rho = r \sqrt{E_n}$

$$F_n \approx \frac{\pi^2}{2E_n^2} \left[\frac{5\rho^2}{4} + \frac{7}{4} (\cos 2\rho - 1) + \frac{5}{4} \rho \sin^2 \rho - \frac{\rho^4}{4} \cos^2 \rho \right] \quad (\text{III.3.18})$$

It can be proved to be positive and the error is less than 15% for $n = 9$.

IV. Summary and conclusion

We first established that, for $n = 0$

$$Y = \frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{dV}{dr} \geq 0 \Rightarrow E(n, \ell+1) \leq \frac{1}{2} [E(n, \ell) + E(n+1, \ell)] \quad (\text{IV.1})$$

For $n > 0$, no counterexample has been found, and, in fact, this result is proved both in the limit of very large ℓ and, with any ℓ and n , for perturbations around h.o.

Next, we studied the spacing between levels at fixed $\ell = 0$. For perturbation around h.o.

$$Z = \frac{d}{dr} \left[r^5 \frac{d}{dr} \left(\frac{1}{r} \frac{dv}{dr} \right) \right] \geq 0 \text{ and } \lim r^3 v = 0 \Rightarrow \Delta_n \geq 0 \quad (\text{IV.2})$$

when $\Delta_n = E_{n+1} + E_{n-1} - 2 E_n$. Explicit (and even analytic) counterexamples show that this property does not hold beyond perturbation theory. The WKB approximation and the study of the pure linear and perturbed linear potentials strongly suggest that $V'' < 0$ should suffice to ensure $\Delta_n < 0$, but no rigorous proof has been found.

We have the feeling that dramatic simplifications or generalizations are still possible in this field, since some studies involve tedious calculations for relatively simple results.

As far as physics is concerned, tests of inequalities can be done with the P level of Charmonium and 1P and 2P states of Upsilon (χ_b and $\chi_{b'}$). In the future, antiproton - proton experiments will hopefully provide new levels of ($c\bar{c}$), such as 1D_2 and 3D_2 . Electromagnetic cascades in Upsilon factories should also give the mass of ($b\bar{b}$) D states. The spacing of $\ell = 0$ levels is directly relevant for the sequences of quarkonium poles seen in e^+e^- experiments, and our inequalities could be tested with Toponium levels if the top mass is not too high.

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APPENDIX A

1) Calculation of $I(r)$, $J(r)$ and $K(r)$

The general strategy to calculate

$$I(R) = \int_R^\infty (u_{n+1}^2 + u_{n-1}^2 - 2u_n^2) dr \quad (A-1)$$

and

$$J(R) = \int_R^\infty I(r) dr \quad (A-2)$$

consists of using the raising and lowering operators (III.1.12-13) to express everything in terms of the u_n and u'_n (hereafter noted u and u'). Then, after some partial integration, one has to calculate integrals of the type

$$\int_R^\infty u^2 r^{2n} dr, \quad \int_R^\infty u u' r^{2n} dr, \quad \int_R^\infty u'^2 r^{2n} dr \quad (A-3)$$

The second one is obviously reduced to the first one. The technique to calculate the first and the last ones is inspired by the method used to prove the virial theorem by combining

$$\int_R^\infty r^{2n-1} \left[-u'' + \left(\frac{r^2}{4} - E \right) u \right] dr = 0 \quad (A-4)$$

$$\text{and} \quad \int_R^\infty \left[2r^{2n-1} - u'' + \left(\frac{r^2}{4} - E \right) u \right] dr = 0 \quad (A-5)$$

One can express $\int_R^\infty u^2 r^{2k} dr$ in terms of $\int_R^\infty u^2 r^{2k-2} dr$ and $\int_R^\infty u^2 r^{2k-4} dr$ plus some boundary value terms. By a descending process, one should end with a term $\int_R^\infty u^2 dr$ which cannot be integrated. However these terms should disappear from $I(R)$ and $J(R)$ since $I(0) = J(0) = 0$. The same method holds for $\int_R^\infty u'^2 r^{2k} dr$.

The results are, omitting a factor $[2n(2n+1)(2n+2)(2n+3)]^{-1}$

$$I(R) = 2(4n+3) \left[uu' - u^2 \frac{R^3}{4} - u'^2 R \right] + 2(4n^2+6n+3) \left(\frac{R}{2} u^2 - R^2 uu' \right) \quad (A-6)$$

$$J(R) = (u'R - u) (4EuR + 3u') + REu^2 + E^3 R^3 u^2 \quad (A-7)$$

To calculate now $K(R) = \int_R^\infty \frac{J(r) dr}{r^5}$, one has to calculate $\int_R^\infty \frac{u^2 dr}{r^4}$ and $\int_R^\infty \frac{u'^2 dr}{r^4}$.

A priori, one would think that these integrals would be expressed in terms of $\int_R^\infty \frac{u^2 dr}{r^2}$

and $\int_R^\infty u^2 dr$. However, another miracle is that $\int_R^\infty u^2 dr$ disappears, since

$$3 \int_R^\infty \frac{u^2 dr}{r^4} = u^2 \left(\frac{1}{R^3} + \frac{E}{R} - \frac{R}{4} \right) + \frac{uu'}{R^2} + \frac{u'^2}{R} - 2E \int_R^\infty \frac{u^2 dr}{r^2} \quad (A-8)$$

$$\text{and} \quad -3 \int_R^\infty \left(\frac{u'^2}{r^4} + \frac{Eu^2}{r^4} \right) dr + \frac{u'^2}{R^3} - \frac{u^2}{4R} + \int_R^\infty \frac{u^2 dr}{r^2} = 0 \quad (A-9)$$

Of course, since $K(0) = 0$, $\int_R^\infty \frac{u^2 dr}{r^2}$ disappears from the expression of K which is

$$K(R) \propto \left(\frac{3}{4R^3} u'^2 + \frac{E}{2R} \right) + \left(-\frac{3E}{4R^3} \right) u^2 - \frac{3}{16R} + \frac{E^2}{2R} - \frac{ER}{8} + \left(-\frac{3}{4R^4} + \frac{E}{2R^2} \right) uu' \quad (A-10)$$

2) Positivity of K

i) Sign of the discriminant of the quadratic form

$$\Delta = \left(-\frac{3}{4R^4} + \frac{E}{2R^2} \right)^2 - 4 \left(\frac{3}{4R^3} + \frac{E}{2R} \right) \left(-\frac{3E}{4R^3} - \frac{3}{16R} + \frac{E^2}{2R} - \frac{ER}{8} \right) \quad (A-11)$$

We shall see that in most of the "classical region", which is $0 < R < 2\sqrt{E}$, the discriminant is negative. After introducing $\lambda = ER^2$, we have

$$4R^8 \Delta = R^4 \left(\lambda + \frac{3}{2} \right)^2 - 4 \left(\lambda - \frac{3}{2} \right) \left(\lambda^2 + \frac{5\lambda}{4} + \frac{3}{8} \right) \quad (A-12)$$

For $\lambda = \frac{3}{2}$, Δ is clearly positive. However, if $\lambda > \frac{3}{2}$, we have

$$\lambda^2 + \frac{5\lambda}{4} + \frac{3}{8} > \frac{1}{2}(\lambda + \frac{3}{2})^2 \quad (\text{A-13})$$

hence Δ is negative for $\lambda > \frac{3}{2}$ and

$$R^4 - 2(\lambda - \frac{3}{2}) < 0 \quad (\text{A-14})$$

$$\text{i.e.} \quad R^4 - 2ER^2 + 3 < 0 \quad (\text{A-15})$$

$$\text{or} \quad \frac{3}{E + (E^2 - 3)^{1/2}} < R^2 < E + (E^2 - 3)^{1/2} \quad (\text{A-16})$$

However we also have

$$(\lambda - \frac{3}{2})(\lambda^2 + \frac{5\lambda}{4} + \frac{3}{8}) > (\lambda + \frac{3}{2})^2(\lambda - \frac{11}{4}) \quad (\text{A-17})$$

for $\lambda > \frac{11}{4}$, i.e. $R^2 > \frac{11}{4E}$. Therefore $\Delta < 0$ if $R^4 - 4(\lambda - \frac{11}{4}) < 0$, which gives

$$\frac{11}{2E + (4E^2 - 11)^{1/2}} < R^2 < 2E + (4E^2 - 11)^{1/2} \quad (\text{A-18})$$

From now we shall take $n \geq 2$, i.e. $E \geq \frac{11}{2}$. Then

$$\frac{n}{2E + (4E^2 - 11)^{1/2}} < E + (E^2 - 3)^{1/2} \quad (\text{A-19})$$

so that Δ is negative for

$$\frac{3}{E + (E^2 - 3)^{1/2}} < R^2 < 2E + (4E^2 - 11)^{1/2} \quad (\text{A-20})$$

i.e. in all the classical region except for a small interval in R^2 of the order of $\frac{3}{2E}$ near the origin and a small interval of the order of $\frac{11}{4E}$ near the turning point. Since the coefficient of u'^2 is positive, this means that K is positive in this interval.

ii) Positivity of K beyond the turning point

In fact, it is easier to look directly at the positivity of $u_{n+1}^2 + u_{n-1}^2 - 2u_n^2$. Then, by integration, the positivity of K follows. Using the raising and lowering operators (III.1.10) we can write

$$\begin{aligned} u_{n+1}^2 + u_{n-1}^2 - 2u_n^2 = & \left[\frac{(2n+2-r^2/2)^2}{(2n+2)(2n+3)} + \frac{(2n+1-r^2/2)^2}{2n(2n+1)} - 2 \right] u_n^2 \\ & + \left[\frac{1}{(2n+2)(2n+3)} + \frac{1}{2n(2n+1)} \right] r u_n'^2 \\ & + 2 \left[\frac{2n+2-r^2/2}{(2n+2)(2n+3)} - \frac{2n+1-r^2/2}{2n(2n+1)} \right] r u_n' u_n \end{aligned} \quad (A-21)$$

The discriminant of the quadratic form is proportional to $4(E^2 + \frac{3}{4}) - (2E - r^2)^2$ which is negative for

$$r^2 > 2(E^2 + \frac{3}{4})^{1/2} + 2E \quad (A-22)$$

iii) Neighbourhood of the turning point

First, we study the interval $4E < r^2 < 4E + \frac{3}{8E}$ where $u'' > 0$. Hence for $r > R$

$$u(r) > u(R) - (r-R) |u'(R)| \quad (A-23)$$

since

$$u'^2(R) - (\frac{R^2}{4} - E) u^2(R) = \int_R^\infty \frac{r}{2} u^2 dr \quad (A-24)$$

we have

$$u'^2(R) > \frac{R}{2} \int_R^{\frac{R}{2} + \left| \frac{u(R)}{u'(R)} \right|} \left[u(R) - (r-R) |u'(R)| \right]^2 dr \quad (A-25)$$

Hence we get

$$\left| \frac{u'(R)}{u(R)} \right|^3 > \frac{R}{6} \quad (A-26)$$

so that, for $4E < r < 4E + \frac{3}{8E} < 5E$, we get

$$u^2 K(R) > \left(\frac{R}{6}\right)^{\frac{2}{3}} \left(\frac{R}{10}\right) - \frac{9}{16R} - \frac{1}{8} \left(\frac{R}{6}\right)^{\frac{1}{3}} \quad (\text{A-27})$$

so that, since $E > \frac{11}{2}$ and $R > 4.69$, $K(R)$ is positive.

Consider now the interval $4E - \frac{3}{E} < R^2 < 4E$ where $u'' < 0$. If we call u_T and u'_T the values of u and u' at the turning point, we have

$$u < u_T + |u'_T| (R_T - R)$$

$$|u''| < \left[u_T + |u'_T| (R_T - R) \right] \frac{3}{4E} \quad (\text{A-28})$$

$$|u'| > u'_T - \frac{3}{4E} \left[u_T + |u'_T| (R_T - R) \right] (R_T - R)$$

and hence

$$\left| \frac{u'}{u} \right| > \left(\frac{R_T}{6} \right)^{\frac{1}{3}} - (R_T - R) \frac{3}{4E} \left[1 + \left(\frac{R_T}{6} \right)^{\frac{1}{3}} (R_T - R) \right] \quad (\text{A-29})$$

with $R_T^2 - R^2 < \frac{3}{E}$ and $R_T > 4.69$, $E > \frac{11}{2}$, we get $\left| \frac{u'}{u} \right| > 0.912$ and

$$u^2 K(R) > \frac{E}{2R} \left[\left| \frac{u'}{u} \right|^2 - \frac{3}{2R^2} - \left| \frac{u'}{u} \right| \frac{1}{R} \right] \quad (\text{A-30})$$

which is positive.

iv) Neighbourhood of the origin

Here it is sufficient to prove that $u_{n+1}^2 + u_{n-1}^2 - 2u_n^2$ is negative near the origin, specifically for $R^2 < \frac{1.54}{E}$ if we want the proof to hold for $E \geq \frac{11}{2}$. Then, by successive integrations from 0 to R , we can prove that K is positive.

Using the raising and lowering operators and noting that $u_n \approx r u'_n$ near the origin, we get

$$\frac{u_{n+1}}{u_n} \approx \left(\frac{2n+3}{2n+2} \right)^{\frac{1}{2}} \quad \text{and} \quad \frac{u_n}{u_{n-1}} \approx \left(\frac{2n+1}{2n} \right)^{\frac{1}{2}} \quad (\text{A-31})$$

and hence $u_{n+1}^2 + u_{n-1}^2 - 2u_n^2$ is proportional to $-[(2n+1)(2n+2)]^{-1}$ which is negative.

However the rigorous proof is more difficult. We use again the expression of $u_{n+1}^2 + u_{n-1}^2 - 2u_n^2$ in terms of u_n^2 and $u'_n{}^2$ which becomes, after some manipulation

$$\begin{aligned} X = \frac{u_n^2}{2n(2n+3)} [3 - (4n+3)r^2] + \frac{2(2n+3/2)^2 - 3/2}{2n(2n+1)(2n+2)(2n+3)} \left[\frac{r^4}{4} u_n^2 + (ru'_n)^2 \right] \\ - \frac{2u_n ru'_n}{2n(2n+3)} \left[3 - \frac{r^2(4n+3)}{(2n+1)(2n+2)} \right] \end{aligned} \quad (\text{A-32})$$

Notice that since $r^2 < \frac{1.54}{E} = \frac{1.54}{2n+3/2}$, u_n and u'_n are positive because the first maximum of u_n lies at $r^2 > \frac{\pi^2}{4E}$. The quantity $\frac{X}{u_n^2}$ is a quadratic form in $x = \frac{ru'_n}{u_n}$, where x can be shown to be less than unity for $r^2 E < \pi$. We have

$$\begin{aligned} \frac{d}{dx} \left[\frac{X}{u_n^2} \right] = \\ 2x \frac{2(2n+3/2)^2 - 3/2}{2n(2n+1)(2n+2)(2n+3)} - \frac{2}{2n(2n+3)} \left[3 - \frac{(4n+3)r^2}{(2n+1)(2n+2)} \right] \end{aligned} \quad (\text{A-33})$$

Because of the restriction on r^2 , the sign of this quantity will be negative if

$$\frac{2(2n+3/2)^2 - 3/2}{(2n+1)(2n+2)} < \left[3 - \frac{3.08}{(2n+1)(2n+2)} \right] \quad (\text{A-34})$$

For $n \geq 2$ this inequality is satisfied, hence $\frac{X}{u_n^2}$ is maximized by taking $\frac{ru'_n}{u_n}$ as small as possible.

Comparing

$$-u''_0 + E u_0 = 0 \quad (\text{A-35})$$

and
$$-u'' + \left(E - \frac{R^2}{4}\right)u = 0 \quad (\text{A-36})$$

we prove that, for $r < \frac{\pi}{2\sqrt{E}}$

$$\frac{ru'}{u} > \frac{r\sqrt{E} \cos(r\sqrt{E})}{\sin(r\sqrt{E})} > C = 0.424 \quad (\text{A-37})$$

if $r^2E > 1.54$.

For a fixed value of $\frac{ru'}{u}$, $\frac{X}{u^2}$ is a quadratic form in r^2 . Consider first the interval $1 < r^2E < 1.54$ and replace $\frac{ru'}{u}$ by C .

At $r^2E = 1$ and $r^2E = 1.54$, we find that $\frac{X}{u^2}$ is negative at both extremities, for $n \geq 2$. Then we consider the interval $0 < r^2E < 1$; then, we can take $\frac{ru'}{u} = 0.64$, and we find that $\frac{X}{u^2}$ is negative at both extremities. Hence, $\frac{X}{u^2}$ is negative for $r^2E < 1$ and $n \geq 2$.

For $n = 1$, an explicit calculation gives (see eq. III.2.1)

$$K(r) = \text{const } (r + r^3) \exp - \frac{r^2}{2} \quad (\text{A-38})$$

obviously positive.

Notice that for $n = 1$

$$J(r) = \text{const } (r^9 - 2r^7 - r^5) \exp - \frac{r^2}{2} \quad (\text{A-39})$$

which is positive for r large and negative for r small. Therefore even if $\frac{d}{dr} \frac{1}{r} \frac{dv}{dr}$ has a constant sign, we cannot draw conclusions on the spacing of the levels.

APPENDIX B First levels of the potential $V = r^6 - (5+4k) r^2$

Consider the one-dimensional potential $V = x^6 - (3 + \lambda)x^2$ and let us denote its wave-function as $\phi(x) = P(x) \exp -\frac{x^4}{4}$. The Schrödinger equation becomes

$$P'' - 2x^3 P' + (E + \lambda x^2) P = 0 \quad (\text{B-1})$$

when odd and even solutions can be searched for separately. Polynomials of degree j can be solutions if the terms of highest degree cancel, requiring $\lambda = 2j$. From (B-1) one gets a descending recursion relation for the coefficients of P . Requiring that the recursion terminates leads to an algebraic equation for E of degree $\frac{j}{2} + 1$ or $\frac{j+1}{2}$ depending on the parity of j . These solutions have at most j nodes, hence they are the lowest eigenstates in the even parity (j even) or odd parity (j odd) sector.

Also, if $P(x)$ is a polynomial solution with energy E , $Q(x) = P(ix)$, the polynomial obtained by flipping the sign of every second coefficient is a solution with eigenvalue $-E$, so that the lowest part of the spectrum (in the appropriate parity sector) is symmetric with respect to $E = 0$.

The odd values $j = 2k+1$ are relevant for the $\ell = 0$ sector of the spherically symmetric potential $r^6 - (4k + 5) r^2$. One obtains explicitly

$$k = 0 \quad V = r^6 - 5r^2 \quad E_0 = 0 \quad P_0 = r \quad (\text{B-2})$$

$$k = 1 \quad V = r^6 - 9r^2 \quad E_0, E_1 = \pm 2\sqrt{6} \quad P_0, P_1 = r \left(\pm \frac{3}{\sqrt{6}} + r^2 \right) \quad (\text{B-3})$$

$$k = 2 \quad V = r^6 - 13r^2 \quad E_0, E_2 = \pm 8\sqrt{2}$$

$$E_1 = 0 \quad (\text{B-4})$$

$$k = 3 \quad V = r^6 - 17r^2 \quad E_0, E_3 = \pm \sqrt{200 + 16\sqrt{109}}$$

$$E_1, E_2 = \pm \sqrt{200 - 16\sqrt{109}} \quad (\text{B-5})$$

with for instance

$$P_0, P_3 = \pm \frac{315 (\sqrt{109}-2) r}{\sqrt{200+16\sqrt{109}}} + \frac{105 r^3}{2(\sqrt{109}-2)} \pm \frac{42 (\sqrt{109}+8) (\sqrt{109}-2)r^5}{\sqrt{200+16\sqrt{109}}} + r^7 \quad (\text{B-6})$$

$$k = 4 \quad V = r^6 - 21 r^2 \quad E_0, E_4 = \pm \sqrt{480+96\sqrt{11}}$$

$$E_1, E_3 = \pm \sqrt{480-96\sqrt{11}}$$

$$E_2 = 0 \quad (\text{B-7})$$

In each case the higher levels exhibit increasing spacing.

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