ON THE STABILITY OF STATIC BOSONIC MEMBRANES

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ABSTRACT

We examine the stability of static open or closed bosonic membranes. We find that the quantum mechanically induced Casimir forces are attractive for spherical or various toroidal membranes. On the contrary, in the case of open rectangular membranes the sign of the Casimir forces changes as a function of the side length ratio. For a particular value of this ratio the Casimir force vanishes. The implications for the membrane stability are discussed.

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There have been recently some interesting developments in the study of the physical properties of relativistic membranes and supermembranes [1-11]. One very central issue is the nature of the spectrum of these theories. Due to the intrinsic non-linearities in the membrane interactions, progress in this question has been slow [11]. In a particular gauge, the light-cone gauge, the theory has striking similarities with Yang-Mills (YM) or super YM quantum mechanics [4,12] with $SU(N)$ as gauge group, but with N going to infinity.

In a recent paper, Lüscher, de Wit and Nicolai [13] proved that for every finite N , the $SU(N)$ super YM quantum mechanical system has continuous spectrum starting from zero mass. This is not the case for the bosonic YM system. In order to pass over the supermembrane limit ($N \rightarrow \infty$), one should be able to remove the mode cut-off N , in a non-perturbative way. We remind the reader that the membrane and supermembrane theories are three-dimensional non-renormalizable field theories by power counting.

Our attention has been concentrated to the bosonic membranes which, because of their geometrical interactions (minimal embeddings), they may lead to some new insights in the study of large N , YM theories [4,14].

In our recent work we studied the ground state energy of some particular membranes in the static approximation [15,16]. The static potential of an open membrane with fixed square boundary has been found in the leading $1/d$ approximation, where d is the space-time dimension [15]. For small size of the boundary, the potential was found to be shifted to positive values with respect to the classical one, $V_c = \kappa R^2$, κ membrane tension and R the size of the square boundary. In the case of the spherical membrane [16], we had to use an embedding space-time with topology $S^2 \times R^{d-2}$, in order to keep the membrane static and wrap the membrane in the spherical part of the space-time S^2 with fixed radius R . The transverse fluctuations of the membrane give rise to attractive forces increasing the classical force, ($V_c = 4\pi\kappa R^2$) and below some critical radius R_c , we found that tachyonic instabilities arise for the spherical membrane. If we could take into account the radial fluctuations, it is not excluded that the instabilities would disappear. For this, we should have introduced fluctuations around classical time varying solution, a much more cumbersome calculation. The above $1/d$ calculations included the leading in d contributions to all orders in the expansion around the classical solution. These results can be understood qualitatively already at the first perturbative level, that is, keeping only the quadratic fluctuations or the first term in the $1/R$ expansion (R the size of the membrane). The potential for the square membrane in this approximation turns out to be

$$V(R) = \kappa R^2 + \frac{d-3}{2R} a \quad (1)$$

where a is a positive constant [15]

$$\alpha = \sum_{\eta, \eta=1}^{\infty} (\eta^2 + \eta^2)^{1/2} = 0.0261 \quad (2)$$

and thus there is an equilibrium between the attractive classical force and the repulsive Casimir force at some radius R_0

$$R_0 = \left(\frac{d-3}{4\kappa} \alpha \right)^{1/3} \quad (3)$$

In Ref. [15], it was shown that the $1/d$ resummation shifts R_0 to zero. In the case of the spherical membrane we have [16]

$$V(R) = 4\pi\kappa R^2 + \frac{d-3}{2R} b \quad (4)$$

where

$$b = -0.2651 = \sum_{\ell=0}^{\infty} (2\ell+1) (\ell(\ell+1))^{1/2} \quad (5)$$

So in this case the Casimir force is attractive and the sphere tends to collapse. In the present work, we collect some further results for static open rectangular membranes [9] as well as toroidal static membranes [17]. We find that the Casimir force for the rectangular membranes passes from repulsive to attractive, vanishing at some special value of the two side ratio, while for the various toroidal membranes we find, as in the case of the sphere, attractive Casimir forces. To calculate the quadratic fluctuation correction to the classical potential, we follow the methods developed in Refs. [15,16]. We expand around the static classical solution with the given geometry, fix the gauge in the background classical solution and evaluate the determinant of the fluctuation operator. Then the potential is given as:

$$V(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \int \mathcal{D}X_{\perp} e^{-S} \quad (6)$$

$$S = \kappa \int_0^T dt \int_R d^2\xi \det^{1/2} [g_{\alpha\beta}^{cl} + \partial_{\alpha} X^{\perp} \partial_{\beta} X^{\perp}] \quad (7)$$

Here we have used the background field gauge [15,16]

$$X^0 = t, \quad X^{d-1} = X_{cl}^{d-1}, \quad X^{d-2} = X_{cl}^{d-2}, \quad (8)$$

$$X^{\perp} = (X^1, X^2, \dots, X^{d-3})$$

$$g_{\alpha\beta}^{cl} = \partial_{\alpha} X_{cl}^i \partial_{\beta} X_{cl}^i, \quad i = 0, d-1, d-2 \quad (9)$$

and X_{cl}^i is the classical solution

$$X_{cl}^{d-1} = \xi_1, \quad X_{cl}^{d-2} = \xi_2, \quad X_{cl}^1 = 0 \quad (10)$$

$(\xi_1, \xi_2) \in R = [0, a] \times [0, b]$ rectangle or torus.

The boundary conditions for the transverse fluctuations are of Dirichlet type, on the space-time boundaries:

$$X_{\perp}(t=0, \xi_1, \xi_2) = X_{\perp}(t=T, \xi_1, \xi_2) = 0, \quad \forall (\xi_1, \xi_2) \in Q \quad (11)$$

and $\forall t \in [0, T], \xi_1 \in [0, a], \xi_2 \in [0, b]$

$$X_{\perp}(t, a, \xi_2) = X_{\perp}(t, \xi_1, b) = X_{\perp}(t, 0, \xi_2) = X_{\perp}(t, \xi_1, 0) = 0 \quad (12)$$

for the rectangular open membrane, while we use periodic b.c. for the toroidal one:

$$X_{\perp}(t, 0, \xi_2) = X_{\perp}(t, a, \xi_2) \quad \forall (t, \xi_2) \in [0, T] \times [0, b] \quad (13)$$

$$X_{\perp}(t, \xi_1, 0) = X_{\perp}(t, \xi_1, b) \quad \forall (t, \xi_1) \in [0, T] \times [0, a] \quad (14)$$

After these necessary definitions we find:

$$V(R) = \kappa A + \frac{d-3}{2} Tr \Delta^{1/2} \quad (15)$$

where A is the area of the membrane and Δ is the two-dimensional Laplacian for the surface of the rectangular or toroidal membrane with boundary conditions (11)-(14). The eigenvalues of Δ are correspondingly:

$$\lambda_{m\eta} = \frac{\pi^2 \eta^2}{a^2} + \frac{\pi^2 m^2}{b^2} \quad ; \quad \eta, m : 1, 2, \dots, \infty \quad (16)$$

for the rectangular open membrane and

$$\lambda_{m\eta} = \frac{4\pi^2 \eta^2}{a^2} + \frac{4\pi^2 m^2}{b^2} \quad ; \quad (m, \eta) \neq (0, 0), \quad m, \eta = 0, \pm 1, \pm 2, \dots \quad (17)$$

for the toroidal membrane.

The $Tr \Delta^{\frac{1}{2}}$ is defined through zeta function regularization. We can prove, using Poisson resummation techniques that if A is any 2×2 positive definite symmetric real matrix then $[m = (m_1, m_2), \text{ integers}]$

$$\sum_{\eta \in \mathbb{Z}^2} e^{-\eta^T A \eta} = \pi (\det A)^{-1/2} \sum_{\eta \in \mathbb{Z}^2} e^{-\pi^2 \eta^T A^{-1} \eta} \quad (18)$$

Using (18) we can regularize [9]

$$\begin{aligned} \text{Tr } \Delta^{1/2} &= \sum_{m, \eta=1}^{\infty} \left[\frac{\pi^2 \eta^2}{a^2} + \frac{\pi^2 \eta^2}{b^2} \right]^{1/2} \\ &= \frac{1}{24} \left(\frac{\pi}{a} + \frac{\pi}{b} \right) - \frac{ab}{8\pi} \zeta(3) \left(\frac{1}{a^3} + \frac{1}{b^3} \right) \\ &\quad - \frac{ab}{4\pi} \sum_{m, \eta=1}^{\infty} \frac{1}{[a^2 \eta^2 + b^2 \eta^2]^{3/2}} \equiv F_R(a, b) \end{aligned} \quad (19)$$

for the rectangular membrane and

$$\begin{aligned} \text{Tr } \Delta^{1/2} &= F_T(a, b) \\ &\equiv -\frac{ab}{\pi} \zeta(3) \left(\frac{1}{a^3} + \frac{1}{b^3} \right) - \frac{2ab}{\pi} \sum_{m, \eta=1}^{\infty} \frac{1}{[a^2 \eta^2 + b^2 \eta^2]^{3/2}} \end{aligned} \quad (20)$$

for the toroidal membrane. We introduce two functions of the ratio $\rho = a/b$

$$f_r(\rho) = \frac{1}{24} \left(\rho^{1/2} + \frac{1}{\rho^{1/2}} \right) - \frac{\zeta(3)}{8\pi^2} \left(\rho^{3/2} + \frac{1}{\rho^{3/2}} \right) - \frac{1}{4\pi^2} \sum_{m, \eta=1}^{\infty} \frac{1}{[\eta^2 \rho + \eta^2 / \rho]^{3/2}} \quad (21)$$

and

$$f_T(\rho) = -\frac{\zeta(3)}{\pi^2} \left(\rho^{3/2} + \frac{1}{\rho^{3/2}} \right) - \frac{2}{\pi} \sum_{m, \eta=1}^{\infty} \frac{1}{[\eta^2 \rho + \eta^2 / \rho]^{3/2}} \quad (22)$$

which satisfy the relations,

$$f_r\left(\frac{1}{\rho}\right) = f_r(\rho) \quad (23a)$$

$$f_T\left(\frac{1}{\rho}\right) = f_T(\rho) \quad (23b)$$

So it is sufficient to restrict ρ , $0 < \rho < 1$. The potential (15) is written in terms of f_r , f_T as

$$V(R) = \kappa A + \frac{d-3}{2} \frac{\pi}{\sqrt{A}} f_r(\rho) \quad (24)$$

and

$$V(R) = \kappa A + \frac{d-3}{2} \frac{\pi}{\sqrt{A}} f_T(\rho) \quad (25)$$

We observe that $f_T(\rho)$ is a negative function for any value of ρ while $f_r(\rho)$ changes

sign at some particular value $\rho = \rho_c$. To see that, we remind the reader that for $\rho = 1$, [square, Eq. (2)],

$$f_r(1) = \frac{0.0261}{\pi} > 0 \quad (26)$$

We can extract the small ρ behaviour of the potential (24) and (25) for constant A. Using again Poisson resummation formula we find,

$$f_r(\rho) \xrightarrow{\rho \rightarrow 0} \frac{1}{24} - \frac{1}{\rho^{1/2}} - \frac{5(3)}{8\pi^2} \frac{1}{\rho^{3/2}} + O(\rho^{1/2} e^{-\pi/\rho}) \quad (27)$$

Using relation (27), we can find with two decimal place accuracy the value of ρ_0 where $f_r(\rho)$ vanishes

$$\rho_0 = \frac{3\zeta(3)}{\pi^2} \simeq 0.365 \quad (28)$$

For the toroidal membrane we obtain

$$f_r(\rho) \xrightarrow{\rho \rightarrow 0} -\frac{\rho^{1/2}}{3} - \frac{5(3)}{\pi^2} \frac{1}{\rho^{3/2}} + O(\rho^{1/2} e^{-\pi/\rho}) \quad (29)$$

From (27) and (29), we see that keeping the classical potential constant (constant area), the quantum fluctuations create an energy which is minimized for elongated (string) configurations of the fixed boundary.

We come now to the discussion of these results. The bosonic membranes are described in the light-cone gauge by the Hamiltonian H [2,3,4,18]:

$$H = \frac{1}{2} \int d\sigma_1 d\sigma_2 [\dot{X}^2 + \sum_{i,j} \{X^i, X^j\}^2] \quad (30)$$

and the constraint

$$\sum_{i=1}^{d-2} \{\dot{X}^i, X^i\} = 0 \quad (31)$$

$i, j = 1, 2, \dots, d-2$. In the case of surfaces with non-trivial topology there are additional global constraints [11]. As it has been discussed in Refs. [10] and [13], there are (stringy) static degenerate configurations of the membranes for which the "magnetic" energy vanishes

$$B^{ij} = \{X^i, X^j\} = 0 ; \quad i, j = 1, 2, \dots, d-2 \quad (32)$$

This huge degeneracy of the classical vacuum of the bosonic theory is lifted presumably by quantum fluctuations [13]. This degeneracy comes from the infinite dimensional symmetry group of the area preserving diffeomorphisms, that is, co-ordinate transformations:

$$\sigma_1 \rightarrow \sigma_1'(\sigma_1, \sigma_2), \quad \sigma_2 \rightarrow \sigma_2'(\sigma_1, \sigma_2) \quad (33)$$

with Jacobian equal to one.

The calculations we have presented here and elsewhere [15,16] refer to a modified system because we fix the boundaries in the case of open membranes and change the topology of space-time in the case of closed membrane. We found that the Casimir energy in all cases decreases to more and more negative values as we go to elongated (string) configurations of the classical solution. To find the approximate ground state of the membrane (lifting of the degeneracy) we should permit the classical configurations we used to move slowly and find the energy levels of this non-relativistic system in the potentials which we calculated in the $1/d$ or even simpler in the quadratic approximation. This would correspond to an artificial separation of slow modes from fast modes of the membranes. The obvious possibility of starting with a string configuration and do perturbation theory around this classical minimum [13] requires probably a more profound understanding of the gauge fixing procedure for the infinite dimensional gauge group $SU(\infty)$.

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