Quantum Group Interpretation of Some Conformal Field Theories

L. ALVAREZ-GAUMÉ, C. GOMEZ[★] AND G. SIERRA

*Theory Division, CERN
CH-1211 Geneva 23, Switzerland*

ABSTRACT

We show that the representation theory of the q -deformation of $SU(2)$ provides solutions to the polynomial equations of Moore and Seiberg for Rational Conformal Field Theories whenever q is a root of unity. The q -analogue of the $6j$ -symbols give the duality matrices, and there is a close connection between the modular properties of the Kac-Moody character for $SU(2)_k$ and some simple properties of the q -characters of quantum groups. We show how the quantum group can be considered for most purposes as a rather accurate description of the Wess-Zumino-Witten theory at level k , where k is determined by q .

★ Permanent address: Department of Physics, University of Salamanca. Salamanca, Spain.

1. Rational Conformal Field Theories (RCFT) [1] are characterized by a chiral algebra $A = A_L \times A_R$ with A_L (resp. A_R) containing at least the identity and the left (resp. right) moving part of the Virasoro algebra , and a finite number of irreducible representations of A . The Hilbert space of the theory splits into a finite sum $\bigoplus H_i \otimes H_{\bar{i}}$, where H_i (resp. $H_{\bar{i}}$) is an irreducible representation of the left (resp. right) chiral algebra. Each space $H_i \otimes H_{\bar{i}}$ is generated by acting first with a primary field on the $SL(2)$ invariant vacuum , and then applying the algebra A to this state. For rational theories , the "size" of the irreducible representations is considerably reduced by the decoupling of null vectors. The partition function is constructed in terms of some basic building blocks : the characters of the chiral algebras. The correlation functions on the plane are also obtained in terms of some basic multivalued holomorphic functions : the conformal blocks. The physical amplitudes are represented by monodromy invariant combinations of left and right conformal blocks. If there are N representations of the left moving chiral algebra , and we concentrate only on the left moving part of the theory , each representation is characterized by a primary field ϕ_i . One of the defining properties of the RCFT is its fusion algebra which labels the possible couplings between primary states :

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k \quad (1)$$

where the coefficients N_{ij}^k are non-negative integers. The fusion algebra is commutative and associative as follows from the duality properties of Conformal Field Theories (CFT). The unexpected connection between the fusion algebra (1) and the modular properties of the theory [2] has led to the characterization of modular invariant RCFT in terms of a set of polynomial equations [3] [4]. In [4] it was shown using the properties of group characters, that any group (discrete or continuous) provides a solution to the classical limit of the polynomial equations. The variables entering these equations are the duality matrices of the 4-point blocks :

$$C_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} N_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \quad (2)$$

representing respectively $s - u$ and $s - t$ duality (see fig. 1). From fig.1 we see that the matrix C provides a representation of the braid group in the space of conformal blocks. The matrix N expresses the associativity of the operator product expansion (OPE). When we use the duality properties of the conformal blocks in the monodromy invariant combinations of left and right blocks , we obtain the bootstrap equations for CFT [5] . One procedure for the construction of conformal blocks is to explicitly solve the differential equations defined by the decoupling of null-vectors [5]. Since these are generalized hypergeometric equations in genus zero , the information needed to determine their solutions is the collection of monodromy matrices around the branch points and the leading behavior at the singular points. For example, in [6] one finds a detailed analysis of the Knizhnik-Zamolodchikov equations [7] describing the conformal blocks of the Wess-Zumino-Witten (WZW) theory [8] .

In [9] we began exploring the interpretation of the polynomial equations in terms of Quantum Algebras and open Verlinde operators. The main purpose of this letter is to show that to any quantum group one can associate a solutions to the polynomial equations. We concentrate in particular in the group $SL(2, q)$ and proof that the braiding , monodromy and modular properties of the WZW-model at level k are determined by the representation theory of this quantum group. The rationality condition is met by requiring q to be a root of unity. The relation between q and k is given by :

$$q = \exp\left(\frac{2\pi i}{k+2}\right) \quad (3)$$

The duality matrices (2) are explicitly given by the q -analogue of the $6j$ -symbols computed in [10]. The polynomial equations follow from some simple identities satisfied by the $6j$ -symbols. The modular transformation s has also a simple interpretation in terms of the behavior of the $SL(2, q)$ characters with respect to the transformation $q \mapsto q^{-1}$. We will compare our computations of the matrices (2) with the results in [6] in some particularly simple examples. Since the quantum

enveloping algebra $SL(2, q)$ is finitely generated, it is quite remarkable that it contains so much information about the $SU(2)$ level k WZW theory. Perhaps the following interpretation helps the reader get some intuitive understanding of why these seemingly disparate objects have so much in common. A Kac-Moody algebra is characterized by generators $T_n^a, k \in \mathbb{Z}$, and they provide the generators for the chiral algebra of the WZW-theories. In the study of duality and modular transformation properties of CFT, the fundamental information is provided by the properties of the primary fields. Hence when working with $\hat{C}, \hat{N}$ and modular transformations, we are effectively working in the Hilbert space of the theory modulo the action of the chiral algebra. This is, roughly speaking, similar to "forgetting" the moding in the generators T_n^a, k . However, one should not be led to believe that once the moding is forgotten, we are going to be left with the underlying finite dimensional classical algebra. The central charge k is a manifestation of the fact that we are working with a centrally extended loop algebra. Since there are no central extensions of finite dimensional semi-simple groups, the only way out which allows one to keep the information contained in the non-vanishing central extension k is to envisage that one is left with a deformation of the classical algebra, and the deformation parameter is a function of k . We will comment at the end on how the appearance of the quantum group can also be understood if one analyzes the WZW theory using the topological Chern-Simons action in three dimensions [11].

2. There is a vast literature on quantum groups (for details and references see for example [12]). To keep the discussion as simple as possible, we will mainly concentrate on the q -deformation of $SL(2)$ even though most of the arguments and results work also in the general case [13]. In this case the algebra has three generators J^+, J^-, J_3 satisfying the defining relations :

$$[J^+, J^-] = \frac{q^{J_3/2} - q^{-J_3/2}}{q^{1/2} - q^{-1/2}} [J_3, J^\pm] = \pm 2J^\pm \quad (4)$$

we will use the convention :

$$[\lambda] = \frac{q^{\lambda/2} - q^{-\lambda/2}}{q^{1/2} - q^{-1/2}} \quad (5)$$

this is a deformation of the number λ and in the limit as $q \rightarrow 1$ $[\lambda] \rightarrow \lambda$. For generic q , the representation theory of (4) runs parallel to the classical case. The highest weight finite dimensional irreducible representations are labelled by an integer or half integer j and the representation has dimension $2j + 1$. We can choose a basis for the spin j representation as $|j, m\rangle$. Then

$$J^\pm |j, m\rangle = \sqrt{[j \mp m][j \pm m + 1]} |j, m \pm 1\rangle \quad (6)$$

Virtually if in any formula for the classical case we replace numbers by q -numbers (5) we obtain a correct formula for the q -deformed algebra. The co-multiplication Δ in the algebra (4) is the analog of the composition of angular momenta. It is defined by

$$\Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3 \quad \Delta(J^\pm) = J^\pm \otimes q^{J_3/4} + q^{-J_3/4} \otimes J^\pm \quad (7)$$

and it gives the action of the generators on tensor products of representations. For generic q the standard Clebsch-Gordan decomposition holds

$$V^{j_1} \otimes V^{j_2} = \sum_{|j_1 - j_2|}^{j_1 + j_2} V^j \quad (8)$$

There is a second coproduct which can be introduced $\Delta' = \sigma \cdot \Delta$ where σ is the permutation $\sigma(a \otimes b) = b \otimes a$. The Clebsch-Gordan coefficients $K_{j_3}^{j_1 j_2}$ project the representation V^{j_3} out of the product $V^{j_1} \otimes V^{j_2}$

$$K_{j_3}^{j_1 j_2} : V^{j_1} \otimes V^{j_2} \mapsto V^{j_3} \quad (9)$$

To $SL(2, q)$ (and to any quantum group) we can associate a Yang-Baxter matrix

$$R^{j_1 j_2} : V^{j_1} \otimes V^{j_2} \mapsto V^{j_2} \otimes V^{j_1} \quad (10)$$

satisfying the following relations [10]

$$R^{j_1 j_2} R^{j_3 j_4} R^{j_2 j_3} = R^{j_2 j_3} R^{j_1 j_3} R^{j_1 j_2} \quad (11)$$

$$R^{j_1 j_2} K_{j_1}^{j_3 j_4} = K_{j_1}^{j_3 j_4} R^{j_3 j_2} R^{j_4 j_2} \quad (12)$$

$$K_j^{j_1 j_2} R^{j_2 j_1} = (-1)^{j_1 + j_2 - j} q^{(c_j - c_{j_1} - c_{j_2})/2} K_j^{j_2 j_1} \quad (13)$$

where (11) is defined on $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$, and the indices label the spaces where the R -matrix acts. In (13) $c_j = j(j+1)$. The R -matrix is obtained from the universal R -matrix for $SL(2, q)$ [12]

$$R = q^{\frac{j_3 \otimes j_4}{4}} \sum_{n \geq 0} \frac{(1 - q^{-1})^n}{[n]!} q^{-n(n-1)/4} q^{n j_3/4} (J^+)^n \otimes q^{-n j_3/4} (J^-)^n \quad (14)$$

by evaluating it in the tensor product $V^{j_1} \otimes V^{j_2}$ and then permuting the factors (acting with σ). The symbol $[n]!$ is the q -analog of the factorial of a number. A basic property of (14) is that $\Delta'(Y)R = R\Delta(Y)$ for any element in the quantum algebra. With (13) we can begin to make contact with RCF. If one represents the Clebsch-Gordan coefficient as a three point vertex (see fig. 2), then (13) can be understood graphically as in fig.3. This is analogous to a chiral vertex [6], [14] representing a three point function for three primary fields of spins j_1, j_2, j_3 and (13) amounts to the braiding of the fields with spins j_1, j_2 yielding a factor

$$(-1)^{j_1 + j_2 - j} e^{2\pi i(\Delta_j - \Delta_{j_1} - \Delta_{j_2})}$$

where $\Delta_j = \frac{j(j+1)}{k+2}$ for the $SU(2)_k$ WZW model. The sign $(-1)^{j_1 + j_2 - j}$ is a group theory factor keeping track of whether the representation j appears symmetrically

or antisymmetrically in the tensor product of the j_1, j_2 representations. Comparing the previous equations with (13) (fig.3), we are led to the identification (3). This identification will be more fully justified later when we compute the duality matrices (2).

Notice that in (3) the deformation parameter is a root of unity. In this case the representation theory is significantly different from the classical case. The first surprise is that when $q^N = 1$ there are only $N-1$ distinct finite dimensional irreducible representations with spins $j = 0, 1/2, 1, \dots, (N-1)/2$ the representation with $j = (N-1)/2$ is singular, and as we will see later, it does not appear in the conformal blocks. The proof of the periodicity of the finite dimensional representations is very simple. Let $|0\rangle$ be the highest weight state, and choose a basis $|m\rangle = (J^-)^m |0\rangle$ then taking $J_3 |0\rangle = \lambda |0\rangle$ we easily obtain

$$J_3 |m\rangle = (\lambda - 2m) |m\rangle \quad (15)$$

$$J^- |m\rangle = (m+1) |m+1\rangle \quad (16)$$

$$J^+ |m\rangle = \frac{[m][\lambda - m + 1]}{m} |m-1\rangle \quad (17)$$

To have a finite dimensional representation we have to impose the condition $|m+1\rangle = 0$. There are two cases : 1) If $m \leq N-2$ the representation has dimension $2j+1$, $\lambda = 0, 1, 2, \dots, N-2$, and the matrix elements of the generators are as in (6). 2) When $m = N-1$ the representation has dimension N and there is no restriction on λ . The latter is a singular representation and it decouples. If we define $N-2 = k$ and the q -dimension of the spin j representation as

$$\dim_q V^j = [2j+1] \quad (18)$$

we obtain

$$[2j_1+1][2j_2+1] = \sum_{|j_1-j_2|}^{max(j_1+j_2, k-j_1-j_2)} [2j+1] \quad (19)$$

which agrees with the fusion rules of $SU(2)_k$ [15]. There are also q -analogues of the

3j and 6j symbols [10]. In terms of them one can explicitly check (12), (13). Their explicit form exhibits the restrictions on the possible non-singular representation. Since we will not need here the 3j symbols we will not write them down explicitly. The 6j symbols are equivalent to the N matrices, or $s-t$ duality transformations. If we consider the recoupling diagram for angular momenta shown in fig.4:

$$S_f^{j_1 j_2 j_3 j_4} = \sum_{j'} \begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{pmatrix} T_{j'}^{j_1 j_2 j_3 j_4} \quad (20)$$

This equation corresponds in RCFT to

$$S_f^{j_1 j_2 j_3 j_4} = \sum_{j'} N_{jj'} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} T_{j'}^{j_1 j_2 j_3 j_4} \quad (21)$$

with a diagrammatic interpretation as in fig. 4. The explicit form of the 6j-symbols is [10]

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{pmatrix} &= \sqrt{[2j_{12} + 1][2j_{23} + 1]} (-1)^{j_1 + j_2 - j - j_3 - 2j_{12}} \\ &\Delta(j_1, j_2, j_{12}) \Delta(j_3, j, j_{23}) \Delta(j_1, j, j_{23}) \Delta(j_3, j_2, j_{23}) \\ \sum_{z \geq 0} (-1)^z [z + 1]! [z - j_1 - j_2 - j_{12}]! [z - j_3 - j - j_{23}]! [z - j_3 - j_2 - j_{23}]! \\ [j_1 + j_2 + j_3 + j - z]! [j_1 + j_3 + j_{12} + j_{23} - z]! [j_2 + j + j_{12} + j_{23} - z]!^{-1} \end{aligned} \quad (22)$$

where the sum is only over those z such that the arguments of $[\cdot]$ are non-negative, $[0]! = 1$, and

$$\Delta(a, b, c) = \sqrt{\frac{[-a + b + c]! [a - b + c]! [a + b - c]!}{[a + b + c + 1]!}} \quad (23)$$

This quantity always appears in the 6j as well as in the 3j symbols associated to the vertices in fig.2. If $N = k + 2$ then $[k + 2] = 0$ and $[n]! = 0$ if $n \geq k + 2$ hence

the 6j-symbols, and in general the $SL(2, q)$ representation theory blows up unless one imposes the constraint

$$j_1 + j_2 + j \leq k \quad (24)$$

but this is precisely the condition satisfied by the fusion rules of the $SU(2)_k$ WZW theory. In particular the representation with $2j = N - 1$ is excluded when we impose the restriction to regular representations. The justification of this truncation is related via Weyl's representation theory to the properties of the centralizer of the action of the quantum group in the space $(V^j)^{\otimes n}$ (n tensor copies of the representation of spin j). Equations (20) and (21) allow us to identify the 6j symbols with the N duality matrices

$$N_{jj'} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} = \begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{pmatrix} \quad (25)$$

the pentagon equation is [3]

$$\sum_s N_{as} \begin{bmatrix} j & k \\ i & b \end{bmatrix} N_{bp} \begin{bmatrix} s & l \\ i & m \end{bmatrix} N_{sq} \begin{bmatrix} k & l \\ j & p \end{bmatrix} = N_{ap} \begin{bmatrix} j & q \\ i & m \end{bmatrix} N_{bq} \begin{bmatrix} k & l \\ a & m \end{bmatrix} \quad (26)$$

and using (25) it reduces to the q -analogue of the Biedenharn-Elliott identity for 6j symbols [10]. Next, using (13) and (20) one obtains the $s-u$ or C duality matrix to be

$$C_{jj'} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} = (-1)^{j+j'-j_1-j_4} q^{(e_{j_1} + e_{j_4} - e_j - e_{j'})/2} \begin{pmatrix} j_2 & j_1 & j \\ j_3 & j_4 & j' \end{pmatrix} \quad (27)$$

The relation between C and N is as expected in RCFT [4] [9] if we recall that the dimension of the primary field of spin j in the level k $SU(2)$ WZW theory is $\Delta_j = j(j+1)/(k+2)$; the hexagon identity again follows from the properties of 6j symbols and the Yang-Baxter equation satisfied by the R -matrix. As a final

check of the precise relation between the level k $SU(2)$ WZW theory and $SL(2, q)$ we compute the C matrix for the special block $S_{pp'}^{j1/2i/2j}$. We want to compute

$$C_{pp'} \begin{bmatrix} 1/2 & 1/2 \\ j & j \end{bmatrix} \quad (28)$$

p, p' can only range over the values $j + 1/2, j - 1/2$ and we obtain a 2×2 matrix. Using the explicit form of the $6j$ symbols (22) we obtain

$$C_{pp'} \begin{bmatrix} 1/2 & 1/2 \\ j & j \end{bmatrix}_{sl(2,q)} = \begin{pmatrix} -\frac{q^{-(j+3/4)}}{[2j+1]} & \frac{\sqrt{q^{-1/2}[2j][2j+2]}}{[2j+1]} \\ \frac{\sqrt{q^{-1/2}[2j][2j+2]}}{[2j+1]} & q^{j+1/4} \end{pmatrix}_{[2j+1]} \quad (29)$$

whereas the corresponding matrix for $SU(2)_k$ computed in [6] is

$$C_{pp'} \begin{bmatrix} 1/2 & 1/2 \\ j & j \end{bmatrix}_{su(2)_k} = \begin{pmatrix} \gamma_+^{-1} & 0 \\ 0 & \gamma_-^{-1} \end{pmatrix} C_{pp'} \begin{bmatrix} 1/2 & 1/2 \\ j & j \end{bmatrix}_{sl(2,q)} \begin{pmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{pmatrix} \quad (30)$$

as long as (3) holds. In (30) the entries of the similarity matrix are

$$\gamma_{\pm} = \frac{\Gamma(\pm \frac{2j+1}{k+2})}{\left[\Gamma(\pm \frac{3j+2}{k+2}) \Gamma(\pm \frac{2j}{k+2}) \right]^{1/2}} \quad (31)$$

The relation between the two matrices is a similarity transformation which obviously preserves the polynomial equations. It is also clear that we cannot obtain the factors $\gamma_{\pm}$ from the quantum group alone, because we are not looking at the differential equations satisfied by the blocks, and these normalization factors are the normalization conditions of [5] for the blocks. As a side remark, it follows from (29) and (30) that the identification (3) is absolutely necessary for the two braiding matrices to agree. Another simple exercise is to check the Moore-Seiberg representation [3] of the Verlinde operators [2]. This is reproduced in fig.5. The result

of these operations implies, using the pentagon identity (26)

$$N_{0j} \begin{bmatrix} k & i \\ k & i \end{bmatrix} N_{i0} \begin{bmatrix} k & k \\ j & j \end{bmatrix} = N_{00} \begin{bmatrix} k & k \\ k & k \end{bmatrix} N_{ijk} \quad (32)$$

Using the quantum $6j$ symbols (25)

$$\begin{aligned} N_{0j} \begin{bmatrix} k & i \\ k & i \end{bmatrix} &= (-1)^{i+k-j} \sqrt{\frac{[2j+1]}{[2k+1][2i+1]}} \\ N_{i0} \begin{bmatrix} k & k \\ j & j \end{bmatrix} &= (-1)^{j+k-i} \sqrt{\frac{[2i+1]}{[2j+1][2k+1]}} \end{aligned}$$

and

$$N_{00} \begin{bmatrix} k & k \\ k & k \end{bmatrix} = \frac{(-1)^{2k}}{[2k+1]}$$

we conclude that (32) is correct.

The one loop polynomial equations [3] [4] are also easily checked. Before we explain the reason why the truncation of representations when q is a root of unity is consistent, we would like to digress on an intriguing connection between the modular properties of the $SU(2)_k$ characters and the behaviour of the $SL(2, q)$ characters under the transformation $q \rightarrow q^{-1}$.

3. We define the $SL(2, q)$ character for the spin j representation as

$$\chi_j(q) = N_q q^{ctr(j)} q^{H/2} = N_q [2j+1]_q q^{c_j} \quad (33)$$

where N_q is a constant to be fixed presently. It is useful to think of (33) as a regularized version of the Kac-Moody characters in the limit when the modular parameter $\tau \rightarrow 1$.

The modular transformation $S : \tau \rightarrow -1/\tau$ will be represented here by the transformation $q \rightarrow q^{-1}$. Its effect on (33) is

$$\chi_j(q^{-1}) = \sum_{l=0}^{k/2} S_{jl} \chi_l(q) \quad (34)$$

where

$$\chi_j(q^{-1}) = N_{q^{-1}} [2j+1]_{q^{-1}} q^{-c_j} = N_{q^{-1}} [2j+1]_q q^{-c_j} \quad (35)$$

We next show that with an appropriate choice for N_q we can identify the matrix S_{jl} with the modular transformation matrix S for Kac-Moody characters

$$S_{jl} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(2j+1)(2l+1)}{k+2} \quad (36)$$

the result follows directly if one uses the properties of gaussian sums. In particular

$$\sum_{k=0}^{4N-1} e^{2\pi i k^2/4N} = 2\sqrt{2N} e^{i\pi/4} \quad (37)$$

as long as

$$N_q = e^{-i\pi c/8} = q^{-3k/16} \quad (38)$$

where c is the central extension of the Virasoro algebra for the WZW model considered.

A more indirect, but also more illuminating way of illustrating this result is as follows : first notice that

$$[2j+1] = S_{j0}/S_{00}, q^{c_j} = e^{2\pi i \Delta_j}$$

$$\Delta_j = j(j+1)/(k+2)$$

then (34) turns into

$$\frac{N_{q^{-1}}}{N_q} S_{j0} = \sum_{l=0}^{k/2} e^{2\pi i \Delta_l} S_{jl} e^{2\pi i \Delta_l} S_{l0} \quad (39)$$

recalling that the matrix T implementing the modular transformation $\tau \rightarrow \tau + 1$ acts on the characters according to

$$T_{jl} = \delta_{jl} e^{2\pi i \Delta_l - c/24} \quad (40)$$

we can rewrite (39) in matrix notation

$$(ST)^3_{j0} = \frac{N_{q^{-1}}}{N_q} \delta_{j0} e^{-i\pi c/4}$$

In our case $(ST)^3 = 1$ because all the fields are self-conjugate (the OPE of any primary field with itself contains the identity operator), and we obtain (38). This suggests that in general we can define a quantum character for a primary field ϕ_a :

$$\chi_a = e^{-i\pi c/8} [a] e^{2\pi i \Delta_a}$$

with the "quantum" dimension of the primary field given by

$$[a] = S_{a0}/S_{00}$$

This works not only for the WZW model but also for the discrete unitary series of Virasoro representations [16]. The full significance of these remarks remains to be elucidated. There are however some operations which can be performed on quantum groups which closely resemble the modular operations S, T [13].

4 . We now briefly comment on the special group theoretical features of the representations of $SL(2, q)$ when q is a root of unity. Even though we are only considering this case explicitly, most of our conclusions also hold for other

simple groups. Using Weyl's approach to representation theory, the irreducible representations (irreps) of $SL(2, q)$ can be obtained by decomposing an arbitrary tensor product of the basic representation of spin $1/2$, $(V^{1/2})^{\otimes N}$. For a spin j irrep we have defined a Yang-Baxter matrix (10) $R^{jj} : (V^j)^{\otimes 2} \rightarrow (V^j)^{\otimes 2}$. Using this matrix, one can define a representation of the braid group on N strands B_N in $Aut(V^j)^{\otimes N}$:

$$g_i^{(j)} = 1 \otimes \dots \otimes R^{jj} \otimes 1 \dots \otimes 1 \quad (41)$$

where R acts on the $i, i \pm 1$ spaces in the tensor product. The elements (41) are also the generators of the centralizer $C_N^j(q)$ of $SL(2, q)$ in $Aut(V^j)^{\otimes N}$ [10]. For $j = 1/2$, the algebra $C_N^{1/2}(q)$ is isomorphic to the Temperley-Lieb-Jones algebra [17] and therefore a factor of the Hecke algebra $H_{N+1}(q)$ (see below). For generic q the algebra $C_N^j(q)$ is simple [10] and the representation theory of the q -deformed version of the classical algebra has few surprises. The centralizer of the quantum group plays a role similar to the Weyl group in the classical case. If we decompose $(V^{1/2})^{\otimes N}$ into irreducible components, we obtain

$$(V^{1/2})^{\otimes N} = \sum_i W_i \otimes V^i \quad (42)$$

where the W_i 's are irreps of $C_N^{1/2}(q)$. The decomposition (42) can be generalized to arbitrary spin

$$(V^j)^{\otimes N} = \sum_i W_i^{(j)} \otimes V^i \quad (43)$$

and now $W_i^{(j)}$ is an irrep. of $C_N^j(q)$. The multiplicity of V^i equals $\dim W_i^{(j)}$. To understand the irreps. of the quantum group we have to understand the irreps. of its centralizers. In (43) we are taking a product of N spin j irreps. to obtain as a result a spin- l irrep. Hence a basis of $W_i^{(j)}$ can be represented as a conformal block (fig. 6). In this basis, the braid group B_N acts on the external j -legs leaving the l -leg fixed. The representation of the generator $\sigma_i \in B_N$ braiding the

i and $i + 1$ legs is the C matrix (27)

$$C_{p_i p_i'} \begin{bmatrix} j & j \\ p_{i-1} & p_{i+1} \end{bmatrix}$$

and σ_1 acts diagonally (multiplication by $(-1)^{2j-p_2}(c_{p_2}-2c_j)/2$). Since $C_N^j(q)$ can be constructed in terms of $C_N^{1/2}(q)$ we will mainly study blocks W_j with N spin $1/2$ legs and one spin j leg. Now the R matrix takes a particularly simple form

$$q^{3/4} R^{1/2, 1/2} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-1 & q^{1/2} & 0 \\ 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad (44)$$

(from now on we remove the super-index j denoting the representation). Combining (41) and (44) one can show that

$$e_i = \frac{1 + q^{3/4} g_i}{1 + q} \quad (45)$$

satisfies the Jones algebra [17]

$$e_i e_{i \pm 1} e_i = \beta^{-1} e_i$$

$$e_i e_j = e_j e_i, |i - j| \geq 2$$

$$e_i^2 = e_i$$

$$\beta = 2 + q + q^{-1}$$

easily related to the Hecke algebra $H_n(q)$

$$\hat{g}_i = q^{3/4} g_i$$

$$\hat{g}_i \hat{g}_{i+1} \hat{g}_i = \hat{g}_{i+1} \hat{g}_i \hat{g}_{i+1}$$

$$\hat{g}_i \hat{g}_j = \hat{g}_j \hat{g}_i, |i - j| \geq 2$$

$$\hat{g}_i^2 = (q - 1) \hat{g}_i + q \quad (46)$$

For generic q the representations of $H_n(q)$ are classified by Young diagrams (as for the permutation group, obtained from (46) by setting $q = 1$). If $\lambda = [\lambda_1, \dots, \lambda_k]$ is a tableau with n boxes; λ_1 boxes in the first row etc., we can associate a representation of $H_n(q)$ to each λ . When q is an l -th root of unity, most of this representations are singular. Wenzl [18] has studied which among all the tableaux furnish regular representations. His conclusion is that (k, l) -tableaux with $k \leq l+1$, at most k rows and $\lambda_1 - \lambda_k \leq l - k$ define regular irreducible representations $\pi_{(k,l)}$ of $H_N(q, q^l = 1)$ and that the truncation to this subset gives a consistent representation theory. From the point of view of Hecke algebras, there is no a priori restriction on the values of $k \leq l+1$. However if we are interested in $C_N^{1/2}$ (to understand the restrictions on the representations of $SL(2, q)$) we obtain the constraint $k = 2$. The origin of this constraint is as follows: $C_N^{1/2}$ is isomorphic to Jones algebra, which in turn is a quotient of H_{N+1} by a cubic polynomial relation between the g_i 's ([19] equation 11.6), and only representations of H_{N+1} with at most two rows satisfy this constraint [19]. For $q^l = 1$ the representations of $C_N^{1/2}$ are reduced to tableaux with two rows, and such that $\lambda_1 - \lambda_2 \leq l - 2$. To get some feeling for these restrictions, we analyze the example with $l = 5$.

The $(2, 5)$ -Young tableaux with 3, 4 and 5 boxes appear in fig.7. From the point of view of $SU(2)$ these are representations with spins $j = 1/2, 3/2, 0, 1, 1/2, 3/2$ respectively. Hence $(2, 5)$ diagrams correspond to $j \leq 3/2$. In general $(2, l)$ tableaux satisfy $j \leq (l - 2)/2$. In conclusion, for $q^l = 1$ the only irreps. of the centralizer are given by W_j 's with $j \leq (l - 2)/2$. (There are similar generalizations for $SU(N)$). This restriction is the same as the constraint on the possible integrable highest weight representations describing the $SU(2)_k$ WZW theory (see [15] for details). In particular the conformal blocks of these

theories define representations of $H_N(q)$ with $q^{k+2} = 1$ something already known [6]. To determine the dimensions of the spaces W_j and the fusion rules requires more work. By considering explicitly the sequence of embeddings :

$$\dots \pi_{(k,l)} H_n(q) \subset \pi_{(k,l)} H_{n+1}(q) \subset \dots, q^l = 1 \quad (47)$$

it is possible to show that the dimensions of the W 's are exactly the ones one would obtain using the fusion rules for $SU(2)$ level k . More precisely, to compute $\dim W_j$ for different tableaux, it suffices to look at the integers appearing in the Bratteli diagram associated to the inclusion (47) computed in [18, 17]. Details will be presented elsewhere [13].

5. In conclusion, we have shown that the representation theory of $SL(2, q)$ with q a root of unity provides solutions to the polynomial equations for RCFT [3, 4], and it also provides a rather efficient way to compute the duality matrices (2). Furthermore, there seems to be a rather close connection between the modular properties of the Kac-Moody characters, and the behavior of the q -characters under the transformation $q \rightarrow q^{-1}$. Similar conclusions also hold for other quantum groups, and preliminary results show that one can extend the construction to q -homogeneous spaces [13] as well. There is an interesting puzzle if we use the q has unit modulus, but it is not a root of unity, the hexagon and pentagon identities still hold, however it does not seem easy to construct a local CFT whose C, N matrices are determined by the corresponding $6j$ symbols. It is not clear yet however whether the one loop part of the polynomial equations will still be satisfied. If this were the case, this might imply that for irrational Conformal Field Theories extra conditions should be required in order to have modular invariance and factorization for all genera. This may be similar to the potential modular anomalies that one should find in the Coulomb gas approach to the minimal theories [5] constructed by Dotsenko and Fateev [20] when the background charge α_0 does not have a rational square. Another possibility is that

what fails is the possibility of reconstruction the Hilbert space of conformal blocks with reasonable properties. When q is a root of unity, the relevant factors of the centralizer are C^* -algebras, and for these one can use the GNS reconstruction theory to reconstruct a Hilbert space with a positive definite metric. We find very attractive the fact that a second quantized two dimensional field theory (which could represent a string moving on a classical space, the group manifold) can be described rather accurately in terms of a quantum system with a finite number of degrees of freedom "moving" on a quantum space: the quantum group. These questions are currently being investigated.

It may be possible to relate the three dimensional topological Chern-Simons theory [11] with the results given here. If one considers three manifolds with boundaries and three dimensional Wilson lines going through them, it was argued by Witten [11] that the constraints on the possible gauge connections were equivalent to having a quantum group. In his study of the WZW theory, Fröhlich (see [14]) finds a connection satisfying the classical Yang-Baxter equation and giving the monodromy properties of the solutions to the Knizhnik-Zamolodchikov differential equations. We believe that if one constructed explicitly the global holonomy for this connection, one would be led to the braiding matrices constructed in this paper. This might provide an indirect way to solve these constraints in the Chern-Simons theory.

REFERENCES

- [1] The first time that the notion of a RCFT appeared is in the formulation of Conformal Field Theories in the Universal Moduli Space initiated by D. Friedan and S. Shenker : Nucl. Phys. **B281**(1987)509.
- [2] E. Verlinde : "Fusion Algebra and Modular Transformations in Conformal Field Theory" . Utrecht preprint THU-88/17.
- [3] G. Moore and N. Seiberg : " Polynomial Equations in Rational Conformal Field Theory" . IASSNS-HEP-88/8.
- [4] G. Moore and N. Seiberg : " Classical and Quantum Conformal Field Theories " . IASSNS-HEP-88/35. To appear in Comm. Math. Phys.
- [5] A. A. Belavin , A. M. Polyakov and A. B. Zamalodchikov : Nucl. Phys. **B241** (1984)333.
- [6] A. Tsuchiya and Y. Kanie : Lett. Math. Phys. **13** (1987)303.
- [7] V. I. Knizhnik and A. B. Zamalodchikov : Nucl. Phys. **B247**(1984)83.
- [8] E. Witten : Comm. Math. Phys. **92**(1984)455.
- [9] L. Alvarez-Gaumé , C. Gomez and G. Sierra CERN-TH.5129/88.
- [10] A. N. Kirillov and N. Yu. Reshetikhin : LOMI-Preprint-E-9-88 , N. Yu. Reshetikhin : LOMI-Preprint-E-4-87.
- [11] E. Witten : " Quantum Field Theory and the Jones Polynomials" . To appear in the Proc. of the IAMP Congress. Swansea, July 88.
- [12] V. G. Drinf'eld : "Quantum Groups" . In Proc. of the International Congress of Mathematicians. MSRI, Berkeley, 1986 , and L. D. Faddeev , N. Yu. Reshetikhin and L. A. Takhtajan : Quantization of Lie Groups and Lie Algebras" . LOMI-Preprint E-14-87 , and references therein.
- [13] L. Alvarez-Gaumé , C. Gomez and G. Sierra , to appear.

[14] B. Schroer : " Algebraic Aspects of Non-Perturbative Quantum Field Theory". Como Lecture Notes. K. Bleuler ed. to appear in Springer-Verlag. See also J. Fröhlich : " Statistics of Fields , the Yang-Baxter equation and the Theory of Knots and Links". Lectures at the Cargèse Summer Institute. July 87. A. Jaffe et al. eds. Plenum Press to appear.

[15] D. Gepner and E. Witten : Nucl. Phys **B278**(1986)493 .

[16] D. Friedan , Z. Qiu and S. Shenker : Phys. Rev. Lett. **52** (1984)1575.

[17] V. Jones : Inv. Math. **72**(1983)1 , and references therein.

[18] H. Wenzl : Inv. Math. **92**(1988)349.

[19] V. Jones : Ann. of Math. **126**(1987)335.

[20] V. S. Dotsenko and V. A. Fateev : Nucl. Phys. **B240**(1984)312 , ibid **B251**(1985)691 , Phys. Lett. **154B**(1985)291.

$$\begin{array}{c} j \\ | \\ i \end{array} \begin{array}{c} k \\ | \\ p \end{array} \begin{array}{c} l \\ | \\ i \end{array} = \sum_{p'} c_{pp'} \begin{array}{c} j \ k \\ | \quad | \\ i \quad i \end{array} \begin{array}{c} j \\ | \\ p' \end{array} \begin{array}{c} l \\ | \\ i \end{array}$$

$$\begin{array}{c} j \\ | \\ i \end{array} \begin{array}{c} k \\ | \\ p \end{array} \begin{array}{c} l \\ | \\ i \end{array} = \sum_{p'} n_{pp'} \begin{array}{c} j \ k \\ | \quad | \\ i \quad i \end{array} \begin{array}{c} j \ k \\ | \quad | \\ p' \quad p' \end{array} \begin{array}{c} l \\ | \\ i \end{array}$$

Fig. 1

Fig. 2

$$= (-1)^{j_1+j_2-j} \frac{1}{q} \sum (c_j - c_{j_1} - c_{j_2})$$

Fig. 3

$$= \sum_{j'} N_{j'} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}$$

Fig. 4

Fig. 5

Fig. 6

Fig. 7