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ON THE APPROACH TO THE COSMOLOGICAL SINGULARITY
IN QUADRATIC THEORIES OF GRAVITY: THE KASNER REGIMES

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ABSTRACT

We study the Kasner-type solutions of the vacuum field equations of D -dimensional theories of gravity which derive from Lagrangians that are non-linear in the curvature. We treat both the generic quadratic case which yields field equations of fourth order in the metric and the Lovelock theory whose field equations remain second order. We show that in four-dimensional space-times the standard Kasner solution is always a solution (and often the only one) whereas in $D > 4$ all spatial dimensions can always contract as one goes backward in time, contrary to what happens in Einsteinian gravity. The bearing of these results on the conjecture that the chaotic approach to the Big Bang singularity is specific to four-dimensional space-times is briefly discussed.

INTRODUCTION

The question as to whether the curvature singularity of the Friedmann solution of Einstein's gravitational field equations is specific to its spatial homogeneity and isotropy or is a common feature of a more general class of cosmological models has been extensively studied, either in the context of the singularity theorems of general relativity (see, e.g., [1,2]) or in the course of actually building the "general" solution of Einstein's equations near a spacelike singularity [3,4] where a chaotic approach to the singularity of, e.g., the curvature invariants could be shown [5,6].

These results that pertain to highly curved regions of space-time were obtained in the framework of Einstein's theory of gravity. However, in these regimes, general relativity must probably be replaced by some unified theory of the interactions and it appears that some of the modifications to be expected deal with the dimensionality of space-time which need no longer be four (as in Kaluza-Klein theories), or the Lagrangian of the theory, which need no longer be linear in the curvature. In particular, it may include the so-called Lanczos or Gauss-Bonnet term, as suggested by the low-energy limit of superstring theories [7,8]. It seems therefore worth looking at whether the above-mentioned chaotic approach to the Big Bang singularity is altered by such modifications.

The rôle in this problem of the dimensionality D of space-time was studied in a series of papers [9-13] with the interesting result that when $D > 11$ the approach loses its chaotic character. The rôle now in this problem of the higher order terms in the Lagrangian was, as far as we know, only touched upon in [14,14a,15] and a thorough analysis in the spirit of [9-13] remains to be done.

The purpose of this paper is to initiate such an analysis. Because they are the starting point of any further study [16], we investigate in some detail the Kasner-type cosmological models of D -dimensional theories of gravity in vacuo derived from non-linear Lagrangians. We begin with the Lovelock theory and show that there always exists a range of values of the Kasner parameters such that all spatial dimensions contract as one goes towards the singularity, in sharp contrast with D -dimensional Einsteinian gravity where there is always at least one dimension expanding. We then turn to the generic quadratic case and first consider the linear combination $\alpha R^2 + \beta R_{AB} R^{AB}$ of the scalar curvature R and the Ricci tensor R_{AB} , the only one to be considered in four-dimensional space-times because of the Gauss-Bonnet theorem. We show that apart from two particular theories [purely "scalar" ($\beta = 0$) and Weyl's ($\beta = -3\alpha$)], the standard Kasner solution (" $\Sigma p_i = \Sigma p_i^2 = 1$ ") is always a solution to the field equations; moreover it is the only one (of that type) if $\beta = k\alpha$, $k \notin [-3, -1]$. Finally, we add a term quadratic in the Riemann

tensor to the Lagrangian ($\propto R_{ABCD} R^{ABCD}$), in order to render the theory generic in $D > 4$ dimensional space-times where the Gauss-Bonnet theorem no longer holds, and we recover the behaviour already encountered in the Lovelock theory: there is a range of values of the Kasner parameters such that all spatial dimensions contract.

We conclude with some speculations about the chaotic or non-chaotic character of the approach to the singularity that these results may imply.

BIANCHI I UNIVERSES IN THE LOVELOCK THEORY IN $D = 5$, $D = 6$ DIMENSIONS

Consider the Lanczos Lagrangian for pure gravity:

$$\mathcal{L} = \sqrt{-g} (\alpha_0 + \alpha_1 R + L_{(2)}) \quad (1)$$

with

$$L_{(2)} = R_{ABCD} R^{ABCD} - 4 R_{AB} R^{AB} + R^2 \quad (2)$$

to which the Lovelock theory of gravity based on all possible Euler forms in D dimensions reduces in $D \leq 6$. (We refer to [17,18] for notations and definitions.) When the geometry is restricted to be of the Bianchi type I:

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} e^{2\sigma_i(t)} dx_i dx^i \quad (3)$$

the non-vanishing components of the Riemann tensor are ($\cdot \equiv d/dt$):

$$R_{i0i0} = -e^{2\sigma_i} (\ddot{\sigma}_i + \dot{\sigma}_i^2) ; R_{ijij} = e^{2(\sigma_i + \sigma_j)} \dot{\sigma}_i \dot{\sigma}_j \quad (4)$$

so that we have:

$$\mathcal{L}_{(1)} \equiv \sqrt{-g} R = 2 \left(e^{\sum \sigma} \sum_i \dot{\sigma}_i \right)' + e^{\sum \sigma} \bar{L}_1 \quad (5a)$$

with

$$\bar{L}_1 = -2 \sum_{i < j} \dot{\sigma}_i \dot{\sigma}_j \quad (5b)$$

and, after some algebra:

$$\mathcal{L}_{(2)} \equiv \sqrt{-g} L_{(2)} = 8 \left(e^{\Sigma\sigma} \sum_{i < j < k} \dot{\sigma}_i \dot{\sigma}_j \dot{\sigma}_k \right) + e^{\Sigma\sigma} \bar{L}_2 \quad (6a)$$

with

$$\bar{L}_2 = -8 \sum_{i < j < k < l} \dot{\sigma}_i \dot{\sigma}_j \dot{\sigma}_k \dot{\sigma}_l . \quad (6b)$$

Thus when $D = 4$, $\bar{L}_{(2)} = 0$ for shortage of available indices, and we recover that in four-dimensional space-times the Gauss-Bonnet combination is a pure divergence. In $D = 5$, $\bar{L}_{(2)}$ has only one term: $\bar{L}_{(2)} = -8\dot{\sigma}_1\dot{\sigma}_2\dot{\sigma}_3\dot{\sigma}_4$, in $D = 6$ it has five and so on.

The corresponding field equations can be obtained either from the Lanczos field equations derived from Eqs. (1) and (2) (see, e.g., [19,20]) or by a Euler variation of Eqs. (5) and (6), the constraint equation in the latter case being obtained as usual by introducing in $\bar{\mathcal{L}} = \sqrt{-g} (\alpha_0 + \alpha_1 \bar{L}_1 + \alpha_2 \bar{L}_2)$, a lapse function σ_0 : $d/dt \rightarrow e^{-\sigma_0} d/dt$, $\sqrt{-g} \rightarrow e^{\Sigma\sigma + \sigma_0}$, and varying with respect to σ_0 which is then set equal to zero. We checked explicitly that the two methods yield the same field equations:

$$\alpha_0 - \alpha_1 \bar{L}_1 - 3\alpha_2 \bar{L}_2 = 0 , \quad (\text{constraint}) \quad (7)$$

$$\begin{aligned} \alpha_0 + 2\alpha_1 \left[\sum_{\substack{j=1 \\ j \neq i}}^{D-1} (\ddot{\sigma}_j + \dot{\sigma}_j^2) + \sum_{\substack{j,k \neq i \\ j < k}} \dot{\sigma}_j \dot{\sigma}_k \right] + \\ + 8\alpha_2 \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{D-1} \left[(\ddot{\sigma}_j + \dot{\sigma}_j^2) \sum_{\substack{k,l \neq i,j \\ k < l}} \dot{\sigma}_k \dot{\sigma}_l \right] + 3 \sum_{\substack{j,k,l \neq i \\ j < k < l < m}} \dot{\sigma}_j \dot{\sigma}_k \dot{\sigma}_l \dot{\sigma}_m \right\} = 0 \end{aligned} \quad (8i)$$

As required by the Gauss-Bonnet theorem, the Einstein and Lanczos tensors [terms in α_1 and α_2 respectively in (7) and (8)] vanish identically in two and four dimensions respectively.

We now specialize even further by looking for Kasner-type solutions of the system (7) and (8i):

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} t^{2p_i} dx_i dx_i ; \quad (\sigma_i \equiv p_i \ln t) \quad (9)$$

where $p_i \in \mathbb{R}$. Close to the singularity (if the solution admits one) the leading term in the field equations is that of highest order in the curvature. When $D = 5$ or $D = 6$ that is the Lanczos term, so that Eqs. (7)-(9) reduce to:

$$\sum_{i < j < k < l} p_i p_j p_k p_l = 0 , \quad (10)$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{D-1} \left[p_j (p_j - 1) \sum_{\substack{k < \ell \\ k, \ell \neq i, j}} p_k p_\ell \right] + 3 \sum_{\substack{j < k < \ell < m \\ j, k, \ell, m \neq i}} p_j p_k p_\ell p_m = 0. \quad (11i)$$

Let us first consider the case $D = 5$. Then Eq. (10) reduces to $p_1 p_2 p_3 p_4 = 0$ so that p_1 , say, is zero. In Eqs. (11i), the last term vanishes for want of available indices and they all reduce, taking into account that $p_1 = 0$, to:

$$p_2 p_3 p_4 (p_2 + p_3 + p_4 - 3) = 0. \quad (12)$$

There are therefore two classes of Kasner-type metrics which solve the Lanczos field equations (7) and (8) to leading order in $1/t$ in $D = 5$ dimensions:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + t^{2p_3} dx_3^2 + t^{2p_4} dx_4^2 \quad (13)$$

and

$$ds^2 = -dt^2 + dx_1^2 + t^{2(3-p_3-p_4)} dx_2^2 + t^{2p_3} dx_3^2 + t^{2p_4} dx_4^2 \quad (14)$$

with p_3, p_4 arbitrary. The metric (13) is singular at $t = 0$ unless $p_3 = p_4 = 0$ or ($p_3 = 0, p_4 = 1$) in which case it is flat (a fact which seems to have been overlooked in Ref. [15]); the metric (14) is singular whatever p_3 and p_4 .

We note now that there is a whole range of values for the parameters ($p_3 > 0, p_4 > 0$ for the first class, $p_3 > 0, p_4 > 0$ and $p_3 + p_4 < 3$ for the second class) for which all the scale factors contract (or stay constant) as one goes towards the singularity. This is in sharp contrast with the Kasner solution of D -dimensional Einstein's equations where the conditions on the p_i are $\sum p_i = \sum p_i^2 = 1$ so that necessarily one of them is negative [16]. Going now forward in time, we also note that at least one spatial dimension does not expand, a fact that may indicate a way to compactify dynamically the extra dimension.

Let us now consider the case $D = 6$. The field equations (10) and (11) can, after some algebra, be written as:

$$a_4 = 0, \quad (15)$$

$$(a_1 - 3)(p_i^3 - p_i^2 a_1 + p_i a_2 - a_3) = 0 \quad (16i)$$

where

$$a_1 \equiv \sum_i a_i ; a_2 \equiv \sum_{i < j} a_i a_j ; \dots ; a_m \equiv \sum_{i_1 < \dots < i_m} p_{i_1} \dots p_{i_m} . \quad (16b)$$

Summing Eq. (16i) on i we obtain:

$$(a_1 - 3) a_2 = 0 . \quad (16c)$$

The solutions therefore fall into two classes. The first is such that

$$a_3 = a_4 = 0 \quad (17)$$

$$p_i (p_i^2 - p_i a_1 + a_2) = 0 ; \quad (18)$$

In order to find the metrics satisfying the condition (17) and (18) we first note that some p_i must be zero. Indeed if they are all non-zero, then summing (18i) on i implies $a_2 = 0$ so that (18i) reduces to $p_i = a_1$ whose only solution is $p_i = 0$ corresponding to a flat metric. Now if one p_i at least is zero, then at least three are zero in order that (17) be satisfied, and the remaining two p_i which may be non-zero identically satisfy (18i). The most general non-flat Kasner-type metric satisfying (17) and (18) is therefore:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + t^{2p_4} dx_4^2 + t^{2p_5} dx_5^2 \quad (19)$$

p_4 and p_5 arbitrary. The second class of solutions is such that:

$$a_4 = 0 ; a_1 = 3 . \quad (20)$$

[Eq. (16i) is then identically satisfied.] Now the p_i which solve (20) are either all non-zero or one of them at least is zero. In the first case, (all $p_i \neq 0$), some of the p_i must be negative in order that $a_4 = 0$ be satisfied. In the second case (one p_i at least zero) one sees from $a_4 = 0$ that not one but two p_i at least must be zero so that the metric then reads:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + t^{2(3-p_4-p_5)} dx_3^2 + t^{2p_4} dx_4^2 + t^{2p_5} dx_5^2 \quad (21)$$

p_4 and p_5 arbitrary. The set of metrics (19) and (21), together with the Kasner-type metrics such that $a_4 = 0$, $a_1 = 3$ with all $p_i \neq 0$ (and hence some negative), is then the general Kasner-type solution of the Lanczos field equations in $D = 6$ to leading order in $1/t$ and the solutions such that all $p_i > 0$ are of the form (19) (with $p_4 > 0$, $p_5 > 0$) or (21) (with furthermore $p_4 + p_5 < 3$); they cover a whole range of values of the parameters. However, they form a set of measure zero in the space of all solutions [since in the second class the condition $p_i = 0$ must be added to (20)].

Let us summarize the results of this section: in $D = 5$ dimensions the Kasner type metrics which solve the Lanczos field equations to leading order in $1/t$ are given by (13) and (14); in $D = 6$ dimensions they are given by (19) and (21) together with the Kasner metrics (9) with all $p_i \neq 0$ such that the conditions (20) are fulfilled. In both cases there exist subclasses of solutions such that all parameters p_i are positive (and hence all spatial dimensions contract as one goes towards the singularity). In the second class however ($D=6$), this subclass is a set of measure zero.

GENERALIZATION TO SPACE-TIMES OF ARBITRARY DIMENSION

In D dimensions the Lovelock Lagrangian for pure gravity is a sum of $([(D-1)/2]+1)$ terms, $[]$ denoting the integer part, (two terms in four dimensions, three terms in five and six dimensions, etc.), of degree 0, 1, 2, ... $[(D-1)/2]$ in the Riemann tensor. See Ref. [18] for its explicit expression up to quartic order. When the metric is now chosen to be Bianchi I [Eq. (3)] it is only a matter of insight to deduce from the expression (6) of the quadratic (Lanczos) Lagrangian in that universe that the Lovelock Lagrangian of degree p is:

$$\mathcal{L}_{(p)} = 2^{2p-1} \left(e^{\Sigma \sigma} \sum_{i_1 < \dots < i_{2p-1}} \dot{\sigma}_{i_1} \dot{\sigma}_{i_2} \dots \dot{\sigma}_{i_{2p-1}} \right) + e^{\Sigma \sigma} \bar{\mathcal{L}}_{(p)} \quad (22a)$$

with

$$\bar{\mathcal{L}}_{(p)} = -2^{2p-1} \sum_{i_1 < \dots < i_{2p}} \dot{\sigma}_{i_1} \dot{\sigma}_{i_2} \dots \dot{\sigma}_{i_{2p}}. \quad (22b)$$

The Euler variation of $\bar{\mathcal{L}}_{(p)}$ with respect to σ_i then is:

$$\frac{1}{2^{2p-1}} G_{(p)i} = \sum_{\substack{j=1 \\ j \neq i}}^{p-1} \left[(\ddot{\sigma}_j + \dot{\sigma}_j^2) \sum_{\substack{i_1 < \dots < i_{2p-2} \\ i_k \neq i, j}} \dot{\sigma}_{i_1} \dots \dot{\sigma}_{i_{2p-2}} \right] + (2p-1) \sum_{\substack{i_1 < \dots < i_{2p} \\ i_k \neq i}} \dot{\sigma}_{i_1} \dots \dot{\sigma}_{i_{2p}}. \quad (23)$$

The Lovelock field equations in a Bianchi I universe therefore read:

$$\sum_{p=0}^{[(D-1)/2]} (2p-1) \alpha_p \bar{\mathcal{L}}_{(p)} = 0, \quad (24)$$

$$\sum_{p=0}^{[(D-1)/2]} \alpha_p G_{(p)i} = 0 \quad (25i)$$

(with $G_{(0)i} = 1$). Close to the singularity the leading-order term is the last one, of degree $[(D-1)/2]$ in the Riemann tensor (we suppose the coefficient $\alpha_{[(D-1)/2]}$ to

be different from zero). The exponents of the Kasner-type metrics (9) must therefore obey the relations:

$$\sum_{i_1 < \dots < i_{2p}} p_{i_1} p_{i_2} \dots p_{i_{2p}} = 0, \quad (26)$$

$$0 = \sum_{\substack{j=1 \\ j \neq i}}^{D-1} [p_j (p_j - 1) \sum_{\substack{i_1 < \dots < i_{2p-2} \\ i_k \neq i, j}} p_{i_1} \dots p_{i_{2p-2}}] + (2p-1) \sum_{\substack{i_1 < \dots < i_{2p} \\ i_k \neq i}} p_{i_1} \dots p_{i_{2p}} \quad (27i)$$

The analysis then closely follows from the previous section. In odd dimension the general solution falls into two classes:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + \sum_{j=3}^{D-1} t^{2p_j} dx_j dx^j, \quad (28)$$

$$ds^2 = -dt^2 + dx_1^2 + t^{2[D-2 - \sum_{j=3}^{D-1} p_j]} dx_2^2 + \sum_{j=3}^{D-1} t^{2p_j} dx_j dx^j \quad (29)$$

with $(D-3) p_j$ arbitrary. In even dimension ($D = 2n+2$) it is a fairly safe guess to assess (but we did not show it rigorously) that the field equations (26) and (27) that generalize (15) and (16) can be written as:

$$a_{2n} = 0, \quad (30)$$

$$[a_1 - (2n-1)] \sum_{k=0}^{2n-1} (-1)^k p_i^k a_{2n-1-k} = 0. \quad (31i)$$

Summing Eq. (31i) on i we have:

$$[a_1 - (2n-1)] a_{2n-1} = 0 \quad (32)$$

so that as before the solutions fall into three classes:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \sum_{i=4}^{D-1} t^{2p_i} dx_i dx^i, \quad (33)$$

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + t^{2[2n-1 - \sum_{i=4}^{D-1} p_i]} dx_3^2 + \sum_{i=4}^{D-1} t^{2p_i} dx_i dx^i \quad (34)$$

with $p_4, p_5, \dots, p_{D-1}$ arbitrary, and:

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} t^{2p_i} dx_i dx^i \quad (34a)$$

with all p_i different from zero and satisfying:

$$a_{2n} = 0 ; \quad a_1 = 2n - 1 . \quad (34b)$$

We are therefore led to generalize the conclusions of the previous section: in $D = 2n+1$ dimensions the Kasner-type metrics which solve the Lovelock field equations to leading order in $1/t$ are given by (28) and (29); in $D = 2n+2$ dimensions they are given by (33) and (34). In both cases there exist subclasses of solutions for which all p_i are positive: in even dimension however this subclass is a set of measure zero.

QUADRATIC THEORIES IN D-DIMENSIONAL SPACE-TIMES

When the condition that the gravitational field equations remain second order is relaxed, the Lagrangian becomes a priori quite arbitrary, since it can then include any scalar density formed with the Riemann tensor. In particular, its degree in the Riemann tensor is no longer bounded by the dimension of space-time so that a study of the behaviour of the solution near a cosmic singularity becomes very model-dependent. We shall therefore content ourselves with an analysis of quadratic theories in D-dimensional space-times.

Near the singularity the linear Einstein-Hilbert term is negligible and to leading order the Lagrangian is:

$$\mathcal{L}_Q = \alpha \mathcal{L}_R + \beta \mathcal{L}_{Ric} + \gamma \mathcal{L}_{Rie} \quad (35a)$$

with

$$\mathcal{L}_R \equiv \sqrt{-g} R^2 ; \quad \mathcal{L}_{Ric} \equiv \sqrt{-g} R_{AB} R^{AB} ; \quad \mathcal{L}_{Rie} \equiv \sqrt{-g} R_{ABCD} R^{ABCD} . \quad (35b)$$

The corresponding field equations then read:

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_Q}{\delta g^{AB}} \equiv H_{AB} = \alpha H_{AB}^R + \beta H_{AB}^{Ric} + \gamma H_{AB}^{Rie} = 0 \quad (36)$$

with

$$H_{AB}^R = 2RR_{AB} - \frac{1}{2}g_{AB}R^2 + 2g_{AB}\Box R - 2\nabla_{AB}R, \quad (36a)$$

$$H_{AB}^{Ric} = 2R_{AMBN}R^{MN} - \frac{1}{2}g_{AB}R_{MN}R^{MN} + \Box R_{AB} + \frac{1}{2}g_{AB}\Box R - \nabla_{AB}R, \quad (36b)$$

$$H_{AB}^{Rie} = 2R_{AMNP}R_B^{MNP} - 4R_{AM}R_B^M + 4R_{AMBN}R^{MN} - \frac{1}{2}g_{AB}R_{MNPQ}R^{MNPQ} + 4\Box R_{AB} - 2\nabla_{AB}R . \quad (36c)$$

(we note that only the Lanczos Lagrangian ($\alpha = \gamma$; $\beta = -4\alpha$) yields second-order field equations). When now the metric is imposed to be Bianchi I [Eq. (3)] the Lagrangian densities become:

$$\begin{aligned}\mathcal{L}_R &= e^{\Sigma\sigma} [4(\Sigma\ddot{\sigma})^2 + 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma}^2 + 4\Sigma\ddot{\sigma}(\dot{\Sigma}\dot{\sigma})^2 + (\dot{\Sigma}\dot{\sigma}^2)^2 + (\dot{\Sigma}\dot{\sigma})^4 + 2\dot{\Sigma}\dot{\sigma}^2(\dot{\Sigma}\dot{\sigma})^2], \\ \mathcal{L}_{Ric} &= e^{\Sigma\sigma} [(\dot{\Sigma}\ddot{\sigma})^2 + \dot{\Sigma}\ddot{\sigma}^2 + 2\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma}^2 + 2\Sigma\ddot{\sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma} + (\dot{\Sigma}\dot{\sigma}^2)^2 + \dot{\Sigma}\dot{\sigma}^2(\dot{\Sigma}\dot{\sigma})^2], \\ \mathcal{L}_{Rie} &= e^{\Sigma\sigma} [4\Sigma\ddot{\sigma}^2 + 8\Sigma\ddot{\sigma}\dot{\sigma}^2 + 2(\dot{\Sigma}\dot{\sigma}^2)^2 + 2\Sigma\dot{\sigma}^4].\end{aligned}\quad (37)$$

As for their Euler variation (36) they read:

$$\begin{aligned}H_{ii}^R &= -4\ddot{\sigma}_i [2\Sigma\ddot{\sigma} + \dot{\Sigma}\dot{\sigma}^2 + (\dot{\Sigma}\dot{\sigma})^2] - 4\dot{\sigma}_i [2\Sigma\ddot{\sigma} + 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma} + 2\Sigma\ddot{\sigma}\dot{\sigma} + (\dot{\Sigma}\dot{\sigma})^3 + \\ &\quad + \dot{\Sigma}\dot{\sigma}^2\dot{\Sigma}\dot{\sigma}] + 8\Sigma\ddot{\sigma}^{(4)} + 16\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma} + 8\Sigma\ddot{\sigma}\dot{\sigma} + 12(\dot{\Sigma}\ddot{\sigma})^2 + 8\Sigma\ddot{\sigma}^2 + (\dot{\Sigma}\dot{\sigma}^2)^2 + \\ &\quad + 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma}^2 + 12\Sigma\ddot{\sigma}(\dot{\Sigma}\dot{\sigma})^2 + 8\Sigma\ddot{\sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma} + (\dot{\Sigma}\dot{\sigma})^4 + 2(\dot{\Sigma}\dot{\sigma})^2\dot{\Sigma}\dot{\sigma}^2,\end{aligned}\quad (38a)$$

$$\begin{aligned}H_{00}^R &= 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma} - 2(\dot{\Sigma}\dot{\sigma})^2 - 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma}^2 + 4\Sigma\ddot{\sigma}(\dot{\Sigma}\dot{\sigma})^2 + 4\Sigma\ddot{\sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma} - \frac{3}{2}(\dot{\Sigma}\dot{\sigma}^2)^2 - \\ &\quad - \dot{\Sigma}\dot{\sigma}^2(\dot{\Sigma}\dot{\sigma})^2 + \frac{1}{2}(\dot{\Sigma}\dot{\sigma})^4,\end{aligned}$$

$$\begin{aligned}H_{ii}^{Ric} &= 2\ddot{\sigma}_i^{(4)} + 4\ddot{\sigma}_i\Sigma\ddot{\sigma} + 2\ddot{\sigma}_i[(\dot{\Sigma}\dot{\sigma})^2 - 2\Sigma\dot{\sigma}^2] - 2\dot{\sigma}_i[\dot{\Sigma}\ddot{\sigma} + 4\Sigma\ddot{\sigma}\dot{\sigma} + \dot{\Sigma}\ddot{\sigma}\dot{\Sigma}\dot{\sigma} + \\ &\quad + 2\Sigma\dot{\sigma}^2\dot{\Sigma}\dot{\sigma}] + 2\Sigma\ddot{\sigma}^{(4)} + 2\Sigma\ddot{\sigma}\dot{\sigma} + 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma} + 3(\dot{\Sigma}\ddot{\sigma})^2 + 3\Sigma\ddot{\sigma}^2 + \\ &\quad + 2\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma}^2 + 2\Sigma\ddot{\sigma}(\dot{\Sigma}\dot{\sigma})^2 + 4\Sigma\ddot{\sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma} + (\dot{\Sigma}\dot{\sigma}^2)^2 + \dot{\Sigma}\dot{\sigma}^2(\dot{\Sigma}\dot{\sigma})^2,\end{aligned}\quad (38b)$$

$$\begin{aligned}H_{00}^{Ric} &= \dot{\Sigma}\ddot{\sigma}\dot{\Sigma}\dot{\sigma} + \dot{\Sigma}\ddot{\sigma}\dot{\sigma} - \frac{1}{2}\dot{\Sigma}\dot{\sigma}^2 - \frac{1}{2}(\dot{\Sigma}\ddot{\sigma})^2 + \dot{\Sigma}\dot{\sigma}(\dot{\Sigma}\dot{\sigma})^2 + 2\Sigma\ddot{\sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma} - \\ &\quad - \dot{\Sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma}^2 - \frac{3}{2}(\dot{\Sigma}\dot{\sigma}^2)^2 + \frac{1}{2}\dot{\Sigma}\dot{\sigma}^2(\dot{\Sigma}\dot{\sigma})^2,\end{aligned}$$

$$\begin{aligned}H_{ii}^{Rie} &= 8\ddot{\sigma}_i^{(4)} + 16\ddot{\sigma}_i\Sigma\ddot{\sigma} - 24\ddot{\sigma}_i\dot{\sigma}_i^2 + 16\ddot{\sigma}_i\dot{\sigma}_i\Sigma\ddot{\sigma} + 8\ddot{\sigma}_i[\dot{\Sigma}\ddot{\sigma} + (\dot{\Sigma}\dot{\sigma})^2 - \dot{\Sigma}\dot{\sigma}^2] - \\ &\quad - 8\dot{\sigma}_i^3\dot{\Sigma}\dot{\sigma} + 8\dot{\sigma}_i^2[\dot{\Sigma}\ddot{\sigma} + (\dot{\Sigma}\dot{\sigma})^2] - 8\dot{\sigma}_i[2\Sigma\ddot{\sigma}\dot{\sigma} + \dot{\Sigma}\dot{\sigma}^2\dot{\Sigma}\dot{\sigma}] + 4\Sigma\ddot{\sigma}^2 + \\ &\quad + 8\Sigma\ddot{\sigma}\dot{\sigma}^2 + 2(\dot{\Sigma}\dot{\sigma}^2)^2 + 2\Sigma\dot{\sigma}^4,\end{aligned}\quad (38c)$$

$$H_{00}^{Rie} = 4\Sigma\ddot{\sigma}\dot{\Sigma}\dot{\sigma} - 2\Sigma\dot{\sigma}^2 + 4\Sigma\ddot{\sigma}\dot{\sigma}\dot{\Sigma}\dot{\sigma} - 3\Sigma\dot{\sigma}^4 - 3(\dot{\Sigma}\dot{\sigma}^2)^2 + 4\dot{\Sigma}\dot{\sigma}^3\dot{\Sigma}\dot{\sigma}.$$

The H_{ii} can equivalently be obtained from a variation of (37) with respect to σ_i . The constraint terms H_{00} , on the other hand, cannot be obtained by the usual trick of introducing a lapse function in an effective Lagrangian which would be first order in σ_i as there is no such object in a generic quadratic theory.

We remark in passing that maximally symmetric space-times [such that $R_{ABCD} =$

$K(g_{AC}g_{BC}-g_{AD}g_{BC})$, K arbitrary] are solutions to the field equations (36) for any $(\alpha\beta\gamma)$ in $D = 4$ dimensional space-times only; if $D \neq 4$ the relation $2\gamma = -(D-1)(\alpha D + \beta)$ must hold.

We now specialize to Kasner-type metrics [Eq. (9)]. The field equations (36) and (38) then somehow simplify as we have:

$$H_{ii}^R = 4p_i (3-A_1) [A_1^2 - 2A_1 + A_2] + A_1^4 + 2A_1^2 A_2 + A_2^2 - 12A_2 A_1 - 12A_1^3 + 44A_1^2 + 24A_2 - 48A_1, \quad (39a)$$

$$H_{00}^R = -\frac{3}{2}A_1^2 - A_2 A_1^2 + \frac{1}{2}A_1^4 - 4A_1^3 + 6A_1^2,$$

$$H_{ii}^{Ric} = 4p_i (3-A_1)(A_2-1) + A_2^2 + A_2 A_1^2 - 2A_1^3 - 6A_2 A_1 + 7A_2 + 11A_1^2 - 12A_1,$$

$$H_{00}^{Ric} = -\frac{3}{2}A_1^2 + \frac{1}{2}A_2 A_1^2 - A_1^3 - A_2 A_1 + \frac{3}{2}A_2 + \frac{3}{2}A_1^2, \quad (39b)$$

$$H_{ii}^{Rie} = 8p_i (3-A_1) (p_i^2 - p_i A_1 + A_1 + A_2 - 2) + 2A_4 + 2A_1^2 - 8A_3 + 4A_2,$$

$$H_{00}^{Rie} = -3A_4 - 3A_2^2 + 4A_3 A_1 - 4A_2 A_1 + 6A_2, \quad (39c)$$

where $A_i \equiv \sum_k^n p_k^n$. Moreover some algebra shows that:

$$H_{ii}^{Rie} = -H_{ii}^R + 4H_{ii}^{Ric} + 8[-3a_4^i + a_3^i(a_1^i - 3)], \quad (40a)$$

$$H_{00}^{Rie} = -H_{00}^R + 4H_{00}^{Ric} + 12a_4, \quad (40b)$$

where a_n^i is defined as a_n [Eq. (16b)] except for the fact that the sums are taken over all indices except i . In $D = 4$ dimensions the indices i can take three values (123) only, so that the last term in Eqs. (40a,b) vanishes and one recovers the Gauss-Bonnet theorem.

Let us consider first the subset of theories that satisfy $\gamma = 0$ (the only one to be considered in $D = 4$ dimensions) and look for their anisotropic Kasner-like solutions (such that the p_i are not all equal). Then the factor independent of p_i and the coefficient of p_i in the equation $\alpha H_{ii}^R + \beta H_{ii}^{Ric} = 0$, H_{ii}^R and H_{ii}^{Ric} given by (39a,b) must be equal to zero; and the constraint equation $\alpha H_{00}^R + \beta H_{00}^{Ric} = 0$ must

also be satisfied. It is then very easy to see (cf. [14a]) that the necessary and sufficient condition for these three conditions to hold is, when $\beta = 0$ and $\beta = k\alpha$, $k \notin [-3, -1]$:

$$A_1 \equiv \sum_{i=1}^{D-1} p_i = 1 \quad ; \quad A_2 \equiv \sum_{i=1}^{D-1} p_i^2 = 1. \quad (41)$$

Hence the Kasner-type solutions of the field equations (36) of quadratic theories of gravity when $\gamma = 0$ are then the Kasner solutions (41) that also solve Einstein's equations $R_{AB} = 0$. This result generalizes [15] where it was only shown that (41) was a solution. It shows that in these quadratic theories the Kasner-like solutions are always such that at least one dimension expands as one goes towards the singularity (as Eq. (41) implies that at least one p_i is negative [16]).

There are special cases however: the R^2 theory ($\beta = 0$) where the p_i are only restricted to satisfy:

$$A_2 = A_1 (2 - A_1) \quad (42)$$

a condition which is certainly compatible with $p_i > 0$, as special examples (e.g., $p_1 = p_2 = 1/4$, $6p_3 = 1 + \sqrt{7}$ in $D = 4$) show. Another special case is the Weyl theory, $\beta = -3\alpha$, where the p_i must satisfy:

$$2A_2 = A_1^2 - A_1 + 3 \quad (43)$$

which is also compatible with $p_i > 0$. Finally, when $\beta = k\alpha$, $k \in [-3, -1]$ the field equations admit, besides (41), another family of solutions (see [14a]):

$$A_1 = 3 \quad ; \quad A_2 = \frac{\beta - 3\alpha + 2\sqrt{-2\beta(\beta + 3\alpha)}}{\alpha + \beta}.$$

One can easily verify that when $k \in [-2, -1]$ ($D = 4$) one of the p_i is necessarily negative as in the Kasner solution (41); when $k \in [-2, -3]$ then all the p_i can be positive. To be complete we should now analyse the generic theory in $D > 4$ dimensional space-times, α , β and γ arbitrary, but this would induce us into rather pointlessly involved algebra. To see, however, how the inclusion of $\mathcal{L}_{\text{Rie}}$ may modify the preceding conclusions, we shall consider the case $\alpha = \beta = 0$, $\gamma \neq 0$ and remark that the equations $H_{00}^{\text{Rie}} = H_{ii}^{\text{Rie}} = 0$ [cf. (39c)] admit a subclass of solutions such that

$$A_1 = 3, \quad A_4 + A_2^2 - 4A_3 + 2A_2 = 0. \quad (44)$$

It is then a matter of straightforward algebraic manipulation to show that the metrics:

$$ds^2 = -dt^2 + \sum_{i=1}^m dx_i dx^i + \sum_{i=m+1}^{m+k} t^{2p} dx_i dx^i + t^{2q} dx_{D-1} dx^{D-1} \quad (45)$$

with $2+m+k = D$, $p \neq q$, that satisfy the conditions (44) are such that both p and q are positive. We have hence exhibited a subset of solutions to the field equations of the quadratic theory based on $\mathcal{P}_{\text{Kie}}$ alone such that all the spatial dimensions contract as one goes towards the singularity. Since we do not have the general solution however, we do not know the measure of this subset among the set of all solutions.

CONCLUSION

In Einsteinian gravity the Kasner solution [that is the metric (9) satisfying (41)] is the starting point of the construction of the general (or at least a large class of) solutions of Einstein's equations near a space-like singularity. Indeed as an in-depth analysis has shown ([3-6], [9-13] and [16]), if one of the exponents

$$\alpha_{ijk} = 2p_i + \sum_{l \neq i,j,k} p_l \quad (i \neq j \neq k) \quad (46)$$

is negative (which is the case when $D < 11$ [9-13]), then the terms in the general field equations which describe inhomogeneities and deviations from Bianchi-I type anisotropies, and which were ignored in order to obtain the Kasner solution, start to grow towards the singularity and force the "general" solution to switch from one set of Kasner parameters p_i to another. This oscillatory approach to the singularity is endless and chaotic.

A contrario, if the Kasner solution admitted a subset of solutions such that all p_i were positive, then after possibly a finite number of switches, the exponents α_{ijk} would fall in the range where they are all positive and the solution would settle in this regime as the perturbative terms would not grow.

Now in this paper, we showed that when the Einstein-Hilbert Lagrangian is modified by the addition of non-linear terms in the curvature, and when moreover space-time is more than four-dimensional, then the parameters of the Kasner-like solutions to the field equations near the singularity can indeed all be positive. On the other hand, we showed that in four-dimensional space-times (where $\alpha_{ijk} = 2p_i$) the only Kasner-like solution to most quadratic theories of gravity is the standard one (9) and (41) with hence some negative parameters p_i and exponents

α_{ijk} , the exceptions being the theories $\beta = 0$ and $\beta = k\alpha$, $k \in [-2, -3]$. We therefore have grounds to make the conjecture that the oscillatory approach to the singularity may be maintained in higher order theories of gravity when space-time is four-dimensional but disappears in higher dimensions. It is clear, however, that in order to prove or disprove this claim, an analysis of the perturbative terms in the field equations that were here neglected should be undertaken. This is left to further work.

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