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A GRASSMANNIAN FORMULATION OF THE WZW AND MINIMAL MODELS

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ABSTRACT

Grassmannians provide a geometrical interpretation of infinite dimensional Kac-Moody and Virasoro algebras. Here we show how to obtain, using this formalism, the values of the central charges and the conformal dimensions (weights) corresponding to the WZW and minimal models. This is achieved by suitably generalizing the procedure which gave the central charge and the conformal dimension of the b-c system. In analogy with this previous case we get a prescription to compute the correlation functions of such models on Riemann surfaces of arbitrary genus.

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In order to understand better the nature of conformal field theories (CFT's), it is useful to develop tools that can help us in relating the algebraic aspect of the theory to the geometric one: the infinite dimensional Grassmannian manifold is just one of these tools. In the context of the study of CFT's, Grassmannians were first introduced by Alvarez-Gaumé and collaborators and by Witten.^[1]

Here we will follow a somewhat different point of view^[2,3] using Grassmannians to explicitly build Virasoro algebras. From the Borel-Weil theorem, we know that sections of the dual of the determinant line bundle, built over the Grassmannian, are irreducible representations of an infinite dimensional Kac-Moody algebra.[†] The content of the previous sentence is tautological, in the sense that it is so by construction.

Now, as it was shown by Krichever,^[4] to every point of the Grassmannian we can associate a geometrical datum constituted by a Riemann surface, a point on it, a coordinate system in the neighbourhood of such point, a line bundle and a trivialization : this makes the geometrical content even more explicit. So, in principle, we can build any irreducible representation of the Virasoro algebra and attach a geometrical meaning to it. This kind of reasoning has an additional bonus, because it gives us a prescription for computing correlation functions. It is in fact well known that an equivalent way of describing a Virasoro algebra is via the so-called vertex functions.* Then any correlation function of an arbitrary number of conformal fields is equivalent, after bosonization, to the action of the corresponding vertices upon a certain function called the τ function of the theory. The vertices corresponding to the Kac-Moody algebras are summarized in Ref.3 where the interested reader can also find a discussion of the τ functions corresponding to these algebras.

The purpose of this letter is to build explicitly those representations of the Virasoro algebra corresponding to the WZW and minimal models and give a Grassmannian interpretation, i.e. build the geometrical datum. All this will be done by generalizing the construction which gave us the central charge and weight of the b-c system (see Ref.2). If we manage to do this then we will be able to decide immediately the vertices and τ 's needed to perform the computation of the correlation functions.[‡] Historically, the first examples of Virasoro

† These are the positive energy representations. With very little extra work it can be demonstrated that these representations are also acted upon by the Virasoro algebra.

* This fact is better known among physicists by the name of boson-fermion equivalence.

‡ This doesn't imply that this computation is trivial. It only means that we have a general principle to perform a feasible computation of correlators in the WZW and minimal models.

algebras were given in terms of Heisenberg oscillators. More recently the same algebras have been built using the semi-infinite wedge representation: for an up-to-date overview see Kac and Raina.^[5]

Using the notations of Ref.5 we now discuss the general setting. If we define the Virasoro generators as:

$$L_n^a = \frac{1}{2} \sum_{j \in \mathcal{Z}} a_{-j} a_{j+n} + \eta_a n Q a_n, \quad (1)$$

where the a_n 's are Heisenberg oscillators satisfying the commutation relations $[a_m, a_n] = m \delta_{m, -n}$, $[a_n, L_m] = n a_{m+n}$, $\eta_a \equiv (1, i)$, $Q \in \mathcal{R}$ and use the well-known commutation rule of the L_n 's, we get the central charges $c_a = 1 + 12(\eta_{3-a})^2 Q^2$. Notice that $(L_n^2)^\dagger = L_{-n}^2$, implying that this representation is unitary.

As Fock space we introduce $C[x_1, x_2 \dots]$ the space of polynomials in infinitely many variables $x_1, x_2, \dots$.

We now define:

$$\begin{aligned} a_n &= \frac{\partial}{\partial x_n}, \\ a_{-n} &= n x_n. \end{aligned} \quad (2)$$

These operators clearly satisfy the commutation rules of the Heisenberg algebra. It is now possible to build an isomorphism between this representation and the semi-infinite wedge representation. This is the so-called bosonization.

The semi-infinite wedge representation is very well suited to fit the Grassmannian formalism, as the states of this representation are also sections of the determinant line bundle built over the Grassmannian. Given an infinite dimensional complex vector space V and a fixed basis $\{v_j\}$ with $j \in \mathcal{Z}$, the infinite wedge space is the vector space whose basis are the sets of the type:

$$v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \dots \quad (3)$$

with $i_0 > i_{-1} > \dots$ and $i_k = k$ for $k \ll 0$.[‡] We now briefly recall from Ref.2 how to build semi-infinite wedge representations of the Virasoro algebra and compute their central

[‡] In the bosonic representation this state corresponds to $S_{i_0, i_{-1}+1, \dots}(x) = \det(S_{i_k+j-k}(x))$ where $S_i(x)$ is a Shur polynomial.

charges and weights: a basis in Hilbert space is given by the monomials $\{z^j\}$ with $j \in \mathcal{Z}$. If we define the Virasoro generators as :

$$L_n = z^{n+1} \frac{\partial}{\partial z} + \lambda(n+1)z^n, \quad (4)$$

put the basis $\{z^j\}$ back into (3) and have $L_0 = 2[L_1, L_{-1}]$ and $[L_2, L_{-2}]$ act on it, we obtain the central charge $c = -2(6\lambda^2 - 6\lambda + 1)$ and the conformal dimension $h = \frac{1}{2}(2\lambda - Q)(1 - Q)$, where Q is the external charge that is, the index of the Grassmannian. This construction gives the representations of the Virasoro algebra over the space V_λ of regular differentials of degree λ on C^* . The basis of V_λ is the set $\nu_k = \{z^k dz^\lambda\}$ with $k \in \mathcal{Z}$.^[6] This gives a complete description of the algebra generated by a system of free fermions of weight λ with $\lambda \in \frac{1}{2}\mathcal{Z}$. Employing this formalism, we can also compute the correlation functions, recovering the results found in literature.^[7]

How can we obtain a description of the WZW and minimal models along these lines? The first relevant modification is to build our representations not on V_λ but on the space $V_{\alpha,\lambda}$ of "densities" of the form $P(z)z^\alpha dz^\lambda$ where $P(z)$ are arbitrary polynomials in z, z^{-1} and $\alpha, \lambda \in \mathcal{C}$, according to Ref.5. A basis in this space is given by $\{\nu_k = z^{k+\alpha} dz^\lambda\}$ with $k \in \mathcal{Z}$. Let's now look at the transformation properties of these vectors under an infinitesimal transformation of coordinates $\zeta(z) = z + \varepsilon(z)$ with $\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1}$:

$$\begin{aligned} \nu'_k(\zeta^{-1}(z)) &\approx (1 - (k + \alpha) \sum_{n=-\infty}^{\infty} \varepsilon_n z^n) (1 - \lambda \sum_{n=-\infty}^{\infty} \varepsilon_n (n+1) z^n) z^{k+\alpha} = \\ &= -\varepsilon(z) \frac{\partial \nu_k(z)}{\partial z} - \lambda \frac{\partial \varepsilon(z)}{\partial z} \nu_k(z). \end{aligned} \quad (5)$$

These vectors transform as conformal fields.^[8] Moreover:

$$[\nu_k, L_n] = z^{n+1} \frac{\partial \nu_k}{\partial z} + \lambda(n+1)z^n \nu_k. \quad (6)$$

A point of paramount importance for the rest of our argument is that (6) holds even if we neglect the second term on the r.h.s. of (4). The weights now become:

$$h = \frac{1}{2}(\alpha - Q)(\alpha + 2\lambda - 1 - Q). \quad (7)$$

Let's suppose to choose the geometry of the sphere; then $Q = (2\lambda - 1)^\diamond$ and we recover the

[◊] We remind the reader that on surfaces with genus $g > 1$ the external charge is given by the Riemann-Roch theorem : $Q = (2\lambda - 1)(g - 1)$

conformal dimensions of Dotsenko and Fateev.^[9]

Now we have to match the values of the central charges. This can be achieved by taking $\lambda = \frac{p}{2p'} \in \mathcal{Q}$ with $p - p' = 1$; then $Q = \frac{(p-p')}{p'}$.^[10] Defining the Virasoro generators as:

$$L_n = z^{n+1} \frac{\partial}{\partial z} + \frac{1}{2} (\sqrt{\frac{p'}{p}} Q + 1) (n+1) z^n, \quad (8)$$

we get the correct central charges after a rescaling of α and Q by $\sqrt{2}$. Putting the coefficient of the second term on the r.h.s. of (8) in (7) and rescaling α once again leads to the correct conformal dimensions. How to connect a geometrical datum to this procedure will be dealt with later on.

Recovering the central charges and weights of the WZW models is more complicated. We start by setting notations according to Ginsparg.^[11] The central charge and weights are:

$$c = \frac{k \dim G}{(k+x)}, \quad (9)$$

$$h_{(r)} = \frac{C_{(r)}}{\psi^2(k+x)} = \frac{l_{(r)} \dim G}{\psi^2 d_{(r)}(k+x)},$$

where $\dim G$ is the dimension of the group, x the Coxeter number, ψ the highest root, k is the level, $C_{(r)}$ the Casimir of the r representation and $d_{(r)}$ its dimension. Moreover if $t_{(r)}^a$ is a generator of such representation: $\text{Tr } t_{(r)}^a t_{(r)}^b = l_{(r)} \delta^{ab}$ and $C_{(r)} d_{(r)} = l_{(r)} \dim G$. We now introduce the Virasoro generators:

$$L_n = z^{n+1} \frac{\partial}{\partial z} + \frac{1}{2} (i \sqrt{\frac{p'}{p}} Q + 1) (n+1) z^n, \quad (10)$$

where i is the imaginary unit and $p - p' = 1$. We remark that in terms of oscillators, this algebra was first studied by Chodos and Thorn.^[12] The Virasoro generators in (10) satisfy the unitarity condition $L_n^\dagger = L_{-n}$ and give a central charge $c = 1 + 3Q^2$. To get the WZW values of the central charge we can now tensor this representation with those obtained by (8) with $\lambda = 0$ and $\lambda = \frac{1}{2}$ and then tensor again all this $\dim G$ $k(p-1)$ times. Setting $p = 6(k+x)$ and tuning α in the formula of the weights for this representation we then recover (9).

This way of proceeding is not the most convenient if we have in mind the computation of correlators. The representations given by (8) are not in fact unitary. An alternative is

given by tensoring $\dim G \, k \, (p-1)$ times the representation given by (10) and computing its correlator. The central charge of this representation is now $c = c_{WZW} + \dim G \, k \, (p-1) c_{SU(2)}$ where the last $SU(2)$ is in level 1. All the representations involved in this last formula are now unitary. We now remind the reader that, given two complex spaces V_1, V_2 , if two representations in V_1, V_2 are unitary, we can define a Hermitian form on $V_1 \otimes V_2$ by:

$$\langle v_1 \otimes v_2 | w_1 \otimes w_2 \rangle = \langle v_1 | w_1 \rangle \langle v_2 | w_2 \rangle. \quad (11)$$

As the correlators for the $SU(2)$ level-one WZW models are known, they can be modded out leading to the correlator we were looking for.

The last delicate point is how to cancel correctly the charge Q in the correlators of fields with the same conformal dimension given by the representation (10). It is easy to be convinced that this can be done by a slight modification of the Dotsenko-Fateev procedure employing two types of vertices: one of conformal dimension one and the other with charge given by the α parameter set to reproduce the correct dimension of the field.

We now turn our attention towards a problem of a more mathematical nature: we want to assign to the representations discussed above a geometrical datum. To do that we have to build a line bundle whose sections are dz^λ differentials with $\lambda \in \mathcal{Q}$. Let's restrict ourselves, for the sake of simplicity and definiteness, to a simple geometry: the sphere S^2 . In no case will this choice affect the generality of our discussion: all our arguments and conclusion can then be extended to more complicated geometries. As is well known, if we denote by K the canonical line bundle, on S^2 it is only possible to define a line bundle $\mathcal{L}$ such that $\mathcal{L}^2 = K$. Let's briefly review how to do it: the starting point is the definition of K . On S^2 we may choose just two neighbourhoods which cover the northern and southern hemispheres with complex homogeneous coordinates $U_0 = (1, z)$ and $U_1 = (z', 1)$.^[13] To determine the divisor of K (which is described by dz in U_0) we change coordinates to get $dz' = d(z^{-1}) = -z^{-2}dz$. Then, denoting by $O(-2)$ the sheaf of germs of holomorphic cross-sections obtained in this case,^[14] the existence of $O(-1)$, that is, of $K^{\frac{1}{2}}$, is immediately inferred. Before generalizing all this to show the existence of a line bundle $K^{-\frac{1}{q}}$, we need to recall a few notions about algebraic curves.^[15]

• This is not the only way to write this equation. Remembering (for example) that at level 1 $c_{SU(n)} = n-1$ we could rewrite it as $c = c_{WZW} + \dim G \, k \, c_{SU(p)}$.

We focus our attention on the intersection, at $P = (a, b)$, of a curve C with equation $f(z_0, z_1) = 0$ and a line L with parametric equations $z_0 = a + \gamma t, z_1 = b + \beta t$. Substituting the parametric equations of the line in the equation of the curve and expanding in a Taylor series in t , we get:

$$(f_{z_0}\gamma + f_{z_1}\beta)t + \frac{1}{2}(f_{z_0 z_0}\gamma^2 + 2f_{z_0 z_1}\gamma\beta + f_{z_1 z_1}\beta^2)t^2 + \dots = 0, \quad (12)$$

where $f_{z_0}, f_{z_1}, \dots$ are the derivatives of f evaluated at P . We have now two possibilities:

1. $f_{z_0}, f_{z_1} \neq 0$. L has a single intersection at P with C , with the exception of the tangent given by $f_{z_0}\gamma + f_{z_1}\beta = 0$.
2. All the $(r - 1)$ -th derivatives of f at P vanish but at least one r -th derivative at P is $\neq 0$. The equations of the tangents are given by the roots of:

$$f_{z_0^r}\gamma^r + \binom{r}{1}f_{z_0^{r-1}z_1}\gamma^{r-1}\beta + \dots + \binom{r}{r}f_{z_1^r}\beta^r = 0.$$

Multiple solutions of this last equation define coincident tangents with the same multiplicities. P is said to be a point of multiplicity r . If $r > 1$, P is said to be singular. A point of C with multiplicity one is called a simple point, one of multiplicity two a double point etc. A point of multiplicity r is ordinary if its r tangents are distinct. Before going back to our original argument we need to define the genus g of C :

$$g = \frac{(n-1)(n-2)}{2} - \sum_i \frac{r_i(r_i-1)}{2}, \quad (13)$$

where i runs over all singular points (of multiplicity r_i) and n is the degree of the polynomial defining C .

We can now go back to our original problem of defining a line bundle $\mathcal{L}$ such that $\mathcal{L}^p = K$ on a curve C with $g = 0$, that is a sphere. To define such a line bundle we need a curve described by a polynomial of degree p but, at the same time, we have to satisfy the constraint $g = 0$. This can be achieved by assigning to C one or more singular points with multiplicities sufficient to cancel the first term on the r.h.s. of (13). The simplest example of such a curve can be found in CP^2 :

$$z_0^p - z_1^{p-1}z_2 = 0. \quad (14)$$

A suitable parametrization of this curve is given by $(1, t, t^{-p+1})$. By repeating the argument given before in the CP^1 case we get: $d(t^{-p+1}) = (-p+1)t^{-p}dt$, which is what we wanted, a

pole of order p .^{*} We can now ask ourselves which are the new basis vectors that span the Hilbert space of this new vacuum of our theory. Doing a Puiseux expansion^[16] we obtain the set $\{z^{\frac{1}{p}}, z^{\frac{2}{p}}, \dots, z, z^{1+\frac{1}{p}}, \dots\}$.[†] Substituting this basis in (3) and computing the central charge using the set of Virasoro generators defined in (4) (or in (8) and (10)) we get p times the values of the central charges computed in the first part of this work. This result is not a surprise, as we already found it in Ref.3 where we considered a p -covering of the Hilbert space on which we defined our Grassmannian. We also remark that similar formulas appear in the study of correlation functions on hyperelliptic surfaces.^[17] The main difference with respect to this work is that our approach is obviously local.

Before concluding we would like to make a final observation. To compute the correlators of the WZW model other groups^[18] have formulated this problem in terms of the Riemann monodromy problem or the Riemann-Hilbert problem. As discussed in a series of papers by a Japanese group,^[19] all these different formulations can naturally be compounded in the Grassmannian formalism. It would be interesting to explicate this observation in more detail.

Summary

Central charges $c = 1 \pm 3Q^2$ can be recovered by a little modification of the standard description of the Virasoro algebra by Heisenberg's oscillators (see (1)). We then build the semi-infinite wedge representations isomorphic to these oscillators representations. This allows us to exploit the Grassmannian formalism to get prescriptions to compute the correlation functions of the WZW and minimal models.

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^{*} We warn the reader that our argument has to be regarded only as showing the plausibility of a rigorous proof along these lines.

[†] The Puiseux expansion is basically a Taylor expansion around a singular point of multiplicity r .

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