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INTERMITTENCY AND CRITICAL BEHAVIOUR

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Abstract

Following a brief introduction to intermittent behaviour, we show that the Ising model leads to intermittency at the critical point. The intermittency indices are given in terms of the critical exponents. This result is expected to hold for second order phase transitions in general; for finite temperature $SU(2)$ gauge theory, it follows from universality.

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The concept of intermittency, originally developed in the study of turbulent flow¹, has recently become of considerable interest in statistical particle physics², in particular as a tool to investigate fluctuations. In a short numerical study, Wosiek³ has found hints for intermittent behaviour in the critical region of the two-dimensional Ising model. This suggests the general question: what, if any, relation is there between intermittency and critical behaviour? The answer is of interest also for the study of high energy nuclear collisions, in which one hopes to find evidence for the transition from hadronic matter to a quark-gluon plasma. In such reactions, there are indications of intermittency⁴.

In this note, we first illustrate in very simple terms the idea of intermittency; we then show that the critical behaviour of the Ising model indeed leads to intermittency, with indices determined by the critical exponents. Finally, we briefly mention the extension of the argument to second-order phase transitions in general.

Intermittency basically means random deviations from smooth or regular behaviour. To illustrate this, we imagine putting N balls into a box of size R , which is subdivided into n cells of size L ($n = R/L$). Let k_m denote the number of balls put into the m -th cell, so that $\sum k_m = N$. We want to see what happens when we vary n at fixed R and N , i.e., when the balls are partitioned among an increasing number of cells. The l -th moment for a given configuration (a given distribution of the balls) is defined as

$$f_l(n) = [(1/n) \sum_{m=1}^n k_m^l] / [(1/n) \sum_{m=1}^n k_m]^l, \quad (1)$$

i.e., it is the average of k_m^l over all m cells, normalized to the l -th power of the average k_m , which is just N/n :

$$f_l(n) = n^{l-1} N^{-l} \sum_{m=1}^n k_m^l. \quad (2)$$

The corresponding moments for an ensemble are then obtained by averaging $f_l(n)$ with the relevant weights over all the configurations of the ensemble; we denote this by $\langle f_l(n) \rangle$. By definition, $\langle f_1(n) \rangle = f_1(n) = 1$.

Now let us see what $f_l(n)$ looks like for some specific configurations. In the case of equidistribution (N/n balls in each box), we get with

$$k_m = N/n \quad \forall m, \quad (3)$$

in eq.(1) that

$$f_l(n) = 1 \quad \forall l. \quad (4)$$

On the other hand, if we put all the balls into one box,

$$k_m = \begin{cases} N, & m = r \\ 0, & m \neq r \end{cases} \quad (5)$$

and hence

$$f_l(n) = n^{l-1}. \quad (6)$$

Eq.(6) tells us that for an extreme fluctuation (from a thermodynamic point of view), the moments increase as a power with the number of cells into which the system is subdivided. We can rewrite eq.(6) in the form

$$\ln f_l(n) = -(l-1) \ln L + (l-1) \ln R, \quad (7)$$

which equivalently says that the logarithm of the moments varies linearly with the logarithm of the cell size. Such behaviour is called intermittent, and the coefficients of $\ln L$ are referred to as intermittency indices.

We can now easily formulate a more general definition of intermittency. Consider a d -dimensional volume R^d , subdivided into $n = (R/L)^d$ cells of equal size L^d . Distribute N objects into the array of cells, with k_m objects in the k -th cell. The l -th moment for a given configuration is again given by eq.(1)*; in the ensemble average, we shall in general not keep N fixed, however, but average with the relevant weights over all possible configurations. We speak of intermittency if

$$\ln \langle f_l(L) \rangle = -\lambda_l \ln L + g_l(R), \quad (8)$$

with constants $\lambda_l > 0$ and $g_l(R)$ independent of L . The set λ_l are the intermittency indices.

In the case of the Ising model, we start from a d -dimensional lattice of R sites in each dimension. To each of the R^d sites, we associate a spin variable $s_i = \pm 1$. The relative weights for the different possible spin configurations are determined by the Boltzmann factor

$$\exp(-\beta \sum_{i,j} s_i s_j), \quad (9)$$

with $i \neq j$ running over all sites of the lattice; β denotes the inverse temperature. We subdivide the lattice into $n = (R/L)^d$ blocks of size L^d , for a given block size L , the average moments are then defined as

$$\langle f_l(L) \rangle = \langle (1/n) \sum_{\text{blocks}} \left[\sum_{i=1}^{L^d} (s_i/L^d) \right]^l / \left[(1/n) \sum_{i=1}^{R^d} (s_i/L^d) \right]^l \rangle. \quad (10)$$

The average over all possible configurations is carried out with the weights (9); it thus depends on the temperature β^{-1} . The limiting case, just one spin per cell, gives us

$$\langle f_l(L=1) \rangle = \begin{cases} \langle s^{-l} \rangle, & l \text{ even} \\ \langle s^{-(l-1)} \rangle, & l \text{ odd} \end{cases} \quad (11)$$

Here we have defined

$$s = \sum_{i=1}^{R^d} s_i / R^d \quad (12)$$

* In the case of finite size systems, one often considers factorial moments^{2,3}, in order to eliminate statistical fluctuations. We are here, however, interested in the thermodynamic limit and therefore retain the simple moments (1).

and used $s_i^l = 1$ (l even) and $s_i^l = s_i$ (l odd). In terms of s , we can rewrite eq.(10) in the form

$$\langle f_l(L) \rangle = \langle (1/n) \sum_{\text{blocks}} \left[\sum_{i=1}^{L^d} (s_i/L^d) \right]^l / s^l \rangle. \quad (13)$$

For obtaining intermittency, the crucial feature of the Ising model is its invariance under scale transformations as we approach the transition point. As $\beta \rightarrow \beta_c$, the correlation length ξ diverges. This allows a subdivision of the lattice into blocks of linear size L , with $\xi \gg L \gg 1$; here ξ is measured in lattice spacings. The critical behaviour of the “blocked” system is the same as that of the original model; we can therefore define block spins⁵ $\tilde{s}_\alpha = \pm 1$,

$$Q \tilde{s}_\alpha = \sum_{i=1}^{L^d} s_i / L^d, \quad (14)$$

in terms of a function Q depending only on the block size L . With this, eq.(13) becomes

$$\langle f_l(L) \rangle = \begin{cases} Q^l \langle f_l(L=1) \rangle, & l \text{ even} \\ Q^{(l-1)} \langle f_l(L=1) \rangle, & l \text{ odd} \end{cases} \quad (15)$$

where we have used relation (11). To keep the field-dependent term of the Ising Hamiltonian,

$$h \sum_{i=1}^{R^d} s_i, \quad (16)$$

invariant under the block transformation (14), we must have

$$\tilde{h} = (L^d Q) h \quad (17)$$

where h denotes the external field. It transforms under blocking as

$$\tilde{h} = L^x h \quad (18)$$

with an exponent x which can be expressed in terms of the critical exponents of the Ising model. Eqs. (17) and (18) give

$$Q = L^{-(d-x)}. \quad (19)$$

Inserting eq. (14) in the expression (13) for the moments, and using eqs. (11) and (19), we obtain

$$\langle f_l(L) \rangle / \langle f_l(L=1) \rangle = \begin{cases} L^{-l(d-x)}, & l \text{ even} \\ L^{-(l-1)(d-x)}, & l \text{ odd} \end{cases} \quad (20)$$

With

$$\ln \langle f_l(L) \rangle = -\lambda_l \ln L + \ln \langle f_l(L=1) \rangle, \quad (21)$$

$$\lambda_l = \begin{cases} -l(d-x), & l \text{ even} \\ -(l-1)(d-x), & l \text{ odd} \end{cases} \quad (22)$$

we have thus arrived at intermittency for the Ising model. Note that this result is valid only for $\xi \gg L$, since otherwise relation (14) becomes untenable. Therefore, if we measure L in units of ξ , with $\Delta = L/\xi$,

$$\ln \langle f_l(\Delta) \rangle = -\lambda_l \ln \Delta + \ln \langle f_l(\Delta = 1/\xi) \rangle \xi^{-\lambda_l}, \quad (23)$$

then we obtain intermittency as $\Delta \rightarrow 0$. It remains to fix the intermittency indices in terms of the critical exponents. The factor $(d-x)$ in eq.(22) governs the behaviour of the correlation function $\Gamma(r, \beta)$ of two spins separated by a distance r at the critical point $\beta = \beta_c$:

$$\Gamma(r, \beta_c) \sim r^{-2(d-x)}. \quad (24)$$

Scaling arguments give⁵

$$2(d-x) = d-2+\eta, \quad (25)$$

where η is the anomalous part of the critical scale dimension. For the two-dimensional Ising model, only the non-zero value of η therefore leads to intermittency. In the classical (mean field) theory of exponents, which neglects the effect of fluctuations, η vanishes; it is thus the presence of non-uniform spin configurations for all scales which provides the intermittent behaviour. The exponent x is also directly related to the critical exponent δ , governing the variation of the magnetization with the external field on the critical isotherm:

$$x = d [\delta/(\delta+1)]. \quad (26)$$

For $d=2$ and $d=3$, we have $\delta=15$ and 5 , respectively, so that we obtain

$$\lambda_l = \begin{cases} l/8, & d=2 \\ l/2, & d=3 \end{cases} \quad (27)$$

when l is even and

$$\lambda_l = \begin{cases} (l-1)/8, & d=2 \\ (l-1)/2, & d=3 \end{cases} \quad (28)$$

when l is odd, for the intermittency indices in the two most interesting cases.

The crucial aspect of the argument just given is the divergence of the correlation length, with the resulting invariance under block spin transformations. This property remains true for all second-order phase transitions, and hence such critical behaviour is expected to lead quite generally to intermittency. For finite temperature SU(2) gauge theory, there is moreover the proposed universality relation⁶ to the Ising model, implying the same critical exponents for gauge and spin systems⁷. As a consequence, the SU(2) gauge system should show intermittency with the indices (22).

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