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STRING THEORY CONSTRUCTIONS: AN INTRODUCTION TO MODERN METHODS

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ABSTRACT

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STRING THEORY CONSTRUCTIONS: AN INTRODUCTION TO MODERN METHODS

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Using the Batalin-Fradkin-Vilkovisky formalism for constrained systems, BRST quantisation of bosonic and spinning string theories is introduced. As a warm-up exercise, the case of the scalar point-particle is first considered.

1. INTRODUCTION

String theory, with its wide range of different possible constructions, is the only known viable and consistent proposal for a unified description of all interactions among all particles, including also quantum gravity. So far, it passes with flying colours all quantum consistency tests that theorists love to impose as guiding principles in their quest for an "understanding of everything".

Although so much remains to be understood, not only in establishing definitely the consistency of these theories, but also, and this even more important from a physicist's point of view, in relating (at least one of) these different string constructions to the world of elementary particles as we know it, string theory certainly is a domain of theoretical physics which is there to stay with us. Developments over the last years have led to fundamental new insights not only in string theory itself, but also in two dimensional statistical physics and conformal field theories, and in pure mathematics. This alone already explains why it is so fascinating to work in this field. And after all, some string theory may well turn out to be indeed ... a theory of our Universe !

These notes contain part of and are in complement to the lectures presented in

anticipation of the School, during which recent developments in the construction of four-dimensional strings were discussed by Bert Schellekens. They are meant to be read in parallel with and as a follow-up on notes published in Refs.[1,2], which were the basis for the lectures. Ref.[1] contains a detailed discussion of the old covariant and light-cone gauge quantisations of bosonic strings, spinning strings, superstrings and ten dimensional heterotic strings, and of their spectra. Other useful reviews on string theory are found in Refs.[3-7].

As an introduction to modern methods used in constructions of string theories, the present notes discuss the BRST quantisation of bosonic strings and spinning strings, following Hamiltonian methods developed for constrained systems by Batalin, Fradkin and Vilkovisky (BFV) (for a review and references, see Ref.[8]). As a warm-up exercise, we first discuss the BRST quantisation of the free relativistic scalar particle (for references, see Refs.[9,10]). This system already shares, but in a much simpler manner, most of the technicalities which appear for string theory.

String theories however, possess much richer structures which we shall describe as we proceed through the discussion. By lack of

space, we shall only describe the construction of the BRST formulation for bosonic and spinning strings, and derive some standard results. The interested reader may nevertheless find here what is a necessary basis in understanding recent constructions of four-dimensional strings, and some essential points in Bert Schellekens's lectures at the School [11].

We use the same notations as in Ref.[1]. In particular, our space-time Minkowski metric has signature $(- + \dots +)$, and we usually take $\hbar=1=c$.

2. THE SCALAR POINT-PARTICLE

2.1. Lagrangian formulation

Considering the non-linear action

$$S[x] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\dot{x}^2(\tau)}, \quad (2.1)$$

where the particle trajectory $x^\mu(\tau)$ is given by D functions of the world-line parameter τ , it is well known [1,2] that the equation of motion for x^μ simply corresponds to the conservation of the energy-momentum P^μ of the particle:

$$P_\mu(\tau) = \frac{dL(\dot{x})}{d\dot{x}^\mu(\tau)} = \frac{m\dot{x}_\mu(\tau)}{\sqrt{-\dot{x}^2(\tau)}}, \quad \frac{d}{d\tau} P_\mu(\tau) = 0. \quad (2.2)$$

For an infinitesimal reparametrisation

$$\tilde{\tau}(\tau) = \tau - \eta(\tau), \quad \eta(\tau_1) = 0, \quad i=1,2, \quad (2.3)$$

one obtains:

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \dot{\eta}(\tau) [\dot{x}^\mu P_\mu - L] = \int_{\tau_1}^{\tau_2} d\tau \dot{\eta}(\tau) \frac{\sqrt{-\dot{x}^2}}{m} [P^2 + m^2] \quad (2.4)$$

The action being invariant under such transformations, (2.4) shows that $(P^2 + m^2)$ is, up to a factor, the generator of world-line diffeomorphisms, so that in this description of the system, the constraint

$$\phi(\tau) = \frac{1}{2} [P^2(\tau) + m^2] = 0 \quad (2.5)$$

is a consequence of reparametrisation invariance. This gauge freedom allows us to fix the parametrisation through some gauge-fixing

condition, such as the covariant proper-time gauges [12]:

$$-\dot{x}^2(\tau) = m^2 \lambda_0^2, \quad \lambda_0 > 0. \quad (2.6)$$

For such a choice, the equation of motion becomes linear, and its solution is simply:

$$x^\mu(\tau) = q^\mu + P^\mu \lambda_0 \tau, \quad P^\mu(\tau) = P^\mu, \quad (2.7)$$

where the two integration constants q^μ and P^μ (with P^μ being the total momentum of the particle) are determined from the boundary conditions, say $x^\mu(\tau_i) = x_i^\mu$ ($i=1,2$). As a consequence of gauge-fixing in (2.6) however, we must have:

$$P^2 + m^2 = 0. \quad (2.8)$$

Similar considerations follow from the linear action

$$S[x, e] = \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau [e^{-1} \dot{x}^2 - e m^2] = \int_{\tau_1}^{\tau_2} d\tau L(\dot{x}, e) \quad (2.9)$$

where the 1-dimensional scalar "fields" x^μ are coupled to a world-line einbein $e(\tau) > 0$ ($\tau_1 < \tau < \tau_2$). The equation of motion for x^μ expresses again [1,2] the conservation of the particle energy-momentum P^μ :

$$P_\mu = \frac{\partial L(\dot{x}, e)}{\partial \dot{x}^\mu} = \frac{\dot{x}_\mu}{e}; \quad \frac{d}{d\tau} P_\mu = 0. \quad (2.10)$$

In addition, we now have the equation of motion for the einbein, which is algebraic since $e(\tau)$ is an auxiliary field:

$$\dot{x}^2 + m^2 e^2 = 0. \quad (2.11)$$

That this constraint follows from reparametrisation invariance can be seen, either by considering the variation of the action under infinitesimal reparametrisations (as in (2.4)), or by the following. In general, for a d -dimensional field theory (here $d=1$) coupled to a gravitational field $g_{\alpha\beta}$ ($\alpha, \beta, \dots = 0, 1, \dots, d-1$) with the Einstein-Hilbert action, we have, as a consequence of reparametrisation invariance, the covariant conservation of the matter energy-momentum tensor $T_{\alpha\beta}$ ($D_\alpha T^{\alpha\beta} = 0$), which in turn leads to Einstein's equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \quad (2.12)$$

where Λ is a cosmological term. For $d=1$, this reduces to

$$\Lambda g_{\tau\tau} = \kappa T_{\tau\tau}. \quad (2.13)$$

Since in the present case

$$T_{\tau\tau} = \frac{1}{2} \dot{x}^2, \quad (2.14)$$

(2.12) indeed corresponds to the constraint (2.11).

As we shall not discuss the formulation of the bosonic string with a linear action [13], let us remark here that similar considerations show that the constraints $T_{\alpha\beta}=0$ ($\alpha, \beta=0,1$) one obtains in that case, are a consequence of diffeomorphic and Weyl invariance of the linear action. Indeed, for $d=2$, (2.12) reduces to $\Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$. Since Weyl invariance requires $T_{\alpha\alpha}=0$, we must then have $\Lambda=0$ and thus $T_{\alpha\beta}=0$. These are the constraints for string theory, in a formulation where an intrinsic world-sheet metric is coupled to the other degrees of freedom.

Reparametrisation invariance of (2.9) allows one to "fix a gauge". For example, with the proper-time gauge-fixing conditions

$$e(\tau) = \lambda_0, \quad \lambda_0 > 0 \quad (2.15)$$

the solution for $x^\mu(\tau)$ is as in (2.7), with the constraint (2.8) now following from (2.11).

That the two actions describe the same system classically (and actually also quantum mechanically) when $m=0$, can easily be seen by solving in the linear action for $e(\tau)$ as given from (2.11).

2.2. Covariant quantisation

To quantise covariantly this constrained system, let us first use the general formulation developed by Dirac [14] (for a résumé, see Ref.[1]). Considering the non-linear action (2.1), we have the degrees of freedom $x^\mu(\tau)$ and their conjugate momenta $P_\mu(\tau)$, with the Poisson brackets:

$$\{x^\mu(\tau), P_\mu(\tau)\} = \delta_\mu^\nu. \quad (2.16)$$

These phase-space variables are not all independent. We have the primary constraint $\phi(\tau)=0$ in (2.5). Moreover, the canonical Hamiltonian

$$H_0 = \dot{x}^\mu P_\mu - L = 0 \quad (2.17)$$

vanishes identically, as a consequence of reparametrisation invariance (see (2.4)). Hence, the total Hamiltonian is given by

$$H_T = H_0 + \lambda(\tau)\phi(\tau) = \lambda(\tau)\phi(\tau), \quad (2.18)$$

where $\lambda(\tau)$ is an arbitrary function of τ , and $\phi(\tau)$ is the only first-class constraint.

The corresponding equations of motion

$$\dot{x}^\mu = (x^\mu, H_T) = \lambda P^\mu, \quad \dot{P}^\mu = (P^\mu, H_T) = 0 \quad (2.19)$$

also follow from the first-order action

$$S[x, P, \lambda] = \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu P_\mu - \frac{1}{2} \lambda (P^2 + m^2)]. \quad (2.20)$$

The solutions to (2.19) may be expressed in terms of integration constants related to boundary conditions. For example:

$$x^\mu(\tau) = x_1^\mu + \frac{\Delta x^\mu}{c} \int_{\tau_1}^{\tau} d\tau' \lambda(\tau'); \quad P^\mu(\tau) = \frac{\Delta x^\mu}{c}, \quad P^2 + m^2 = 0 \quad (2.21a)$$

where

$$c = \int_{\tau_1}^{\tau_2} d\tau \lambda(\tau), \quad \Delta x^\mu = x_2^\mu - x_1^\mu, \quad x^\mu(\tau_i) = x_i^\mu \quad (i=1,2). \quad (2.21b)$$

The action (2.20) inherits from (2.1) an invariance under transformations generated by $\phi(\tau)$ through its Poisson brackets. For an infinitesimal function $\varepsilon(\tau)$ with $\varepsilon(\tau_i)=0$ ($i=1,2$), consider the variations:

$$\delta_\varepsilon x^\mu = \varepsilon P^\mu, \quad \delta_\varepsilon P^\mu = 0, \quad \delta_\varepsilon \lambda = \dot{\varepsilon}. \quad (2.22)$$

For (2.20), we then obtain:

$$\delta_\varepsilon S = \int_{\tau_1}^{\tau_2} d\tau \frac{1}{2} \frac{d}{d\tau} [\varepsilon (P^2 + m^2)] = 0. \quad (2.23)$$

With the identifications

$$\varepsilon(\tau) = \lambda(\tau)\eta(\tau), \quad P^\mu(\tau) = \frac{\dot{x}^\mu(\tau)}{\lambda(\tau)} \quad (2.24)$$

and η defined in (2.3), it can easily be seen that the variations (2.22) correspond to world-line reparametrisations in this formalism of the system, i.e. the constraint $\phi(\tau)$ is the generator for these local gauge transformations.

As a consequence of (2.22), we also have

$$\delta_0 c - \delta_0 \int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) = 0 \quad (2.25)$$

Hence, the quantity c is the Teichmüller parameter labelling the gauge orbits of the system for gauge transformations connected to the identity transformation. For positive energy solutions, we have forward propagation in time for $c > 0$ (say a particle) and backward propagation for $c < 0$ (an antiparticle). Considering all reparametrisations, including those reversing the world-line orientation, the space of modular parameters reduces to $\mathbb{R}/\mathbb{Z}_2 \approx (0, \infty]$ [10].

The freedom in the choice of the Lagrange multiplier $\lambda(\tau)$ thus corresponds to the freedom in the choice of world-line parametrisation. Hence, gauge-fixing of the system is tantamount to a choice for $\lambda(\tau)$. However, different functions $\lambda(\tau)$ leading to the same value of c lie on the same gauge orbit, i.e. they describe identically the same *physical* situation, as seen in (2.21). All such functions belonging to the gauge orbit characterized by a value c can be parametrised as [10]:

$$\lambda(\tau) = \frac{c}{\Delta\tau} \frac{d}{d\tau} h(\tau) \quad \text{with } h(\tau_i) = \tau_i, \Delta\tau = \tau_2 - \tau_1 \quad (i=1,2) \quad (2.26)$$

The choice corresponding to proper-time parametrisations in (2.6) is $\lambda(\tau) = \lambda_0$ ($\lambda_0 > 0$).

Solving for P^μ in (2.20), one recovers the linear action (2.9), where $\lambda(\tau)$ is identified with the world-line einbein. Indeed, the variation $\delta_0 \lambda$ in (2.22), with the identification (2.24), corresponds to the transformation of such a quantity under reparametrisations. The parameter c thus measures the total proper-time of the particle.

It is left as an exercise to perform the analogous analysis for the linear action (2.9), with the result that it eventually leads to the same discussion as above [9].

Through the correspondence principle, the quantised system is defined by the commutation relations (we always give only the non-vanishing ones):

$$[x^\mu, p^\mu] = i\hbar \delta^\mu_\nu \quad (2.27)$$

and the Schrödinger equation

$$i\hbar \frac{d}{d\tau} |\psi; \tau\rangle = H_T |\psi; \tau\rangle = -\frac{1}{2} \lambda(\tau) [p^2 + m^2] |\psi; \tau\rangle \quad (2.28)$$

The corresponding space of states is spanned by eigenstates $|q^\mu\rangle$ (resp. $|p^\mu\rangle$) of the position (resp. momentum) operator x^μ (resp. p^μ):

$$x^\mu |q^\mu\rangle = q^\mu |q^\mu\rangle, \quad p^\mu |p^\mu\rangle = p^\mu |p^\mu\rangle \quad (2.29a)$$

with

$$\langle q|q'\rangle = \delta^{(D)}(q-q'), \quad \langle p|p'\rangle = \delta^{(D)}(p-p') \quad (2.29b)$$

Space-time covariance of the quantised system is readily established by checking that the energy- and angular-momentum operators satisfy the Poincaré algebra.

Physical states are defined by the constraint

$$[p^2 + m^2] |\psi; \tau\rangle = 0 \quad (2.30)$$

which thus implies, by (2.28), that they are independent of τ , as it should. Note that in a wave-function representation of (2.27), in configuration space say:

$$|\psi; \tau\rangle = \int d^D q \, \psi(q; \tau) |q\rangle, \quad (2.31)$$

physical states are determined by

$$[-\hbar^2 \square + m^2] \psi(q) = 0 \quad (2.32)$$

which is the Klein-Gordon equation for a scalar particle.

2.3. BFV formulation

In the BFV approach [8] (for a brief review, see Ref.[10]), phase-space is extended as follows. First, one introduces a degree of freedom $\pi(\tau)$ conjugate to $\lambda(\tau)$:

$$(\lambda(\tau), \pi(\tau)) = 1 \quad (2.33)$$

with the first-class constraint $\pi(\tau) = 0$. Associated to the two constraints $G_1(\tau) = \pi(\tau)$ and $G_2(\tau) = \phi(\tau)$,

one introduces respectively two Hermitian Grassmann variables $\eta^1(\tau)$ and $\eta^2(\tau)$ and their conjugate momenta $\mathcal{P}_1(\tau)$ and $\mathcal{P}_2(\tau)$ (anti-Hermitian Grassmann variables) with

$$(\eta^a(\tau), \mathcal{P}_b(\tau)) = -\delta_a^b, \quad a, b=1, 2. \quad (2.34)$$

From the algebra of constraints $(G_a(\tau), G_b(\tau)) = 0 = (H_0(\tau), G_a(\tau))$, $a, b=1, 2$ (2.35) one obtains the BRST generator

$$Q_B = \eta^a G_a = \eta^1 \pi + \frac{1}{2} \eta^2 (p^2 + m^2), \quad (2.36)$$

and the BRST invariant Hamiltonian

$$H_B = H_0 = 0. \quad (2.37)$$

The ghost charge is defined by $Q_c = i\eta^a \mathcal{P}_a$. By construction, Q_B is nilpotent, i.e. $(Q_B, Q_B) = 0$.

In the present formulation, gauge invariance generated by the constraint $\phi(\tau)$ in the Hamiltonian formalism is now traded for BRST invariance, generated by Q_B . BRST transformations of any quantity X can be obtained from $\delta_B X = (X, Q_B)$, leading to a choice of BRST invariant boundary conditions, say:

$$x^i(\tau_i) = x_i^i, \quad \pi(\tau_i) = 0, \quad \eta^2(\tau_i) = 0 = \mathcal{P}_1(\tau_i), \quad i=1, 2 \quad (2.38)$$

Time-evolution of the system in extended phase-space is determined by an effective Hamiltonian

$$H_{\text{eff}} = H_B = (\Psi, Q_B) = -(\Psi, Q_B), \quad (2.39)$$

where Ψ is a function over extended phase-space of ghost number (-1) . Its role is to fix a gauge for the system, and it has to be chosen appropriately [9,10].

The first-order action leading to the corresponding equations of motion and the boundary conditions (2.38) is:

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \mathcal{P}_\mu - \lambda \dot{\pi} + \dot{\eta}^1 \mathcal{P}_1 - \eta^2 \dot{\mathcal{P}}_2 - H_{\text{eff}}] \quad (2.40)$$

Considering the choice [10]

$$\Psi = \mathcal{P}_1 f(\lambda) + \mathcal{P}_2 \lambda, \quad (2.41)$$

with $f(\lambda)$ some function of λ , one has:

$$H_{\text{eff}} = \frac{1}{2} \lambda (p^2 + m^2) + \pi f(\lambda) - \frac{d}{d\tau} f(\lambda) \mathcal{P}_1 \eta^1 - \mathcal{P}_2 \eta^1. \quad (2.42)$$

The corresponding equation of motion for $\lambda(\tau)$ is

$$\dot{\lambda} = f(\lambda). \quad (2.43)$$

Given a solution for $f(\lambda)$, the remaining equations can then be solved with the boundary conditions (2.38) [10]. One then recovers the solution (2.21a), together with

$$\pi(\tau) = 0 = \eta^a(\tau) = \mathcal{P}_a(\tau), \quad a=1, 2, \quad (2.44)$$

describing thus the same physical situation. Note that we then have $Q_B = 0$ and $Q_c = 0$, ensuring the gauge invariance of the solution.

The function Ψ , i.e. in the present case the function $f(\lambda)$, has to be chosen such that all gauge orbits of the system can be obtained. Such good gauge-fixing functions exist for the particle. For example, $f(\lambda) = 0$ [12] corresponds to the proper-time gauge introduced above. Other choices are also possible [10]. It may happen however, that only some subset of Teichmüller space (where multiplicities are counted with their orientation) is obtained [9,10]. For example, $f(\lambda) = \alpha \lambda^3$ ($\alpha > 0$) is such a case of a bad gauge-fixing function [10].

If no good gauge-fixing function exists at all, the formulation of the system suffers from a Gribov problem [15]. This is the case for string theory [9,16], as we shall discuss. To illustrate the manner in which one may nevertheless proceed in presence of a Gribov problem [9,10], let us consider the choice

$$f(\lambda) = \frac{1}{\beta} \left(\lambda - \frac{c}{\Delta\tau} \dot{h} \right), \quad (2.45)$$

where $h(\tau)$ is some function such that $h(\tau_i) = \tau_i$ ($i=1, 2$), and β, c are free parameters. In the corresponding effective action and Hamiltonian, let us then redefine $\pi \rightarrow \beta\pi$ and $\mathcal{P}_1 \rightarrow \beta\mathcal{P}_1$ and only then take the limit $\beta \rightarrow 0$. This leads to:

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \mathcal{P}_\mu + \dot{\eta}^2 \mathcal{P}_2 - \frac{1}{2} \lambda (p^2 + m^2) - \pi \left(\lambda - \frac{c}{\Delta\tau} \dot{h} \right) + \mathcal{P}_1 \eta^1 + \mathcal{P}_2 \eta^1]. \quad (2.46)$$

Clearly, this procedure enforces the gauge-fixing condition

$$\dot{\lambda} = \frac{c}{\Delta\tau} \dot{h}(\tau) \quad (2.47)$$

and thus singles out the gauge orbit labelled by c . The degrees of freedom π , η^1 and P_2 have become auxiliary variables. Solving in (2.46) for these quantities (this is consistent with BRST transformations), one obtains:

$$S_{\text{eff}} = \int_{t_1}^{t_2} dt [\dot{x}^\mu P_\mu - \dot{\eta}^2 P_1 - \frac{1}{2} \frac{c}{\Delta t} \dot{h}(P^2 + m^2)] \quad (2.48)$$

with the boundary conditions

$$x^\mu(t_i) = x_i^\mu, P^2(t_i) = m^2 = 0, \eta^2(t_i) = 0 = P_1(t_i), i=1,2. \quad (2.49)$$

The corresponding equations of motion for x^μ , P_μ , η^2 and P_1 then also follow from $H_{\text{eff}} = \frac{c}{2\Delta t} \dot{h}(P^2 + m^2)$, with the Poisson brackets

$$(x^\mu(\tau), P_\nu(\tau)) = \delta_\nu^\mu, (\eta^2(\tau), P_1(\tau)) = 1. \quad (2.50)$$

With the boundary conditions (2.49), the solutions in (2.21a) are reproduced, together with $\eta^2(\tau) = 0 = P_1(\tau)$, describing thus the same physical situation for the fixed value of c defined in (2.45).

In the limit $\beta \rightarrow 0$, i.e. in the gauge (2.47) the BRST and ghost charges reduce to

$$Q_B = \frac{1}{2} \eta^2 (P^2 + m^2), Q_c = -i \eta^2 P_1. \quad (2.51)$$

With (2.49) and (2.50), Q_B generates BRST transformations leaving (2.48) invariant:

$$\delta_B x^\mu = \eta^2 P^\mu, \delta_B P^\mu = 0, \delta_B \eta^2 = 0, \delta_B P_1 = \frac{1}{2} (P^2 + m^2) \quad (2.52)$$

$$\delta_B S_{\text{eff}} = \int_{t_1}^{t_2} dt \frac{1}{2} \frac{d}{dt} [\eta^2 (P^2 + m^2)] = 0. \quad (2.53)$$

Note the analogy with (2.22) and (2.23), and the correspondence between $\alpha(\tau)$ and $\eta^2(\tau)$. Actually, when solving for the conjugate momentum P^μ , these relations constitute the "correspondence principle" which determines, for a specific gauge-fixing choice, the BRST formulation of a gauge invariant system within the Lagrangian approach [17]. Through the limiting procedure just presented [9,10], the BFM formalism thus justifies the usual Lagrangian BRST construction. This is the case not only for the particle [9,10], but also for string theory [9,18],

as we shall show, and actually for any gauge invariant system [19].

2.4. BRST quantisation

The quantum system in the BFM formulation is defined through the (anti)commutation relations

$$[x^\mu, P_\nu] = i\delta_\nu^\mu, [\lambda, \pi] = i, (\eta^a, P_b) = -i\delta_b^a, \quad (2.54)$$

and the time-evolution operator

$$H_{\text{eff}} = -(\Psi, Q_B). \quad (2.55)$$

It is readily checked that

$$(Q_B, Q_B) = 0, [H_{\text{eff}}, Q_B] = 0, \quad (2.56)$$

which establishes the gauge covariance of the quantum description. The space of states is spanned by the eigenstates of x^μ , λ and η^a or of their conjugate momenta. Let us take for example the basis $|q\rangle \otimes |\lambda\rangle \otimes |a\rangle$, with:

$$\begin{aligned} x^\mu |q\rangle &= q^\mu |q\rangle, \lambda |\lambda\rangle = \lambda |\lambda\rangle, \\ \eta^a |a\rangle &= a |a\rangle, \eta^a |a\rangle = 0, \\ P_a |a\rangle &= -a |a\rangle, P_a |a\rangle = -i |a\rangle, \\ \langle q|q'\rangle &= \delta^{(D)}(q-q'), \langle \lambda|\lambda'\rangle = \delta(\lambda-\lambda'), a \langle a|b\rangle = \delta_{ab} \end{aligned} \quad (2.57)$$

An explicit representation of the ghost algebra $(\eta, P) = -i$ is realized for example on the space of functions $\varphi(\theta)$ of a Grassmann variable θ , with:

$$\eta \varphi(\theta) = \theta \varphi(\theta), P \varphi(\theta) = -i \frac{d}{d\theta} \varphi(\theta), \quad (2.58a)$$

and

$$|- \rangle = \varphi(\theta) = 1, |+\rangle = \varphi(\theta) = \theta. \quad (2.58b)$$

Physical states are defined by the conditions:

$$Q_B |\psi; \tau\rangle = 0, Q_c |\psi; \tau\rangle = 0. \quad (2.59)$$

These are solved by

$$|\psi; \tau\rangle = \int d^D q \psi(q; \tau) |q\rangle \otimes \int d\lambda |\lambda\rangle \otimes |-\rangle \otimes |+\rangle, \quad (2.60)$$

with $[-\hbar^2 \square + m^2] \psi(q; \tau) = 0$. These states are thus in one-to-one correspondence with physical states (2.30) in the covariant quantisation.

Imposing the Schrödinger equation, one finds that the function $\psi(q; \tau)$ is independent of τ (this must be the case, at least for good gauge-fixing functions and by appropriate ordering of H_{eff} if necessary), so that the Klein-Gordon equation (2.32) is reproduced for physical states.

A similar discussion applies to the BFM formulation in the fixed gauge $\lambda(\tau) = \frac{c}{\Delta\tau} \hbar$ given in (2.45). The quantum system is then defined by (see (2.50)):

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \quad (\eta^2, p_1) = i, \quad (2.61)$$

and (see (2.51)):

$$H_{\text{eff}} = \frac{c}{\Delta\tau} \hbar [p^2 + m^2], \quad Q_B = \frac{1}{2} \eta^2 (p^2 + m^2), \quad Q_C = -i\eta^2 p_1. \quad (2.62)$$

Poincaré and gauge covariance are readily established. The space of states is spanned for example, by the states $|q \otimes| - \rangle$, with $|q \otimes| - \rangle$ defined as in (2.57).

Physical states are again defined by the conditions

$$Q_B | \psi; \tau \rangle = 0, \quad Q_C | \psi; \tau \rangle = 0. \quad (2.63)$$

These are solved by

$$| \psi; \tau \rangle = \int d^D q \psi(q; \tau) | q \otimes| - \rangle, \quad (2.64)$$

where $\psi(q; \tau)$ satisfies the Klein-Gordon equation. Imposing the time-evolution operator H_{eff} , physical states are again found to be independent of τ , as it should.

To conclude, let us remark that the quantised BFM formulation leads also to a path-integral representation of quantum amplitudes, which is gauge (i.e. BRST) invariant and only depends on the gauge-fixing function Ψ through its gauge-equivalence class [9,10]. For example, for good gauge-fixing functions, the BFM path-integral representation of the propagator of the particle reduces to the well known proper-time integral representation [9,10,12]. One should

also mention that interesting developments have recently been achieved concerning this system [20], in the attempt of understanding the fundamental role played by BRST symmetry in gauge invariant systems better.

3. THE BOSONIC STRING

3.1. Lagrangian formulation

In these notes, only the non-linear Nambu-Goto action [21] will be considered for the bosonic string. A discussion similar to the one below applies to the linear action, where an intrinsic world-sheet metric is introduced. Let us thus consider:

$$S[x] = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma L(\dot{x}, x') \quad (3.1a)$$

with

$$L(\dot{x}, x') = \frac{-1}{2\pi\alpha'} \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}, \quad (3.1b)$$

where the string trajectory $x^\mu(\sigma, \tau)$ depends on the world-sheet coordinates τ and $0 \leq \sigma \leq \pi$, and α' is the Regge slope.

The equations of motion simply correspond [1,2] to the local conservation of the world-sheet energy-momentum current P_μ^α ($\alpha=0,1$):

$$\partial_\sigma P_\mu^\alpha = 0, \quad P_\mu^\alpha = \frac{\partial L}{\partial(\partial_\sigma x^\mu)}. \quad (3.2)$$

We have the boundary conditions:

$$a) x^\mu(\sigma, \tau_1) = x^\mu(\sigma), \quad i=1,2 \quad (3.3a)$$

$$b) i) \text{ open strings: } P_\mu^1(\sigma=0, \pi; \tau) = 0 \quad (3.3b)$$

$$ii) \text{ closed strings: } x^\mu(\sigma=\pi, \tau) = x^\mu(\sigma=0, \tau). \quad (3.3c)$$

The quantities x^μ and P_μ^α are not all independent. They satisfy the constraints:

$$(P_\mu^0 \pm \frac{\dot{x}_\mu}{2\pi\alpha'})^2 = 0, \quad (P_\mu^1 \pm \frac{\dot{x}_\mu}{2\pi\alpha'})^2 = 0. \quad (3.4)$$

The relations (3.4) are a consequence of reparametrisation invariance of the action. As

in (2.3) and (2.4), this can be established by considering the variation of (3.1) under an infinitesimal world-sheet reparametrisation

$$\tilde{\xi}^a = \xi^a + \eta^a(\xi) \quad (3.5a)$$

with

$$a) \eta^0(\sigma, \tau_i) = 0, \quad i=1,2 \quad (3.5b)$$

$$b) i) \text{ open strings: } \eta^1(\sigma=0, \pi; \tau) = 0 \quad (3.5c)$$

$$ii) \text{ closed strings: } \eta^1(\sigma=0, \tau) = \eta^1(\sigma=\pi, \tau). \quad (3.5d)$$

Such an analysis thus shows that the constraints (3.4) are the generators of world-sheet reparametrisations. Actually, the pair of constraints in (3.4) are obtained from each other by a transformation exchanging the role of σ and τ (in a way preserving world-sheet orientation).

Choosing to fix world-sheet parametrisation through proper-time gauge conditions [9]

$$\dot{x}^2 = 0, \quad \dot{x}^2 + \lambda_0^2 \dot{x}'^2 = 0, \quad \lambda_0 > 0, \quad (3.6)$$

the non-linear equations of motion (3.2) simplify to linear Klein-Gordon equations for massless scalar fields in two dimensions. As λ_0 always multiplies τ , let us restrict our choice in (3.6) to the conformal gauge, with $\lambda_0 = 1$.

Such a gauge-fixing does not completely determine the world-sheet coordinates [1,2]. The subgroup of (pseudo)conformal transformations leaves the conformal gauge conditions invariant. A complete gauge-fixing of the system is achieved in the light-cone gauge, which we shall not discuss in these notes [1,2].

As is well known, the solutions to the equations of motion in the conformal gauge are simply [1,2]:

i) open strings:

$$x^\mu(\sigma, \tau) = \sqrt{2\alpha'} \{ q^\mu + \alpha_0^\mu \tau + i \sum_n \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \} \quad (3.7)$$

ii) closed strings:

$$x^\mu(\sigma, \tau) = \sqrt{2\alpha'} \{ q^\mu + (\alpha_0^\mu + \bar{\alpha}_0^\mu) \tau + \frac{1}{2} i \sum_n \frac{1}{n} [\alpha_n^\mu e^{-in(\tau-\sigma)} + \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}] \} \quad (3.8)$$

where the integration constants may be expressed in terms of the boundary conditions (3.3) [9]. Note that q^μ , α_n^μ and $\bar{\alpha}_n^\mu$ have been normalised differently as in Refs.[1,2]. In particular, the total momentum P^μ of the string is given by $\alpha_0^\mu - \sqrt{2\alpha'} P^\mu$ for the open string, and by $\alpha_0^\mu - \frac{1}{2} \sqrt{2\alpha'} P^\mu = \bar{\alpha}_0^\mu$ for the closed string. By $\sum_n$, we denote an infinite summation from $-\infty$ to $+\infty$ with $n=0$ omitted; $\sum_n$ denotes the same infinite summation with $n=0$ included.

These solutions however, have to satisfy the conformal gauge-fixing conditions $(\dot{x} \pm x')^2 = 0$. This translates [1,2] into the Virasoro constraints:

$$i) \text{ open strings: } L_n^{(\alpha)} = 0, \quad (3.9)$$

$$ii) \text{ closed strings: } L_n^{(\alpha)} = 0, \quad \bar{L}_n^{(\alpha)} = 0, \quad (3.10)$$

with the Virasoro generators:

$$L_n^{(\alpha)} = \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m\mu}, \quad \bar{L}_n^{(\alpha)} = \frac{1}{2} \sum_m \bar{\alpha}_m^\mu \bar{\alpha}_{n-m\mu} \quad (3.11)$$

As we shall see, these quantities generate in the conformal gauge (pseudo)conformal transformations, which are the left-over symmetries of the system after gauge-fixing. This is the reason why the Virasoro generators must vanish for physical solutions.

3.2. Old covariant quantisation

Let us briefly review the so-called old covariant quantisation of the bosonic string, using the canonical Hamiltonian methods

developed by Dirac [14]. We have the degrees of freedom x^μ and their conjugate momenta $\pi_\mu = P_\mu^0$, with the Poisson brackets:

$$\{x^\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)\} = \delta_\nu^\mu \delta(\sigma - \sigma'). \quad (3.12)$$

These phase-space variables are not all independent. We have the primary constraints (see (3.4)):

$$\phi_\pm^{(a)} = \frac{1}{2} \pi \alpha' \dot{\phi}_\pm^\mu \phi_{\pm\mu} = 0, \quad (3.13a)$$

with

$$\phi_\pm^\mu = \pi^\mu \pm \frac{x'^\mu}{2\pi\alpha'}, \quad (3.13b)$$

Moreover, the canonical Hamiltonian

$$H_0 = \int_0^\pi d\sigma [\dot{x}^\mu \pi_\mu - L] = 0, \quad (3.14)$$

vanishes identically, as a consequence of reparametrisation invariance (this can be seen as in (2.4) for the particle). Hence, the total Hamiltonian is:

$$H_T = \int_0^\pi d\sigma [\lambda^+ \phi_+^{(a)} + \lambda^- \phi_-^{(a)}], \quad (3.15)$$

where $\lambda^+(\sigma, \tau)$ and $\lambda^-(\sigma, \tau)$ are arbitrary functions of σ and τ . That (3.15) consistently describes the time-evolution of the system and its constraints can be established from the algebra of ϕ_+ and ϕ_- (see Ref.[9] and references therein):

$$\begin{aligned} & \{\phi_\eta^{(a)}(\sigma, \tau), \phi_{\eta'}^{(a)}(\sigma', \tau)\} = \\ & - \eta \delta_{\eta\eta'} [\phi_\eta^{(a)}(\sigma, \tau) + \phi_{\eta'}^{(a)}(\sigma', \tau)] \partial_\sigma \delta(\sigma - \sigma'), \quad \eta, \eta' = +, - \end{aligned} \quad (3.16)$$

This result shows that ϕ_+ and ϕ_- are first-class constraints.

The equations of motion obtained from H_T :

$$\dot{x}^\mu = \pi \alpha' [\lambda^+ \phi_+^\mu + \lambda^- \phi_-^\mu], \quad (3.17a)$$

$$\dot{\pi}^\mu = \frac{1}{2} \partial_\sigma [\lambda^+ \phi_+^\mu - \lambda^- \phi_-^\mu], \quad (3.17b)$$

also follow from the first-order action

$$S[x, \pi, \lambda^+, \lambda^-] = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma [\dot{x}^\mu \pi_\mu - \lambda^+ \phi_+^{(a)} - \lambda^- \phi_-^{(a)}] \quad (3.18)$$

with the boundary conditions [9]:

$$a) x^\mu(\sigma, \tau_i) = x_i^\mu(\sigma) \quad i=1,2 \quad (3.19a)$$

$$b) i) \text{ open strings: } (\lambda^+ \phi_+^\mu - \lambda^- \phi_-^\mu)(\sigma=0, \pi; \tau) = 0 \quad (3.19b)$$

$$ii) \text{ closed strings: same values at } \sigma=0 \text{ and } \sigma=\pi. \quad (3.19c)$$

The first-class constraints generate through their Poisson brackets symmetries of the action (3.18). For infinitesimal functions $\epsilon^+(\sigma, \tau)$ and $\epsilon^-(\sigma, \tau)$ such that [9]:

$$a) (\epsilon^+ + \epsilon^-)(\sigma, \tau_i) = 0, \quad i=1,2 \quad (3.20a)$$

$$b) i) \text{ open strings: } (\lambda^+ \epsilon^+ - \lambda^- \epsilon^-)(\sigma=0, \pi; \tau) = 0 \quad (3.20b)$$

$$ii) \text{ closed strings: same values at } \sigma=0 \text{ and } \sigma=\pi, \quad (3.20c)$$

let us consider the quantity

$$\phi_\epsilon^{(a)} = \int_0^\pi d\sigma [\epsilon^+ \phi_+^{(a)} + \epsilon^- \phi_-^{(a)}] \quad (3.21)$$

and the variations

$$\delta_\epsilon x^\mu = \pi \alpha' [\epsilon^+ \phi_+^\mu + \epsilon^- \phi_-^\mu] \quad (3.22a)$$

$$\delta_\epsilon \pi^\mu = \frac{1}{2} \partial_\sigma [\epsilon^+ \phi_+^\mu - \epsilon^- \phi_-^\mu] \quad (3.22b)$$

$$\delta_\epsilon \lambda^+ = \dot{\epsilon}^+ - \lambda^+ \epsilon^{+'} + \lambda'^+ \epsilon^+ \quad (3.22c)$$

$$\delta_\epsilon \lambda^- = \dot{\epsilon}^- + \lambda^- \epsilon^{-'} - \lambda'^- \epsilon^- \quad (3.22d)$$

It can then be shown, using the equation of motion (3.17a) for x^μ , that these transformations leave the action (3.18) invariant. With the identifications [9]:

$$\epsilon^+ = \lambda^+ \eta^0 + \eta^1, \quad \epsilon^- = \lambda^- \eta^0 - \eta^1 \quad (3.23)$$

and by solving for π^μ from (3.17a), it can be seen that the variations (3.22) correspond to world-sheet reparametrisations as defined in (3.5). The constraints ϕ_+ and ϕ_- are thus the generators of these local gauge transformations in the Hamiltonian formalism.

When solving for π^μ in (3.18), one obtains the linear action

$$S[x, g] = -\frac{1}{4\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu \quad (3.24)$$

with

$$g_{\alpha\beta} = e^\phi \frac{2}{\lambda^+ + \lambda^-} \begin{pmatrix} -\lambda^+ \lambda^- & \frac{1}{2} (\lambda^+ - \lambda^-) \\ \frac{1}{2} (\lambda^+ - \lambda^-) & 1 \end{pmatrix} \quad (3.25)$$

where $\phi(\sigma, \tau)$ is an arbitrary function. From the variations (3.22c,d), it can be shown that $g_{\alpha\beta}$ transforms indeed as a world-sheet metric tensor [9]. As in the case of the particle, the Lagrange multipliers λ^+ and λ^- thus not only parametrise the freedom in the choice of world-sheet coordinates, but they also correspond to components of a world-sheet metric.

This result allows us to discuss the gauge-fixing of the Nambu-Goto string, which in the present formalism is done by specifying λ^+ and λ^- . It can be argued [9] that a complete gauge-fixing of the theory is obtained by the choice

$$\lambda^+(\sigma, \tau) = \lambda_0 = \lambda^-(\sigma, \tau), \quad \lambda_0 > 0, \quad (3.26)$$

which corresponds to the proper-time gauges in (3.6). In particular, for configurations such that λ^+ and λ^- are independent of σ , it can easily be seen from (3.22c,d) that the quantity [9,22]

$$\mu = \frac{1}{\pi^2} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \frac{1}{2} [\lambda^+(\tau) + \lambda^-(\tau)] \quad (3.27)$$

is an invariant for world-sheet reparametrisations, i.e. the Teichmüller parameter characterizing the gauge orbit of the subspace of phase-space passing through $\lambda^+(\tau)$ and $\lambda^-(\tau)$.

Considering the conformal gauge $\lambda_0 = 1$, the equations of motion (3.17) reduce to those in the Lagrangian formulation for that gauge, with the solutions (3.7) and (3.8). World-sheet reparametrisations leaving the gauge condition unchanged are generated by ϕ_ϵ with functions ϵ^+ and ϵ^- such that $\epsilon^\pm(\sigma, \tau) = \epsilon_\pm(\tau \pm \sigma)$ [1,2].

For open strings, we then have:

$$\epsilon_+(\tau) = \epsilon_-(\tau) = \sum_n \epsilon_n e^{in\tau}, \quad \phi_\epsilon = \sum_n \epsilon_n L_n^{(\epsilon)} \quad (3.28a)$$

with

$$L_n^{(\epsilon)} = \int_0^\pi d\sigma [e^{in(\tau+\sigma)} \phi_+^{(\epsilon)} + e^{in(\tau-\sigma)} \phi_-^{(\epsilon)}] = \int_{-\pi}^\pi d\sigma e^{in(\tau+\sigma)} \phi_+^{(\epsilon)} \quad (3.28b)$$

For closed strings, we have:

$$\epsilon_+(\tau) = \sum_n \epsilon_n^+ e^{2in\tau}, \quad \epsilon_-(\tau) = \sum_n \epsilon_n^- e^{2in\tau}, \quad (3.29a)$$

$$\phi_\epsilon^{(\epsilon)} = 2 \sum_n [\epsilon_n^+ \bar{L}_n^{(\epsilon)} + \epsilon_n^- L_n^{(\epsilon)}] \quad (3.29b)$$

with

$$L_n^{(\epsilon)} = \frac{1}{2} \int_0^\pi d\sigma e^{2in(\tau-\sigma)} \phi_-^{(\epsilon)}, \quad (3.29c)$$

$$\bar{L}_n^{(\epsilon)} = \frac{1}{2} \int_0^\pi d\sigma e^{2in(\tau+\sigma)} \phi_+^{(\epsilon)}. \quad (3.29d)$$

The quantities just introduced can be seen to reduce to those defined in (3.11). This proves that the Virasoro generators indeed induce (pseudo)conformal transformations. Note that in the conformal gauge, for open strings we have $H_T = L_0^{(\epsilon)}$, and for closed strings $H_T = 2(L_0^{(\epsilon)} + \bar{L}_0^{(\epsilon)})$, where it can also be shown

that $2(L_0^{(\epsilon)} - \bar{L}_0^{(\epsilon)})$ generates constant shifts in σ .

Finally, from (3.16), we have the Poisson brackets:

$$\text{i) open strings: } \{L_n^{(\epsilon)}, L_m^{(\epsilon)}\} = -i(n-m)L_{n+m}^{(\epsilon)} \quad (3.30)$$

$$\text{ii) closed strings: } \{L_n^{(\epsilon)}, \bar{L}_m^{(\epsilon)}\} = 0,$$

$$\{L_n^{(\epsilon)}, L_m^{(\epsilon)}\} = -i(n-m)L_{n+m}^{(\epsilon)},$$

$$\{\bar{L}_n^{(\epsilon)}, \bar{L}_m^{(\epsilon)}\} = -i(n-m)\bar{L}_{n+m}^{(\epsilon)}. \quad (3.31)$$

Through the correspondence principle, the quantised system is defined by the

commutation relations

$$[x^\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)] = i\delta_\nu^\mu \delta(\sigma - \sigma'). \quad (3.32)$$

and the appropriate Schrödinger equations. For open strings, (3.32) is equivalent to:

$$[q^\mu, \alpha_n^\nu] = i\eta^{\mu\nu} \delta_{n,0}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu} \delta_{n+m,0}, \quad (3.33)$$

and for closed strings:

$$[q^\mu, \alpha_n^\nu] = \frac{1}{2} i\eta^{\mu\nu} \delta_{n,0} = [q^\mu, \bar{\alpha}_n^\nu], \quad (3.34a)$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu} \delta_{n+m,0} = [\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu]. \quad (3.34b)$$

The corresponding space of states is spanned by ground-states $|\Omega\rangle \otimes |p\rangle = |\Omega; p\rangle$ and all their excitations obtained by applying the creation operators α_n^μ and $\bar{\alpha}_n^\mu$ ($n \geq 1$). Here, $|\Omega\rangle$ is annihilated by all operators α_n^μ , $\bar{\alpha}_n^\mu$ ($n \geq 1$), and $|p\rangle$ are eigenstates of the momentum operator P^μ (position eigenstates could also be used instead). They are normalised by $\langle \Omega | \Omega \rangle = 1$ and $\langle p | p' \rangle = \delta^{(D)}(p - p')$.

Having an infinite number of degrees of freedom, a normal ordering prescription for the product of local operators has to be adopted. The one chosen here is to have all creation operators to the left of all annihilation operators, and all position operators to the left of all momentum operators. This normal ordering is denoted by " $::$ ". It is assumed in what follows that all composite operators have been normal ordered in that manner. For example, we now have

$$L_n^{(\alpha)} = \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m}^\mu, \quad \bar{L}_n^{(\alpha)} = \frac{1}{2} \sum_m \bar{\alpha}_m^\mu \bar{\alpha}_{n-m}^\mu. \quad (3.35)$$

As a consequence of this prescription, physical states in the quantum theory are now defined by the Virasoro gauge conditions:

i) open strings:

$$(L_0^{(\alpha)} - a) |\psi\rangle = 0, \quad L_n^{(\alpha)} |\psi\rangle = 0, \quad n \geq 1 \quad (3.36)$$

ii) closed strings:

$$L_n^{(\alpha)} |\psi\rangle = 0 = \bar{L}_n^{(\alpha)} |\psi\rangle, \quad n \geq 1 \quad (3.37a)$$

$$[L_0^{(\alpha)} + \bar{L}_0^{(\alpha)} - 2a] |\psi\rangle = 0, \quad (3.37b)$$

$$[L_0^{(\alpha)} - \bar{L}_0^{(\alpha)}] |\psi\rangle = 0, \quad (3.37c)$$

where "a" is a subtraction constant following from normal ordering in the zero-mode Virasoro generators.

Similarly, one expects a central extension for the quantum conformal algebra in (3.30) and (3.31). Indeed, one finds (see the Appendix for details):

i) open strings:

$$[L_n^{(\alpha)}, L_m^{(\alpha)}] = (n-m)L_{n+m}^{(\alpha)} + \frac{1}{12} Dn(n^2-1)\delta_{n+m,0} \quad (3.38)$$

ii) closed strings:

$$[L_n^{(\alpha)}, L_m^{(\alpha)}] = (n-m)L_{n+m}^{(\alpha)} + \frac{1}{12} Dn(n^2-1)\delta_{n+m,0}$$

$$[\bar{L}_n^{(\alpha)}, \bar{L}_m^{(\alpha)}] = (n-m)\bar{L}_{n+m}^{(\alpha)} + \frac{1}{12} Dn(n^2-1)\delta_{n+m,0}$$

$$[L_n^{(\alpha)}, \bar{L}_m^{(\alpha)}] = 0. \quad (3.39)$$

The relations (3.38) define the Virasoro algebra with central charge $c=D$, each scalar field $x^\mu(\sigma, \tau)$ contributing one unit to this value.

Due to the existence of negative-norm states in the covariant quantisation of the theory, one has to check, as a necessary condition for unitarity, that none of the states which satisfies (3.36) or (3.37) has negative-norm. By the no-ghost theorem [23], we know that this is possible only if $D \leq 26$ for $a=1$, or if $D \leq 25$ for $a < 1$.

Moreover, when considering one-loop string amplitudes, it is found that the correct singularity structure due to intermediate states is obtained only if $D=26$ [24]. Hence, the values for a and D necessary for tree-level and one-loop unitarity are $a=1$ and $D=26$.

But this implies that the physical ground-states for the open string and for the closed string are tachyons, a fact which spells disaster for the theory, which cannot be then unitary. The most dramatic consequence of the existence of these tachyons is in the non-decoupling of zero-norm physical states in higher loop amplitudes [25,26]. For example, in the case of the open bosonic string coupled to gauge degrees of freedom through Chan-Paton factors, when computing one-loop amplitudes with external physical open string states, with at least one of zero-norm, one finds that these amplitudes vanish up to two types of contributions. The first contribution vanishes only for $SO(2^{D/2})$ - $SO(2^{13})$ [25,26,27], but the second one, entirely due to tachyons, does not vanish [25,26] (for open superstrings in $D=10$, only the first contribution exists and it vanishes for $SO(2^{D/2})$ - $SO(32)$. This famous result was established for external gauge bosons [28] and gravitons [29] only). Non-unitarity of tachyonic string theories implies the breakdown of their gauge invariance, i.e. in the conformal gauge, of conformal symmetry.

Further discussion concerning the spectra of these string theories, both in the conformal and in the light-cone gauge, may be found in Refs.[1,2].

3.3. BFV formulation

In the BFV formulation applied to the bosonic string (see Ref.[9] and references therein), phase-space is extended as follows. First, one introduces conjugate momenta $\pi_-(\sigma, \tau)$, $\pi_+(\sigma, \tau)$ to $\lambda^-(\delta, \tau)$, $\lambda^+(\sigma, \tau)$ respectively, with:

$$(\lambda^{\eta_1}(\sigma, \tau), \pi_{\eta_2}(\sigma', \tau)) = \delta_{\eta_2}^{\eta_1} \delta(\sigma - \sigma'), \quad \eta_1, \eta_2 = +, - \quad (3.40)$$

and the first-class constraints $\pi_-(\sigma, \tau) = 0 = \pi_+(\sigma, \tau)$. Associated to the constraints $G_1 = \pi_-$, $G_2 = \pi_+$, $G_3 = \phi_-$, $G_4 = \phi_+$, one introduces respectively Hermitian Grassmann variables $\eta^a(\sigma, \tau)$ ($a=1,2,3,4$) and their conjugate momenta $\mathcal{P}_a(\sigma, \tau)$ ($a=1,2,3,4$) (anti-Hermitian Grassmann variables), with:

$$(\eta^a(\sigma, \tau), \mathcal{P}_b(\sigma', \tau)) = -\delta_b^a \delta(\sigma - \sigma') \quad a, b = 1, 2, 3, 4 \quad (3.41)$$

The ghost charge is defined by:

$$Q_c = \int_0^\pi d\sigma \, i \eta^a \mathcal{P}_a \quad (3.42)$$

From the algebra of these constraints G_a ($a=1,2,3,4$), one obtains the BRST generator:

$$Q_B = \int_0^\pi d\sigma [\eta^1 \pi_- + \eta^2 \pi_+ + \eta^3 \phi_-^{(a)} + \eta^4 \phi_+^{(a)} - \eta^3 \partial_\sigma \eta^3 \mathcal{P}_3 + \eta^4 \partial_\sigma \eta^4 \mathcal{P}_4] \quad (3.43)$$

and the BRST invariant Hamiltonian:

$$H_B = H_0 = 0 \quad (3.44)$$

By construction, we have nilpotency of the BRST charge:

$$(Q_B, Q_B) = 0 \quad (3.45)$$

BRST transformations of all variables can easily be obtained [9], leading to a choice of BRST invariant boundary conditions, which we shall specify below. Note that we have:

$$(\mathcal{P}_3, Q_B) = -\phi_-^{(T)}, \quad (\mathcal{P}_4, Q_B) = -\phi_+^{(T)} \quad (3.46)$$

where

$$\phi_\pm^{(T)} = \phi_\pm^{(a)} + \phi_\pm^{(c)} \quad (3.47)$$

and

$$\phi_-^{(c)} = -2\partial_\sigma \eta^3 \mathcal{P}_3 - \eta^3 \partial_\sigma \mathcal{P}_3 \quad (3.48a)$$

$$\phi_+^{(c)} = 2\partial_\sigma \eta^4 \mathcal{P}_4 + \eta^4 \partial_\sigma \mathcal{P}_4 \quad (3.48b)$$

It can be shown that $\phi_-^{(c)}$ and $\phi_+^{(c)}$ have the same Poisson brackets (3.16) as the bosonic constraints $\phi_-^{(a)}$ and $\phi_+^{(a)}$. The total quantities $\phi_-^{(T)}$ and $\phi_+^{(T)}$ are thus the generators of world-

sheet reparametrisations for all degrees of freedom in the extended phase-space, with $\phi_{\frac{1}{2}}^{(a)}$ inducing the transformations of x^μ and π^μ , and $\phi_{\frac{1}{2}}^{(c)}$ those of the BFV ghosts η^a and $\mathcal{P}_a$. By restriction to the conformal gauge, their Fourier modes are thus also the generators of the conformal group, as was shown before in the case of $\phi_{\frac{1}{2}}^{(a)}$.

Time-evolution in extended phase-space is determined by an effective Hamiltonian

$$H_{\text{eff}} = H_B - \int_0^\pi d\sigma (\Psi, Q_B), \quad (3.49)$$

where Ψ is some gauge-fixing function over extended phase-space of ghost number (-1) . One may easily obtain the effective action corresponding to (3.49).

This function Ψ has to be chosen such that one obtains effectively once and only once each orbit of the gauge group. Since string theory suffers from a Gribov problem [9,16], such a function does not exist, but we may nevertheless proceed by using a limiting procedure which effectively gauge-fixes the system onto one single orbit [9], as we discussed for the particle. Let us thus consider the function [9]:

$$\Psi = \frac{1}{\beta} \mathcal{P}_1(\lambda^+ - \lambda_0) + \frac{1}{\beta} \mathcal{P}_2(\lambda^+ - \lambda_0) + \mathcal{P}_3\lambda^- + \mathcal{P}_4\lambda^+, \quad (3.50)$$

with $\lambda_0 > 0$ taking a specified value and β a free parameter. One may then compute the corresponding effective Hamiltonian, action and equations of motion. Performing then the redefinitions $\pi_\pm \rightarrow \beta\pi_\pm$ and $\mathcal{P}_a \rightarrow \beta\mathcal{P}_a$ ($a=1,2$), and only then taking the limit $\beta \rightarrow 0$, the equations of motion for $\pi_\pm$, η^a ($a=1,2$) and $\mathcal{P}_a$ ($a=3,4$) become constraints which are solved in terms of the remaining degrees of freedom [9]. We then also have $\lambda^+ - \lambda_0 - \lambda^-$. This leads to the effective action [9]:

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma [\dot{x}^\mu \pi_\mu - \dot{\eta}^3 \mathcal{P}_1 - \dot{\eta}^4 \mathcal{P}_2 - H_{\text{eff}}] \quad (3.51)$$

with the effective Hamiltonian [9]:

$$H_{\text{eff}} = \int_0^\pi d\sigma \mathcal{H}_{\text{eff}} = - \int_0^\pi d\sigma [\lambda_0 (\phi_{\frac{1}{2}}^{(a)} + \phi_{\frac{1}{2}}^{(c)}) + 2\lambda_0 \partial_\sigma \eta^3 \mathcal{P}_1 + \lambda_0 \eta^3 \partial_\sigma \mathcal{P}_1 - 2\lambda_0 \partial_\sigma \eta^4 \mathcal{P}_2 - \lambda_0 \eta^4 \partial_\sigma \mathcal{P}_2], \quad (3.52)$$

and the correct equations of motion are reproduced with the Poisson brackets $(\mathcal{P}_3, \mathcal{P}_1, \mathcal{P}_4, \mathcal{P}_2)$:

$$(x^\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)) = \delta_\nu^\mu \delta(\sigma - \sigma'), \quad (3.53a)$$

$$(\eta^3(\sigma, \tau), \mathcal{P}_1(\sigma', \tau)) = \delta(\sigma - \sigma') = (\eta^4(\sigma, \tau), \mathcal{P}_2(\sigma', \tau)). \quad (3.53b)$$

The appropriate boundary conditions are [9]:

$$a) \quad x^\mu(\sigma, \tau_i) = x^\mu(\sigma), \quad i=1,2$$

$$[\phi_{\frac{1}{2}}^{(a)} + 2\partial_\sigma \eta^3 \mathcal{P}_1 + \eta^3 \partial_\sigma \mathcal{P}_1](\sigma, \tau_i) = 0, \quad i=1,2$$

$$[\phi_{\frac{1}{2}}^{(a)} - 2\partial_\sigma \eta^4 \mathcal{P}_2 - \eta^4 \partial_\sigma \mathcal{P}_2](\sigma, \tau_i) = 0, \quad i=1,2$$

$$\eta^3(\sigma, \tau_i) = 0 = \eta^4(\sigma, \tau_i); \quad \mathcal{P}_1(\sigma, \tau_i) = 0 = \mathcal{P}_2(\sigma, \tau_i), \quad i=1,2$$

$$b) \quad i) \text{ open strings: } \partial_\sigma x^\mu(\sigma=0, \pi; \tau) = 0$$

$$(\eta^3 - \eta^4)(\sigma=0, \pi; \tau) = 0 = (\mathcal{P}_1 - \mathcal{P}_2)(\sigma=0, \pi; \tau)$$

$$ii) \text{ closed strings: same values}$$

$$\text{at } \sigma=0 \text{ and } \sigma=\pi. \quad (3.54)$$

Solving for $\pi_\pm$, η^a ($a=1,2$) and $\mathcal{P}_a$ ($a=3,4$) is consistent with BRST transformations, which for the remaining degrees of freedom are obtained from the BRST charge in the gauge $\lambda^+ - \lambda_0 - \lambda^-$ [9]:

$$Q_B = \int_0^\pi d\sigma [\eta^3 \phi_{\frac{1}{2}}^{(a)} + \eta^4 \phi_{\frac{1}{2}}^{(c)} + \eta^3 \partial_\sigma \eta^3 \mathcal{P}_1 - \eta^4 \partial_\sigma \eta^4 \mathcal{P}_2] \quad (3.55)$$

The same notation as before is used for this quantity, although it is different. This should not cause any confusion, since in what follows, only the restricted BRST operator is considered. In particular, the boundary conditions (3.54) are BRST invariant, and S_{eff} in (3.51) is invariant for transformations induced by Q_B in (3.55) (provided one uses the equation of motion for x^μ). It may also be checked that Q_B is nilpotent.

In the foregoing, we shall restrict the discussion to the conformal gauge $\lambda_0=1$, and also redefine the notation as follows:

$$c^- = \eta^3, c^+ = \eta^4, b_{--} = -2i\pi p_1, b_{++} = -2i\pi p_2. \quad (3.56)$$

We then have:

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[\dot{x}^\mu \pi_\mu - (\dot{\phi}_+^{(a)} + \dot{\phi}_-^{(a)}) + \right. \\ \left. + \frac{1}{2\pi} i b_{--} (\partial_\tau + \partial_\sigma) c^- + \frac{1}{2\pi} i b_{++} (\partial_\tau - \partial_\sigma) c^+ \right], \quad (3.57)$$

$$Q_B = \int_0^\pi d\sigma \left[c^- \dot{\phi}_-^{(a)} + c^+ \dot{\phi}_+^{(a)} + \right. \\ \left. + \frac{1}{2\pi} c^- \partial_\sigma c^- b_{--} - \frac{1}{2\pi} c^+ \partial_\sigma c^+ b_{++} \right], \quad (3.58)$$

$$Q_c = \int_0^\pi d\sigma \frac{1}{2\pi} [c^- b_{--} + c^+ b_{++}], \quad (3.59)$$

the Poisson brackets:

$$(x^\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)) = \delta_\nu^\mu \delta(\sigma - \sigma'), \quad (3.60a)$$

$$(c^-(\sigma, \tau), b_{--}(\sigma', \tau)) = -2i\pi \delta(\sigma - \sigma') = (c^+(\sigma, \tau), b_{++}(\sigma', \tau)), \quad (3.60b)$$

and the boundary conditions specified in (3.54).

Note that (3.48) and (3.46) lead to:

$$\dot{\phi}_-^{(c)} = \frac{i}{2\pi} [2a_\sigma c^- b_{--} + c^- a_\sigma b_{--}], \quad (3.61a)$$

$$\dot{\phi}_+^{(c)} = \frac{i}{2\pi} [2a_\sigma c^+ b_{++} + c^+ a_\sigma b_{++}], \quad (3.61b)$$

and

$$(b_{--}, Q_B) = -2i\pi \dot{\phi}_-^{(T)}, (b_{++}, Q_B) = -2i\pi \dot{\phi}_+^{(T)}. \quad (3.62)$$

This shows that the boundary conditions at $\tau=\tau_i$ ($i=1,2$) which involve $\dot{\phi}_\pm^{(T)}$, imply that physical solutions do not depend on any specific world-sheet parametrisation which is used.

Let us also remark that solving for π^μ in (3.57), one recovers the action obtained by Kato and Ogawa [30] in their original treatment of BRST quantisation of the bosonic string within a Lagrangian formalism. This is another example [9,18] of the general result stated after (2.53), that the BFV approach can be used to justify the BRST Lagrangian approach. In particular, one may verify that the BRST transformations of x^μ and π^μ induced by Q_B lead precisely to

(3.22a,b), with ϵ' and ϵ replaced by c^+ and c^- respectively. The BFV ghosts $c^\pm$ and $b_{\pm\pm}$ are thus the usual Faddeev-Popov ghosts associated with reparametrisation invariance of the system.

To conclude, let us now consider the solutions to the equations of motion in the conformal gauge, first for the open string. Imposing the boundary conditions (3.54) at $\sigma=0$ and $\sigma=\pi$ only, one finds the following:

$$x^\mu(\sigma, \tau): \text{ same as in (3.7)}, \quad (3.63a)$$

$$c^\pm(\sigma, \tau) = \sum_n c_n^\pm e^{-in(\tau \pm \sigma)}, \quad (3.63b)$$

$$b_{\pm\pm}(\sigma, \tau) = \sum_n b_n^\pm e^{-in(\tau \pm \sigma)}. \quad (3.63c)$$

Defining the Fourier modes of $\dot{\phi}_\pm^{(a)}$, $\dot{\phi}_\pm^{(c)}$ and $\dot{\phi}_\pm^{(T)}$ as in (3.28b), we have:

$$L_n = L_n^{(T)} - L_n^{(a)} + L_n^{(c)}, \quad (3.64)$$

with $L_n^{(a)}$ given in (3.11) and

$$L_n^{(c)} = \sum_m (n-m) b_{n-m} c_m, \quad (3.65)$$

so that

$$\dot{\phi}_\pm^{(a)}(\sigma, \tau) = \frac{1}{2\pi} \sum_n L_n^{(a)} e^{-in(\tau \pm \sigma)}, \quad (3.66a)$$

$$\dot{\phi}_\pm^{(c)}(\sigma, \tau) = \frac{1}{2\pi} \sum_n L_n^{(c)} e^{-in(\tau \pm \sigma)}. \quad (3.66b)$$

It is also easy to obtain:

$$Q_B = \sum_n \left[L_{-n}^{(a)} + \frac{1}{2} L_{-n}^{(c)} \right] c_n, \quad (3.67)$$

$$Q_c = \sum_n c_{-n} b_n, \quad (3.68)$$

and the following Poisson brackets:

$$(c_n, b_m) = -i \delta_{n+m,0}, \quad (3.69)$$

$$(b_n, Q_B) = -i L_n. \quad (3.70)$$

Clearly, the bosonic, ghost and total contributions to the Virasoro generators each

separately satisfies the Poisson brackets (3.30) for the conformal algebra.

Imposing now the boundary conditions at $\tau = \tau_i$ ($i=1,2$), one obtains precisely the same physical solution as in the canonical discussion of Section 3.2. Indeed, from (3.54), we then have $c^2=0=b_{\pm\pm}$ and $L_n=0$. This implies $L_n^{(c)}=0$ and thus $L_n^{(a)}=0$. Hence, only the bosonic degrees of freedom survive, with precisely the Virasoro constraints of (3.9) or (3.13) for the integration constants. This ensures reparametrisation invariance of the solution, which may equivalently be seen by noticing that the BRST and ghost charges vanish as well.

The situation for the closed string is quite similar. Imposing first the boundary conditions at $\sigma=0$ and $\sigma=\pi$, one finds:

$$x^\mu(\sigma, \tau): \text{ same as in (3.8) }, \quad (3.71a)$$

$$c(\sigma, \tau) = \frac{1}{2} \sum_n c_n e^{-2in(\tau - \sigma)}, \quad (3.71b)$$

$$c^*(\sigma, \tau) = \frac{1}{2} \sum_n \bar{c}_n e^{-2in(\tau + \sigma)}, \quad (3.71c)$$

$$b_{--}(\sigma, \tau) = 4 \sum_n b_n e^{-2in(\tau - \sigma)}, \quad (3.71d)$$

$$b_{++}(\sigma, \tau) = 4 \sum_n \bar{b}_n e^{-2in(\tau + \sigma)}. \quad (3.71e)$$

Thus, the separation into right- and left-movers, which was obtained for the bosonic degrees of freedom (see (3.8)), extends also to the ghost system. This is a consequence of the structure of the conformal algebra in two dimensions, for which such a separation occurs (see (3.31)).

From (3.71), we obtain:

$$\phi_-^{(k)}(\sigma, \tau) = \frac{2}{\pi} \sum_n L_n^{(k)} e^{-2in(\tau - \sigma)}, \quad (3.72a)$$

$$\phi_+^{(k)}(\sigma, \tau) = \frac{2}{\pi} \sum_n \bar{L}_n^{(k)} e^{-2in(\tau + \sigma)}, \quad (3.72b)$$

where $k = \alpha, c$ or T . Here, $L_n^{(a)}, \bar{L}_n^{(a)}$ are given in (3.11), $L_n^{(c)}$ is given in (3.65) and $\bar{L}_n^{(c)}$ has the same expression as $L_n^{(c)}$ in terms of the left-moving modes $\bar{c}_n$ and $\bar{b}_n$. Moreover, we have:

$$Q_B = \sum_n \left\{ \left(L_{-n}^{(a)} + \frac{1}{2} L_{-n}^{(c)} \right) c_n + \left(\bar{L}_{-n}^{(a)} + \frac{1}{2} \bar{L}_{-n}^{(c)} \right) \bar{c}_n \right\}, \quad (3.73)$$

$$Q_c = \sum_n [c_{-n} b_n + \bar{c}_{-n} \bar{b}_n], \quad (3.74)$$

and

$$(c_n, b_m) = -i \delta_{n+m,0} = (\bar{c}_n, \bar{b}_m), \quad (3.75)$$

$$(b_n, Q_B) = -i \bar{L}_n, \quad (\bar{b}_n, Q_B) = -i L_n. \quad (3.76)$$

We thus obtain two commuting identical copies (except for the bosonic zero-modes q^μ and p^μ) of the structure which appears for the open string. As is well known, this is characteristic of closed string theories, and is of fundamental importance in the construction of four dimensional heterotic string theories.

Finally, imposing the boundary conditions at $\tau = \tau_i$ ($i=1,2$), we recover precisely the same gauge invariant physical solutions as in the Lagrangian or covariant Hamiltonian approaches. All ghost degrees of freedom then vanish, and the bosonic solution has to satisfy the usual Virasoro constraints (3.10).

3.4. BRST quantisation

Having developed the BFV formulation of the bosonic string in the conformal gauge, it is now straightforward to proceed to its BRST quantisation. By the correspondence principle, the quantum system is defined through the commutation and anticommutation relations which follow from the Poisson brackets (3.60). These (anti)commutators are equivalent, for the bosonic degrees of freedom, to the relations in

(3.33) and (3.34), and for the ghost system to the following anticommutators:

$$(c_n, b_m) = \delta_{n+m,0}; (\bar{c}_n, \bar{b}_m) = \delta_{n+m,0}. \quad (3.77)$$

both for open and for closed strings, with $c_n^\dagger = c_{-n}$ and $b_n^\dagger = b_{-n}$ and similarly for the left-moving ghost modes of the closed string. In what follows, we explicitly write out the analysis only for the open string or one of the two sectors of the closed string. The complete structure for the closed string is straightforwardly obtained by tensoring the quantum algebras and spaces of states with themselves, except for the bosonic zero-modes q^μ and P^μ .

When compared to the old covariant quantisation of the system, the space of states is thus considerably enlarged. In the bosonic sector, we have the ground-states $|\Omega; p\rangle$ defined in Section 3.2, and all their excitations. In addition, we now have a ghost sector. The algebra of ghost zero-modes c_0 and b_0 is identical to that of the ghost system for the particle. Let us thus introduce two states $|- \rangle$ and $|+ \rangle$ such that:

$$c_0 |- \rangle = |+ \rangle, \quad c_0 |+ \rangle = 0, \quad (3.78a)$$

$$b_0 |+ \rangle = |- \rangle, \quad b_0 |- \rangle = 0. \quad (3.78b)$$

In addition, we also have the ghost excitations c_n and b_n . In this sector, let us introduce a ground-state, also denoted by $|\Omega\rangle$, such that:

$$b_n |\Omega\rangle = 0, \quad c_n |\Omega\rangle = 0, \quad n \geq 1. \quad (3.79)$$

Therefore, the space of states of the quantised system is spanned by the Fock vacua $|\Omega; p\rangle \otimes |\pm\rangle$ and all their excitations in the bosonic modes α_n^μ ($n \geq 1$) and in the ghost modes c_n, b_n ($n \geq 1$). Note the doubling in states due to the algebra of ghost zero-modes.

We also need to specify a normal ordering for all quantum operators. In the bosonic sector, the same prescription as in the old covariant approach is obviously used. In the ghost sector, one again chooses to bring all the annihilation

operators c_n, b_n ($n \geq 1$) to the right of all creation operators c_{-n}, b_{-n} ($n \geq 1$). For the ghost zero-modes, in accordance with their algebra, one takes:

$$:c_0 b_0: = -\frac{1}{2} (c_0 b_0 - b_0 c_0). \quad (3.80)$$

Having adopted these prescriptions, the quantum system is properly defined. In particular, we then have:

$$L_n = L_n^{(a)} + L_n^{(c)} - a \delta_{n,0}, \quad (3.81)$$

$$Q_B = \sum_n \left[L_{-n}^{(a)} + \frac{1}{2} L_{-n}^{(c)} \right] c_n = a c_0. \quad (3.82)$$

and

$$Q_c = -\frac{1}{2} (c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} [c_{-n} b_n - b_{-n} c_n], \quad (3.83)$$

where $L_n^{(a)}, L_n^{(c)}$ are now given by the normal ordered expressions of (3.11) and (3.65) respectively, and "a" is a subtraction constant following from normal ordering in $L_n^{(a)}$. Using techniques explained in the Appendix, it is easy to show that:

$$(b_n, Q_B) = L_n, \quad (3.84)$$

with L_n and Q_B given in (3.81) and (3.82), including the same subtraction constant a.

Due to this procedure of normal ordering of composite operators, gauge invariance in the quantum system is not guaranteed. This is a well known phenomenon, corresponding to an anomaly. As we know, this happens for the present system; a central extension of the conformal algebra generated by $L_n^{(a)}$ was found in the old covariant quantisation (see (3.38) and (3.39)).

The question of gauge invariance may equivalently be investigated by considering the algebra of the total Virasoro generators L_n , or by checking the nilpotency of the BRST charge. Both types of operators indeed generate the symmetries of the system, i.e. conformal transformations in the conformal gauge, and are

related to each other (see (3.84)). Let us first consider the Virasoro algebra.

We already know that the bosonic contributions satisfy:

$$[L_n^{(a)}, L_m^{(a)}] = (n-m)L_{n+m}^{(a)} + \frac{1}{12}D(n^3-n)\delta_{n+m,0} \quad (3.85)$$

One also obtains for the ghost contributions (see the Appendix):

$$[L_n^{(c)}, L_m^{(c)}] = (n-m)L_{n+m}^{(c)} + \frac{1}{12}(-26n^3 + 2n)\delta_{n+m,0} \quad (3.86)$$

so that for the total contribution to the Virasoro generators L_n defined in (3.81):

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}[(D-26)n^3 + (2+24a-D)n]\delta_{n+m,0} \quad (3.87)$$

Hence, conformal invariance of the system is preserved in the quantised theory only if:

$$D = 26 \text{ and } a = 1 \quad (3.88)$$

These are precisely the conditions obtained in the discussion of Section 3.2, following from the no-ghost theorem and the correct pole structure of loop amplitudes. However, as was explained there, they are not sufficient for unitarity of the interacting theory. The value for "a" implies the presence of tachyons in the physical spectrum, thereby spoiling unitarity of the quantum interacting amplitudes. The values (3.88) only ensure that gauge invariance (i.e. conformal invariance) is realized on the space of states, i.e. in tree-level amplitudes, but not necessarily at higher loops.

Let us also remark here that conformal invariance may seem to be broken in the old covariant formulation, even for the critical values (3.88), as the central extension in (3.85) does not vanish. This is only an illusion; the old covariant formulation is incomplete, in that it does not include the reparametrisation ghost fields $c^\pm$ and $b_{\pm\pm}$.

Conformal invariance of the quantised system may also be verified by computing the

square of the BRST charge. One finds (see the Appendix):

$$\begin{aligned} (Q_B, Q_B) &= 2 Q_B^2 = \\ &= \sum_n \left[\frac{1}{12}(D-26)n^3 + (2+24a-D)n + 2(a-1) \right] c_{-n}c_n - \\ &= \sum_n ([L_n, L_m] - (n-m)L_{n+m}) \quad (3.89) \end{aligned}$$

This clearly shows that nilpotency of the BRST operator is obtained under the same conditions (3.88) which ensure the cancellation of the conformal anomaly in (3.87) [30]. This is to be expected, since L_n and Q_B are related, as shown in (3.84). If the BRST charge is nilpotent, the following can then easily be obtained:

$$[L_n, Q_B] = 0 \quad (3.90)$$

$$[L_n, L_m] = (n-m)L_{n+m} \quad (3.91)$$

These results show that nilpotency of the BRST charge, or cancellation of the central extension (with its two terms, one proportional to n , the other to n^3) in the algebra of the total Virasoro generators, are one and the same expression of conformal invariance in the quantised string theory.

In the present formalism, physical states are defined by the BRST conditions

$$Q_B |\psi\rangle = 0, \quad Q_c |\psi\rangle = -\frac{1}{2} |\psi\rangle \quad (3.92)$$

When nilpotency of the BRST charge is preserved at the quantum level, the first condition ensures gauge invariance (in the present gauge: conformal invariance) for physical states. Physical states then correspond to cohomology classes of Q_B , with the trivial class containing all zero-norm states of the form $Q_B |\chi\rangle$. It can be shown [31], when Q_B is nilpotent, that the general non-trivial solution (i.e. up to a null state $Q_B |\chi\rangle$) to the first condition in (3.92) is:

$$|\psi\rangle = -|\psi_{(a)}\rangle \otimes |-\rangle + |\psi_{(a)}^\dagger\rangle \otimes |+\rangle \quad (3.93)$$

where $|\psi_{(a)}^\dagger\rangle$ denotes states with no ghost excitations. Therefore, these states lie entirely in

the bosonic sector of the space of states, sector which is identical to the space of states in the old covariant quantisation. Actually [31], the states $|\psi_{(\bar{a})}^{\pm}\rangle$ turn out to be of strictly positive norm, so called transverse states. The second condition in (3.92), where the value $(-1/2)$ follows from normal ordering in Q_c then implies that states in the non-trivial cohomology class of Q_B are of the form:

$$|\psi\rangle = |\psi_{(\bar{a})}\rangle \otimes |-\rangle. \quad (3.94)$$

In turn, the first condition in (3.92) then becomes:

$$\{c_0[L_0^{(a)} - 1] + \sum_{n=1}^{\infty} c_{-n} L_n^{(a)}\} |\psi_{(\bar{a})}\rangle \otimes |-\rangle = 0, \quad (3.95)$$

which thus leads to identically the same physical conditions for $|\psi_{(\bar{a})}\rangle$ as in the old covariant quantisation of the theory (see (3.36) and (3.37)). Note that if we had defined physical states as having ghost charge $(+1/2)$, the Virasoro condition involving $L_0^{(a)}$ would not have been obtained.

This shows that physical states of strictly positive norm in both approaches are in a one-to-one correspondence. Moreover, zero-norm physical states $|\psi_0\rangle$ in the old covariant quantisation of the system correspond here to states $|\psi_0\rangle \otimes |-\rangle$ in the trivial cohomology class of Q_B (i.e. $|\psi_0\rangle \otimes |-\rangle = Q_B |\chi\rangle$) which satisfy the BRST invariant physical conditions (3.92). Indeed, these states $|\psi_0\rangle \otimes |-\rangle$ are free of ghost excitations, satisfy trivially (3.92) but are of zero-norm. Note that the states $|\psi_0\rangle \otimes |+\rangle$ also belong to the trivial cohomology class of Q_B but that they have ghost charge $(+1/2)$.

In so far as positive norm physical states are concerned, the old covariant approach leads to the same results as the BRST approach. The latter however, is complete in that it includes the ghosts due to gauge-fixing in a reparametrisation invariant system.

The physical state conditions (3.92) and the critical values (3.88) imply that the spectra of positive norm physical states in open and closed bosonic strings are exactly those obtained within the old covariant quantisation. For more details, see Refs.[1,2].

3.5. Comments

This concludes the discussion of BRST quantisation of the bosonic string, within the formulation of constrained systems developed by Batalin, Fradkin and Vilkovisky. We explained how world-sheet reparametrisation invariance of the theory, which in the conformal gauge reduces to conformal symmetry in two dimensions, can be maintained at the quantum level only by having the critical values in (3.88). Note that we may say [9] that we presented here the quantisation of the Nambu-Goto string, as opposed to the Polyakov string [32], since the conformal mode of the world-sheet metric (be it in the non-linear or linear formulation) does not couple as a dynamical degree of freedom [9]. In other words, the BFM formalism applied to the bosonic string, even with the linear action, explicitly preserves the Weyl invariance of the system [9]. Only reparametrisation invariance is spoiled by quantum anomalies, which vanish precisely for the critical values (3.88).

This result is to be contrasted with Polyakov's quantisation of the linear action through path-integral methods [32]. In that case, the quantum system is defined such that it explicitly preserves reparametrisation invariance. It is then found however, that the conformal mode becomes dynamical and couples precisely through the usual conformal (Weyl) anomaly, with the factor $(D-26)$ [32]. In that case, Weyl invariance (or decoupling of the conformal mode) can be restored only by

having the critical values in (3.85) [32,33]. (The fact that by way of different regularisations, a same anomaly may appear in different symmetries, is a well known phenomenon in field theory). In that case, Polyakov's approach is equivalent to the one discussed in these notes.

This does not mean that Polyakov's procedure is inconsistent for other values of D or a . Only, in such a case, the conformal mode becomes dynamical and has to be quantised as well [34]. Such a situation may also be studied within the BV formalism, by extending the classical system in a way such that the conformal mode is also dynamical [35]. Recent progress in the formulation of such theories is discussed in Ref.[36].

The reader will certainly have realized that the (b,c) system described above is universal, in the sense that it would appear in gauge fixing any reparametrisation (and Weyl) invariant system in two dimensions. (Our discussion was made specifically for the conformal gauge. It may be repeated for an arbitrary choice of λ^+ and λ^- , i.e. of the world-sheet metric, in which case the (b,c) system would couple covariantly as anticommuting world-sheet tensor fields c^μ and $b_{\mu\nu}$ [3]). Indeed, in the construction of the ghost sector, of its action and of the BRST charge, we only used the fact that the constraints in the bosonic sector satisfy the algebra (3.16) of two dimensional reparametrisations. Moreover, when restricting the system to the conformal gauge, this was done through a gauge-fixing function Ψ independent of the bosonic sector.

The (b,c) system described above will thus appear for any reparametrisation invariant system in the conformal gauge. From that point of view, the results of the analysis can be restated as follows. Gauge invariance of a reparametrisation invariant two-dimensional

system of "matter" fields will be realized at the quantum level if its contribution to the central extension of the conformal algebra precisely cancels that of the (b,c) ghost system. Hence, the matter sector must have central charge $c=26$, to compensate the contribution $c=-26$ from the ghost sector. This puts a restriction on the number of degrees of freedom of matter fields. The other condition, following from the term linear in n in the central extension, puts a constraint on the lowest eigenvalue of the Virasoro zero-mode L_0 in the matter sector, which in string theory, determines the mass spectrum.

For example, for bosonic strings considered here, since all matter fields x^μ are regarded as space-time coordinates, the first condition fixes the dimension of space-time to be $D=26$ (each scalar field contributes one unit to the central charge), and the second condition implies that the lowest physical state is a tachyon. Some of the scalar fields however, may rather be regarded as internal degrees of freedom, taking their values in some internal ("compactified") space, such as a torus for example (see Ref.[11] and references therein). In this way, it is possible to construct consistent bosonic string theories in any space-time dimension less than 26.

Therefore, the values (3.88) are not critical values for the dimension of space-time and of the physical ground-state, but rather, they are the critical values for a consistent quantum formulation of reparametrisation (and Weyl) invariant bosonic string theories. This remark is at the basis of the recent constructions of four dimensional strings.

4. THE SPINNING STRING

4.1. Introduction

As was discussed at length above, bosonic strings are reparametrisation invariant theories in the world-sheet, or equivalently, two-dimensional quantum gravity theories coupled to world-sheet matter fields. Even when using an action which does not involve a world-sheet metric, such a tensor appears in the covariant Hamiltonian formulation through the Lagrange multipliers of the constraints. Gauge symmetries of those theories can be maintained at the quantum level only for some critical values of the number of matter fields and of the ground-state mass eigenvalue. But on the other hand then, these symmetries are necessary for quantum consistency of these theories, namely in a covariant gauge, for the decoupling of negative- and zero-norm states from physical amplitudes, and in the light-cone gauge, for space-time covariance under the full Poincaré algebra (for references, see Refs.[1,2]).

Spinning strings [13] can easily be understood from a similar point of view. World-sheet symmetries are extended to include local supersymmetry transformations, so that these theories are actually two-dimensional quantum supergravity theories. The world-sheet matter fields are then not only bosonic scalar fields, but also spin 1/2 fermionic fields, coupled to a supergravity multiplet, i.e. a world-sheet graviton and gravitino. Gauge symmetries of the system then correspond to local supersymmetry transformations and local reparametrisations in the world-sheet, or concisely local super-reparametrisations in the super-world-sheet. When realized at the quantum level, they ensure the consistency of the theory, in a covariant or in a light-cone gauge formulation (for references, see Ref.[1]).

The action for the spinning string is a d=2, N=1 (minimal) supergravity action of D matter

fields ($x^\mu(\xi)$, $\lambda^\mu(\xi)$) coupled to a supergravity multiplet ($e_a^\mu(\xi)$, $\psi_a(\xi)$) (for details, see Ref.[1] and references therein). Here, $x^\mu(\xi)$ are the familiar scalar degrees of freedom, and $\lambda^\mu(\xi)$ their fermionic N=1 superpartners, namely world-sheet Majorana spinors, given by Grassmann variables. Also, $e_a^\mu(\xi)$ is a world-sheet zweibein (with $\alpha=0,1$ and $a=0,1$ respectively coordinate and tangent space indices), and $\psi_a(\xi)$ its gravitino N=1 superpartner, a world-sheet Majorana Rarita-Schwinger spin 3/2 field, also given by Grassmann variables. To specify the world-sheet spinor algebra, we shall use in what follows a real representation of the Dirac algebra:

$$\{\rho^a, \rho^b\} = 2\eta^{ab}, \quad a, b=0,1, \quad (4.1)$$

where $\eta^{ab} = \text{diag}(-,+)$ is the world-sheet Minkowski metric, representation given by:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^5 = -\rho^0\rho^1 - \rho^1\rho^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.2)$$

As is the case for the bosonic string where the linear action has an additional local Weyl symmetry (i.e. local rescalings of the world-sheet metric), the action for the spinning string has a local super-Weyl symmetry (for details, see Ref.[1]). In a covariant Hamiltonian formulation however, these Weyl degrees of freedom of the graviton supermultiplet decouple, so that a canonical quantisation of the system, either in the old covariant formulation or in the BFV formulation, will trivially be Weyl invariant. Only super-reparametrisation invariance may be broken through quantum anomalies.

As for the bosonic string, the theory may be (partially) gauge fixed in the superconformal gauge, where superconformal invariance is still an explicit symmetry, generated by the Fourier modes of the constraints of local reparametrisation and supersymmetry

invariance. This should be obvious, by analogy with our discussion for the bosonic string. Actually, since superconformal gauge fixing explicitly preserves a N-1 global world-sheet supersymmetry, it is quite straightforward to discuss the BRST quantisation of the spinning string, without going necessarily through the details of the corresponding BFV Hamiltonian formulation for this system (unless of course, one is primarily interested in that part of the discussion). Indeed, we simply need to supersymmetrise the results for the bosonic string. This can easily be done by means of superfields [4-6], which is the method we shall adopt below.

Let us nonetheless sketch the results which would follow from a detailed discussion of the BFV formalism. Clearly, the action of the spinning string leads to two bosonic constraints, generating local reparametrisations on the world-sheet, and to two fermionic constraints, generating local supersymmetry transformations. The associated two commuting and two anticommuting Lagrange multipliers would then correspond to components of the world-sheet graviton supermultiplet, and indeed transform as such quantities under variations induced by the constraints. In the BFV formalism, an anticommuting BFV ghost system is introduced in the sector corresponding to the bosonic constraints, and an additional commuting BFV ghost system is introduced in the sector corresponding to the fermionic constraints. Under local supersymmetry, all these BFV ghost degrees of freedom transform as irreducible supermultiplets of N-1 supergravity, with the corresponding invariant action.

Here again, no good gauge-fixing function Ψ can be found on the extended phase-space, proving the existence of a super-Gribov problem for the spinning string [9]. One may

nevertheless proceed by restricting the discussion to the superconformal gauge, using the same kind of gauge-fixing function Ψ , with the same redefinitions and limits as explained for the bosonic string. The ghost sector associated to the bosonic constraints then reduces to the (b,c) system discussed previously, whereas the ghost sector associated to the fermionic constraints reduces to the (β, γ) superghost system.

In the superconformal gauge, some specific combinations of the constraints, namely their Fourier modes, generate transformations preserving the gauge conditions. From the bosonic constraints, we have the familiar Virasoro generators of conformal symmetry, and from the fermionic constraints the generators of global N-1 supersymmetry. The complete remaining symmetry of the theory in that gauge is thus a N-1 superconformal invariance, under which the matter fields (x^μ, λ^μ) and the ghost fields (b,c) and (β, γ) transform as N-1 supermultiplets. Knowing the action of the matter fields in the superconformal gauge, and the part of the BRST invariant action associated to the bosonic constraints and its ghost system, it is rather straightforward to guess the correct BRST invariant action and the corresponding BRST charge. Having presented a detailed analysis of these points for the bosonic string, all these points should be clear to the reader.

4.2. Old covariant quantisation

Before considering BRST quantisation itself, let us briefly review the main results from the old covariant quantisation of the spinning string. In the superconformal gauge, we have the action:

$$S = \frac{-1}{4\pi\alpha'} \int d^2\xi \left[\eta^{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + i \bar{\lambda}^\mu \delta_\alpha^\beta p^\mu \partial_\beta \lambda_\mu \right], \quad (4.3)$$

where $\eta_{\alpha\beta}$ is the world-sheet Minkowski metric $\eta_{\alpha\beta} = \text{diag}(-,+)$, and $\lambda^\mu(\xi)$ are world-sheet Majorana spinors, with:

$$\lambda^\mu = \begin{pmatrix} \lambda^\mu_+ \\ \lambda^\mu_- \end{pmatrix}, \quad \bar{\lambda}^\mu = \lambda^{\mu\dagger} \rho^0. \quad (4.4)$$

In the representation (4.2) of the Dirac algebra, a Majorana spinor χ satisfies $\chi^* = \chi$. The action (4.3) then reduces to:

$$S = \frac{-1}{4\pi\alpha'} \int d^2\xi \left[x^\mu (\partial_\tau^2 - \partial_\sigma^2) x_\mu - i\bar{\lambda}^\mu (\partial_\tau + \partial_\sigma) \lambda_{+\mu} - i\bar{\lambda}^\mu (\partial_\tau - \partial_\sigma) \lambda_{-\mu} \right], \quad (4.5)$$

leading to the equations of motion

$$(\partial_\tau^2 - \partial_\sigma^2) x^\mu = 0, \quad (4.6a)$$

$$(\partial_\tau + \partial_\sigma) \lambda^\mu_- = 0, \quad (\partial_\tau - \partial_\sigma) \lambda^\mu_+ = 0. \quad (4.6b)$$

For the bosonic degrees of freedom, we have the same boundary conditions as in (3.3). For the fermionic degrees of freedom, we have (see Ref.[1] for details):

i) open strings: $\lambda^\mu(\sigma=0, \tau) = \lambda^\mu(\sigma=\pi, \tau)$

$$\lambda^\mu(\sigma=\pi, \tau) = \epsilon \lambda^\mu(\sigma=0, \tau), \quad (4.7a)$$

with $\epsilon=+1$ for the Ramond (R) sector, and $\epsilon=-1$ for the Neveu-Schwarz (NS) sector.

ii) closed strings:

$$\lambda^\mu(\sigma=\pi, \tau) = \Delta \lambda^\mu(\sigma=0, \tau), \quad (4.7b)$$

with $\Delta = -1$, $\Delta = \rho^5$, $\Delta = -\rho^5$ and $\Delta = 1$ respectively for the NS-NS, R-NS, NS-R and R-R sectors of the closed string. As usual for closed strings, we see from (4.6b) that the structure of the theory is given by a product of right- and left-movers. The boundary conditions (4.7b) correspond to choosing either a Ramond or a Neveu-Schwarz condition in each sector independently.

The action (4.5) is also invariant for N=1 supersymmetry transformations induced by the supercurrent:

$$J_\alpha = \partial_\mu x_\mu \rho^\beta_\alpha \lambda^\mu, \quad (4.8)$$

where $\rho^\beta_\alpha = \delta^\beta_\alpha \rho^0$ and $\rho_\alpha = \eta_{\alpha\beta} \delta^\beta_\alpha \rho^a$. As a consequence of local supersymmetry and

super-Weyl invariance, the equation of motion for the gravitino field leads to the constraints:

$$J_\alpha = 0. \quad (4.9)$$

Similarly, the equation of motion for the zweibein field e^α_μ implies the constraints:

$$T_{\alpha\beta} = 0, \quad (4.10)$$

where the generator of local world-sheet reparametrisations $T_{\alpha\beta}$ is given by:

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu + \frac{1}{4} i \bar{\lambda}^\mu [\rho_\alpha \partial_\beta + \rho_\beta \partial_\alpha] \lambda_\mu - \frac{1}{2} \eta_{\alpha\beta} [\eta^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x_\mu + \frac{1}{2} i \bar{\lambda}^\mu \rho^\gamma \partial_\gamma \lambda_\mu]. \quad (4.11)$$

Using the equations of motion, one may show that the quantities J_α and $T_{\alpha\beta}$ are conserved currents.

Actually, some of the boundary conditions in (4.7) break part or all of the world-sheet supersymmetry. This happens for a Neveu-Schwarz sector, but not for a Ramond sector. This breakdown of supersymmetry is also related to the presence or absence of tachyons in the corresponding physical spectra [1].

For the open string, the solutions to the equations of motion are:

$$x^\mu(\sigma, \tau): \text{ same as in (3.7)}, \quad (4.12a)$$

$$\lambda^\mu_\pm(\sigma, \tau) = \sqrt{\alpha'} \sum_q c^\mu_q e^{-iq(\tau \pm \sigma)}, \quad c^\mu_q = c^\mu_{-q}, \quad (4.12b)$$

with the following notation:

$$\text{R sector: } c^\mu_q = d^\mu_n, \quad (q-n) \in \mathbb{Z} \quad (4.12c)$$

$$\text{NS sector: } c^\mu_q = b^\mu_r, \quad (q-r) \in \mathbb{Z} + 1/2 \quad (4.12d)$$

Imposing the constraints (4.9) and (4.10), one obtains the super-Virasoro gauge conditions:

$$L^{(X)}_0 = 0, \quad J^{(X)}_0 = 0, \quad (4.13a)$$

with

$$L^{(X)}_0 = L^{(\frac{+}{-})}_0 + L^{(\frac{\lambda}{\bar{\lambda}})}_0, \quad (4.13b)$$

where $L^{(\frac{+}{-})}_0$ is given in (3.11), and

$$L^{(\frac{\lambda}{\bar{\lambda}})}_0 = \frac{1}{2} \sum_q (q - \frac{1}{2}n) c_{n-q}^\mu c_{q\mu}, \quad (4.13c)$$

$$J^{(X)}_0 = \sum_m c_{q-m}^\mu \alpha_{m\mu}, \quad (4.13d)$$

or explicitly:

$$\text{R sector: } L_n^{(\lambda)} = \frac{1}{2} \sum_m \left(m - \frac{1}{2}n\right) d_{n-m}^\mu d_{m\mu}, \quad (4.13e)$$

$$J_q^{(X)} = F_n^{(X)} = \sum_m d_{n-m}^\mu \alpha_{m\mu}, \quad (4.13f)$$

$$\text{NS sector: } L_n^{(\lambda)} = \frac{1}{2} \sum_r \left(r - \frac{1}{2}n\right) b_{n-r}^\mu b_{r\mu}, \quad (4.13g)$$

$$J_q^{(X)} = G_r^{(X)} = \sum_m b_{r-m}^\mu \alpha_{m\mu}, \quad (4.13h)$$

For closed strings, we have:

$$x^\mu(\sigma, \tau) : \text{ same as in (3.8)} \quad (4.14a)$$

$$\lambda_q^\mu(\sigma, \tau) = \sqrt{2\alpha'} \sum_q \alpha_q^\mu e^{-2iq(\tau - \sigma)}, \quad (4.14b)$$

$$\bar{\lambda}_q^\mu(\sigma, \tau) = \sqrt{2\alpha'} \sum_q \bar{\alpha}_q^\mu e^{-2iq(\tau + \sigma)}, \quad (4.14c)$$

with $(\alpha_q^\mu, \bar{\alpha}_q^\mu) = (b_r^\mu, \bar{b}_r^\mu)$, $(d_n^\mu, \bar{d}_n^\mu)$, $(b_r^\mu, \bar{d}_n^\mu)$ or $(d_n^\mu, \bar{b}_r^\mu)$ ($n \in \mathbb{Z}$, $r \in \mathbb{Z} + 1/2$), respectively for NS-NS, R-NS, NS-R and R-R boundary conditions. The constraints then reduce to the super-Virasoro conditions:

$$L_n^{(X)} = 0, \quad \bar{L}_n^{(X)} = 0, \quad (4.15a)$$

$$J_q^{(X)} = 0, \quad \bar{J}_q^{(X)} = 0, \quad (4.15b)$$

where $L_n^{(X)}$, $J_q^{(X)}$ are given in (4.13) and $\bar{L}_n^{(X)}$, $\bar{J}_q^{(X)}$ have the same expressions in terms of the left-moving modes α_n^μ , $\bar{\alpha}_n^\mu$ and $\bar{d}_n^\mu$.

The quantised system is defined by the commutation and anticommutation relations which follow from the classical Poisson brackets. For the bosonic degrees of freedom, we have the same commutators as in (3.33) and (3.34). For the fermionic degrees of freedom, we have second-class constraints which must be reduced by the use of Dirac brackets [14] (see Ref.[1]). Doing so, one obtains the anticommutators:

$$(c_{q_1}^\mu, c_{q_2}^\nu) = \eta^{\mu\nu} \delta_{q_1+q_2,0} = (\bar{c}_{q_1}^\mu, \bar{c}_{q_2}^\nu). \quad (4.16)$$

The corresponding Fock spaces are rather straightforward to describe. In the Ramond sector of the open string, we have Fock vacua $|\Omega; p\rangle \otimes u(p)$, where $|\Omega\rangle$ is annihilated by α_n^μ , d_n^μ ($n \geq 1$), $|p\rangle$ are momentum eigenstates, and $u(p)$ is a spinor representation for the Clifford algebra of fermionic zero-modes d_0^μ :

$$(d_0^\mu, d_0^\nu) = \eta^{\mu\nu}, \quad (4.17)$$

such that:

$$d_0^\mu = \frac{1}{\sqrt{2}} i \gamma^\mu (-1)^{n-1} \sum_{n=1}^{\infty} d_n^\mu d_{n\mu}. \quad (4.18)$$

Here, γ^μ are usual Dirac matrices in D dimensions, which satisfy:

$$(\gamma^\mu, \gamma^\nu) = -2 \eta^{\mu\nu}. \quad (4.19)$$

Consequently, all states in the Ramond sector of the open spinning string are space-time fermions.

In the Neveu-Schwarz sector, we simply have Fock vacua $|\Omega; p\rangle$, with $|\Omega\rangle$ annihilated by α_n^μ , b_r^μ ($n \geq 1$, $r \geq 1/2$). Consequently, all states in this sector of the open spinning string are space-time bosons.

For closed strings, the structure of the space of states is obtained by tensoring these spaces (except for the bosonic zero-modes). We thus obtain space-time bosons in the R-R and NS-NS sectors, and space-time fermions in the R-NS and NS-R sectors.

Normal ordering of these operators is defined as before. All creation operators to the left of all annihilation operators. For fermionic zero-modes in the Ramond sector, we choose again:

$$: d_0^\mu d_0^\nu : = \frac{1}{2} [d_0^\mu d_0^\nu - d_0^\nu d_0^\mu]. \quad (4.20)$$

Defining the quantum super-Virasoro generators by the normal ordered expressions of (4.13) and (4.15), one may compute the corresponding superconformal algebras (see the Appendix). This leads to:

$$\text{R sector: } \{F_n^{(X)}, F_m^{(X)}\} = 2L_{n+m}^{(X)} + \frac{1}{2} D n^2 \delta_{n+m,0}.$$

$$\{L_n^{(X)}, F_m^{(X)}\} = \left(\frac{1}{2}n - m\right) F_{n+m}^{(X)}$$

$$\{L_n^{(X)}, L_m^{(X)}\} = (n-m)L_{n+m}^{(X)} + \frac{1}{8} D n^3 \delta_{n+m,0}, \quad (4.21a)$$

with

$$\{L_n^{(\lambda)}, L_m^{(\lambda)}\} = (n-m)L_{n+m}^{(\lambda)} + \frac{1}{12} D \left(\frac{1}{2}n^3 + n\right) \delta_{n+m,0}. \quad (4.21b)$$

$$\text{NS sector: } \{G_r^{(X)}, G_s^{(X)}\} = 2L_{r+s}^{(X)} + \frac{1}{2} D \left(r^2 - \frac{1}{4}\right) \delta_{r+s,0}$$

$$\{L_n^{(X)}, G_r^{(X)}\} = \left(\frac{1}{2}n - r\right) G_{n+r}^{(X)},$$

$$\{L_n^{(X)}, L_m^{(X)}\} = (n-m)L_{n+m}^{(X)} + \frac{1}{8} D n(n^2 - 1) \delta_{n+m,0} \quad (4.22a)$$

with

$$\{L_n^{(\lambda)}, L_m^{(\lambda)}\} = (n-m)L_{n+m}^{(\lambda)} + \frac{1}{24} D n(n^2 - 1) \delta_{n+m,0}. \quad (4.22b)$$

This shows that quantum effects in the matter sector break the symmetries of the system associated to world-sheet super-reparametrisation invariance. Indeed, we obtain central extensions due to quantum anomalies in the algebras of world-sheet supersymmetry and world-sheet reparametrisations. Note in particular that each Majorana-Weyl fermion contributes a value (+1/2) to the central charge of the Virasoro algebra (in units where a scalar contributes a value (+1)).

Physical states in the old covariant quantisation of the spinning string are defined by:

$$\text{R sector: } F_n^{(X)} |\psi\rangle = 0 = L_n^{(X)} |\psi\rangle, n \geq 1, \quad (4.23a)$$

$$(L_0^{(X)} - \sqrt{a_+}) |\psi\rangle = 0, \quad (4.23b)$$

$$(L_0^{(X)} - a_+) |\psi\rangle = 0. \quad (4.23c)$$

$$\text{NS sector: } G_r^{(X)} |\psi\rangle = 0 = L_n^{(X)} |\psi\rangle, n \geq 1, r \geq 1/2, \quad (4.24a)$$

$$(L_0^{(X)} - a_-) |\psi\rangle = 0. \quad (4.24b)$$

where a_+ (resp. a_-) is the subtraction constant in the Ramond (resp. Neveu-Schwarz) sector.

The corresponding no-ghost theorem [37] then implies:

$$D = 10, a_+ = 0, a_- = \frac{1}{2}. \quad (4.25)$$

Using these results, it is possible to describe [1] the physical spectrum, both for open and for closed strings in all their sectors. In particular, the GSO projection [38] of these theories leads to unitary and space-time supersymmetric spectra, free of any tachyons. For more details, see Ref.[1].

4.3. BRST quantisation

Let us first rewrite the results for the bosonic string in the conformal gauge. After solving for $\pi^\mu(\sigma, \tau)$, we have the action (see (3.24) and (3.57)):

$$S_{\text{(bosonic)}} = \int d^2\xi \left[\frac{-1}{4\pi\alpha'} \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2\pi} b_{--} (\partial_\tau + \partial_\sigma) c^- + \frac{i}{2\pi} b_{++} (\partial_\tau - \partial_\sigma) c^+ \right], \quad (4.26)$$

which is invariant under transformations induced by the BRST charge (see (3.58)):

$$Q_B^{\text{(bosonic)}} = \int_0^\pi d\sigma \left[\frac{1}{8\pi\alpha'} c^-(\dot{x} - x')^2 + \frac{1}{8\pi\alpha'} c^+(\dot{x} + x')^2 + \frac{i}{2\pi} c^- \partial_\sigma c^- b_{--} - \frac{i}{2\pi} c^+ \partial_\sigma c^+ b_{++} \right]. \quad (4.27)$$

For convenience, let us introduce a new notation (which is useful in the conformal gauge but not appropriate when considering generally world-sheet covariance of matter and ghost

fields) by defining world-sheet light-cone coordinates:

$$\sigma_+ = \tau + \sigma, \quad \sigma_- = \tau - \sigma, \quad (4.28a)$$

with

$$\partial_+ = \frac{\partial}{\partial \sigma_+}, \quad \partial_- = \frac{\partial}{\partial \sigma_-}, \quad (4.28b)$$

such that

$$\partial_\tau = \partial_+ + \partial_-, \quad \partial_\sigma = \partial_+ - \partial_-. \quad (4.28c)$$

We also introduce:

$$T_{\pm\pm}^{(\alpha)} = \phi_{\pm}^{(\alpha)} = \frac{1}{2\pi\alpha'} (\partial_\pm x^\mu)^2, \quad (4.29a)$$

$$T_{\pm\pm}^{(\zeta)} = \phi_{\pm}^{(\zeta)} = \frac{-i}{2\pi} [2\partial_\pm c^\pm b_{\pm\pm} + c^\pm \partial_\pm b_{\pm\pm}]. \quad (4.29b)$$

With these redefinitions, (4.26) and (4.27) become:

$$S_{(\text{bosonic})} = \int d\sigma_+ d\sigma_- \left[\frac{1}{2\pi\alpha'} \partial_\pm x^\mu \partial_\pm x_\mu + \frac{i}{2\pi} b_{--} \partial_- c^+ + \frac{i}{2\pi} b_{++} \partial_+ c^+ \right], \quad (4.30)$$

and

$$Q_B^{(\text{bosonic})} = \int_0^\pi d\sigma \left[c^+ (T_{--}^{(\alpha)} + \frac{1}{2} T_{--}^{(\zeta)}) + c^+ (T_{++}^{(\alpha)} + \frac{1}{2} T_{++}^{(\zeta)}) \right]. \quad (4.31)$$

Similarly for the spinning string, one obtains for the action (4.3) or (4.5):

$$S_{(\text{matter})} = \frac{1}{2\pi\alpha'} \int d\sigma_+ d\sigma_- \left[\partial_\pm x^\mu \partial_\pm x_\mu + \frac{i}{2} \lambda_\pm^\mu \partial_\pm \lambda_{\pm\mu} + \frac{i}{2} \lambda_\pm^\mu \partial_\pm \lambda_{\pm\mu} \right]. \quad (4.32)$$

Moreover, the constraints (4.9) and (4.10) may equivalently be expressed by the conditions:

$$J_{\pm}^{(X)} = 0, \quad T_{\pm\pm}^{(X)} = 0, \quad (4.33)$$

where these generators of supersymmetry and reparametrisation transformations are defined by:

$$J_{\pm}^{(X)} = \frac{1}{4\pi\alpha'} \partial_\pm x^\mu \lambda_{\pm\mu}, \quad J_{\pm}^{(X)} = \frac{1}{4\pi\alpha'} \partial_\pm x^\mu \lambda_{\pm\mu}, \quad (4.34a)$$

and

$$T_{\pm\pm}^{(X)} = \frac{1}{2\pi\alpha'} (\partial_\pm x^\mu)^2 + \frac{i}{4\pi\alpha'} \lambda_\pm^\mu \partial_\pm \lambda_{\pm\mu} = T_{\pm\pm}^{(\alpha)} + T_{\pm\pm}^{(\lambda)}, \quad (4.34b)$$

$$T_{\pm\pm}^{(X)} = \frac{1}{2\pi\alpha'} (\partial_\pm x^\mu)^2 + \frac{i}{4\pi\alpha'} \lambda_\pm^\mu \partial_\pm \lambda_{\pm\mu} = T_{\pm\pm}^{(\alpha)} + T_{\pm\pm}^{(\lambda)}, \quad (4.34c)$$

In order to take advantage of the global N=1 supersymmetry explicit in the superconformal gauge, let us introduce two real Grassmann variables θ_+ and θ_- , and define supersymmetry generators Q_- and Q_+ , which act on superfields:

$$Q_- = \frac{\partial}{\partial \theta_-} + 2i\theta_- \partial_-, \quad (4.35a)$$

$$Q_+ = \frac{\partial}{\partial \theta_+} + 2i\theta_+ \partial_+, \quad (4.35b)$$

with the corresponding supercovariant derivatives:

$$D_\pm = \frac{\partial}{\partial \theta_\pm} - 2i\theta_\pm \partial_\pm. \quad (4.36)$$

Let us remark here that it is the doublet

$$\begin{pmatrix} -\theta_+ \\ \theta_- \end{pmatrix}$$

which transforms as a d=2 Majorana spinor under local Lorentz transformations. Since we do not consider here explicitly questions related to this symmetry of the system, this notational inconvenience is not important.

Corresponding to the matter fields of the spinning string, we then introduce the real scalar superfield

$$X^\mu(\xi, \theta) = x^\mu(\xi) + i\theta_- \lambda_-^\mu + i\theta_+ \lambda_+^\mu + i\theta_- \theta_+ B^\mu. \quad (4.37)$$

The action (4.32) may then be rewritten in an explicitly supersymmetric invariant form (we

define $\int d\theta_+ d\theta_- \theta_- \theta_+ = 1$):

$$S_{\text{(matter)}} = \frac{1}{8\pi\alpha'} \int d\sigma_- d\sigma_+ d\theta_- d\theta_+ D_- X^\mu D_- X_\mu$$

$$= \frac{1}{2\pi\alpha'} \int d\sigma_- d\sigma_+ \left[\partial_- X^\mu \partial_- X_\mu + \frac{1}{2} i\lambda_-^\mu \partial_- \lambda_{-\mu} + \frac{1}{2} i\lambda_+^\mu \partial_- \lambda_{-\mu} + \frac{1}{4} B^\mu B_\mu \right], \quad (4.38)$$

The equation of motion for the auxiliary field B^μ being $B^\mu=0$, we may simply ignore it in our discussion. The action (4.38) is then that of the spinning string in the superconformal gauge, and is invariant under the supersymmetry transformations induced on $X^\mu(\xi, \theta)$ by the supercharges Q_- and Q_+ (provided one uses the equations of motion since we set $B^\mu=0$).

The constraints (4.33) can also be expressed in terms of the superfields:

$$\Pi_{\pm\pm}^{(B)} = \frac{1}{8\pi\alpha'} D_\pm X^\mu D_\pm X_\mu = J_{\pm}^{(X)} - \theta_\pm T_{\pm\pm}^{(X)}. \quad (4.39)$$

These relations determine how the bosonic and fermionic constraints transform under world-sheet supersymmetry.

In order to complete the BRST formulation of the spinning string in the superconformal gauge, let us now introduce ghost superfields:

$$B_{\pm\pm}(\xi, \theta) = \beta_{\pm\pm}(\xi) + i\theta_{\pm\pm} b_{\pm\pm}(\xi), \quad (4.40a)$$

$$C^\pm(\xi, \theta) = c^\pm(\xi) + \theta_\pm \gamma^\pm(\xi), \quad (4.40b)$$

where the Hermitian Grassmann fields $c^\pm, b_{\pm\pm}$ are associated to the bosonic constraints (these are the ghosts of the bosonic string), and the Hermitian commuting fields $\beta_{\pm\pm}, \gamma^\pm$ are associated to the fermionic constraints.

The action for the ghost sector of the spinning string in the superconformal gauge is then given by:

$$S_{\text{(ghost)}} = \frac{-i}{4\pi} \int d\sigma_- d\sigma_+ d\theta_- d\theta_+ [B_{--} D_- C^- - B_{++} D_- C^+]$$

$$= \int d\sigma_- d\sigma_+ \left[\frac{i}{2\pi} b_{--} \partial_- c^- + \frac{i}{2\pi} b_{++} \partial_- c^+ + \frac{1}{2\pi} \beta_{--} \partial_- \gamma^- + \frac{1}{2\pi} \beta_{++} \partial_- \gamma^+ \right]. \quad (4.41)$$

Indeed, this quantity is explicitly invariant under world-sheet supersymmetry, and reduces in the (b, c) sector to the ghost action of the bosonic string.

The total action $S_{\text{(spinning)}}$ for the spinning string in the superconformal gauge is then given by the sum of (4.32) and (4.41). The equations of motion for the (b, c) ghost system are as in the bosonic string, namely:

$$\partial_- c^- = 0, \quad \partial_- c^+ = 0, \quad (4.42a)$$

$$\partial_- b_{--} = 0, \quad \partial_- b_{++} = 0, \quad (4.42b)$$

and for the (β, γ) superghost system, we also have:

$$\partial_- \gamma^- = 0, \quad \partial_- \gamma^+ = 0, \quad (4.43a)$$

$$\partial_- \beta_{--} = 0, \quad \partial_- \beta_{++} = 0. \quad (4.43b)$$

The boundary conditions for the (b, c) system are as in (3.54), whereas the boundary conditions for the (β, γ) system are identical to those for the fermionic matter fields λ^μ since these two sets of boundary conditions are correlated through world-sheet supersymmetry.

The generators of world-sheet supersymmetry and world-sheet reparametrisations in the ghost sector are given by:

$$T_{\pm\pm}^{(B)} = \frac{-i}{2\pi} \left[\frac{3}{4} D_\pm^2 C^\pm B_{\pm\pm} - \frac{1}{4} D_\pm C^\pm D_\pm B_\pm + \frac{1}{2} C^\pm D_\pm^2 B_{\pm\pm} \right] \quad (4.44a)$$

$$= J_{\pm}^{(B)} - \theta_\pm T_{\pm\pm}^{(B)}. \quad (4.44b)$$

We have:

$$J_{\pm\pm}^{(B)} = \frac{-1}{2\pi} \left[\frac{3}{2} \partial_\pm c^\pm \beta_{\pm\pm} + c^\pm \partial_\pm \beta_{\pm\pm} + \frac{1}{4} \gamma^\pm b_{\pm\pm} \right], \quad (4.44c)$$

$$T_{\pm\pm}^{(B)} = T_{\pm\pm}^{(c)} + T_{\pm\pm}^{(\gamma)}, \quad (4.44d)$$

with $T_{\pm\pm}^{(c)}$ given in (4.29b) and

$$T_{\pm\pm}^{(y)} = \frac{1}{2\pi} \left[\frac{3}{2} \partial_{\pm} \gamma^{\pm} \beta_{\pm\pm} + \frac{1}{2} \gamma^{\pm} \partial_{\pm} \beta_{\pm\pm} \right]. \quad (4.44e)$$

The total generators for all world-sheet symmetries are thus:

$$J_{\pm} = J_{\pm}^{(X)} + J_{\pm}^{(B)}, \quad (4.45a)$$

$$T_{\pm\pm} = T_{\pm\pm}^{(X)} + T_{\pm\pm}^{(B)} \\ = T_{\pm\pm}^{(a)} + T_{\pm\pm}^{(\lambda)} + T_{\pm\pm}^{(c)} + T_{\pm\pm}^{(y)}. \quad (4.45b)$$

Finally, by analogy with (4.31), the BRST charge is now obtained as:

$$Q_B = \int_0^{\pi} d\sigma \left\{ \int d\theta_- C^- \left[\Pi_{\pm\pm}^{(X)} + \frac{1}{2} \Pi_{\pm\pm}^{(B)} \right] + \right. \\ \left. + \int d\theta_+ C^+ \left[\Pi_{\pm\pm}^{(X)} + \frac{1}{2} \Pi_{\pm\pm}^{(B)} \right] \right\} \quad (4.46a)$$

$$- \int_0^{\pi} d\sigma \left\{ c^- \left[T_{\pm\pm}^{(X)} + \frac{1}{2} T_{\pm\pm}^{(B)} \right] + c^+ \left[T_{\pm\pm}^{(X)} + \frac{1}{2} T_{\pm\pm}^{(B)} \right] + \right. \\ \left. + \gamma^- \left[J_{\pm}^{(X)} + \frac{1}{2} J_{\pm}^{(B)} \right] + \gamma^+ \left[J_{\pm}^{(X)} + \frac{1}{2} J_{\pm}^{(B)} \right] \right\}. \quad (4.46b)$$

All the quantities just introduced generate symmetries of the total spinning string action $S_{\text{(spinning)}}$ in the superconformal gauge. As for the bosonic string (see (3.62)), the Poisson brackets of $b_{\pm\pm}$ and $\beta_{\pm\pm}$ with Q_B lead to the bosonic and fermionic constraints $T_{\pm\pm}$ and $J_{\pm}$ respectively.

For the open spinning string, the solutions to the equations of motion for x^{μ} , $\lambda_{\pm}^{\mu}$, $c^{\pm}$ and $b_{\pm\pm}$ are as given previously (see (3.7), (4.12b), (3.63b) and (3.63c)). For the superghosts, we have:

$$\gamma^{\pm}(\sigma, \tau) = \sum_q \gamma_q e^{-iq\sigma_{\pm}}, \quad (4.47a)$$

$$\beta_{\pm\pm}(\sigma, \tau) = \sum_q \beta_q e^{-iq\sigma_{\pm}}, \quad (4.47b)$$

with $(q-n) \in \mathbb{Z}$ in the Ramond sector and $(q-r) \in \mathbb{Z}+1/2$ in the Neveu-Schwarz sector. One then obtains:

$$T_{\pm\pm}^{(k)} = \frac{1}{2\pi} \sum_n L_n^{(k)} e^{-in\sigma_{\pm}}, \quad k = \alpha, \lambda, c, \gamma \quad (4.48a)$$

$$J_{\pm}^{(K)} = \frac{1}{2\pi} \sum_q \frac{1}{2\sqrt{2}} J_q^{(K)} e^{-iq\sigma_{\pm}}, \quad K = X, B \quad (4.48b)$$

Here, $L_n^{(a)}$, $L_n^{(\lambda)}$, $L_n^{(c)}$ and $J_q^{(X)}$ were defined previously. For the remaining quantities, we have:

$$L_n^{(\gamma)} = -i \sum_q \left(\frac{1}{2} n - q \right) \beta_{n+q} \gamma_{-q}, \quad (4.49)$$

$$J_q^{(B)} = \frac{1}{\sqrt{2}} \sum_n b_{-n} \gamma_{q+n} + 2i\sqrt{2} \sum_n \left(\frac{1}{2} n + q \right) c_{-n} \beta_{q+n}. \quad (4.50)$$

and

$$Q_B = \sum_n c_{-n} \left[L_n^{(a)} + L_n^{(\lambda)} + \frac{1}{2} L_n^{(c)} + \frac{1}{2} L_n^{(\gamma)} \right] + \\ + \sum_q \frac{1}{2\sqrt{2}} \gamma_{-q} \left[J_q^{(X)} + \frac{1}{2} J_q^{(B)} \right]. \quad (4.51)$$

For the closed spinning string, the solutions to the equations of motion for the matter fields and the (b,c) system are as previously (see (3.8), (4.14b,c), (3.71b,c) and (3.71d,e)). For the superghost system, we have:

$$\gamma^-(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_q \gamma_q e^{-2iq\sigma_-}, \quad (4.52a)$$

$$\gamma^+(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_q \bar{\gamma}_q e^{-2iq\sigma_+}, \quad (4.52b)$$

$$\beta_{--}(\sigma, \tau) = 2\sqrt{2} \sum_q \beta_q e^{-2iq\sigma_-}, \quad (4.52c)$$

$$\beta_{++}(\sigma, \tau) = 2\sqrt{2} \sum_q \bar{\beta}_q e^{-2iq\sigma_+}, \quad (4.52d)$$

with as before $(q-n) \in \mathbb{Z}$ or $(q-r) \in \mathbb{Z}+1/2$ correspondingly for a Ramond or a Neveu-Schwarz sector. We then also obtain:

$$T_{\pm\pm}^{(k)} = \frac{2}{\pi} \sum_n L_n^{(k)} e^{-2in\sigma_{\pm}}, \quad T_{\pm\pm}^{(\bar{k})} = \frac{2}{\pi} \sum_n \bar{L}_n^{(\bar{k})} e^{-2in\sigma_{\pm}}. \quad (4.53a)$$

$$J_{-}^{(K)} = \frac{2}{\pi} \sum_q \frac{1}{4} J_q^{(K)} e^{-2iq\sigma}, \quad J_{+}^{(K)} = \frac{2}{\pi} \sum_q \frac{1}{4} \bar{J}_q^{(K)} e^{-2iq\sigma}. \quad (4.53b)$$

with $k=\alpha, \lambda, c, \gamma$ and $K=X, B$. Here, $L_n^{(k)}$ and $J_q^{(K)}$ are defined as in the open string sector, and $\bar{L}_n^{(k)}, \bar{J}_q^{(K)}$ have identical definitions in terms of the left-moving modes $\bar{\alpha}_n^k, \bar{b}_n^k, \bar{d}_n^k, \bar{c}_n, \bar{b}_n, \bar{\gamma}_q$ and $\bar{\beta}_q$. Moreover, we have:

$$Q_B = Q_{B,R} + Q_{B,L}, \quad (4.54)$$

where $Q_{B,R}$ is given in (4.51) and $Q_{B,L}$ has the same expression in terms of the left-moving modes.

To complete the classical BRST formulation of the spinning string in the superconformal gauge, we only need to specify in addition the Poisson brackets for all degrees of freedom. These Poisson brackets were specified as we proceeded with the discussion, except for those of the superghost system, which can simply be obtained from those of the (b,c) system by supersymmetry.

BRST quantisation of the system is defined by the corresponding commutation and anticommutation relations. For the matter sector and the (b,c) ghosts, they are given in (3.33), (3.34), (4.16) and (3.77). For the superghosts, one obtains:

$$[\gamma_{q1}, \beta_{q2}] = i\delta_{q1+q2,0}, \quad \gamma_q^* = \gamma_{-q}, \quad \beta_q^* = \beta_{-q}, \quad (4.55)$$

$$[\bar{\gamma}_{q1}, \bar{\beta}_{q2}] = i\delta_{q1+q2,0}, \quad \bar{\gamma}_q^* = \bar{\gamma}_{-q}, \quad \bar{\beta}_q^* = \bar{\beta}_{-q}, \quad (4.56)$$

both for open and for closed strings.

The corresponding spaces of states, both in the Ramond and the Neveu-Schwarz sector, need not be specified here for our discussion below, and their structure should be clear. Let us however remark that subtleties arise in the superghost sector for the Ramond sector, having to do with the infinite degeneracy of the Fock

space due to the zero-modes γ_0 and β_0 , and with picture changing in a bosonised version of the superghost system [4.5]. Normal ordering prescriptions are as usual. For the superghost zero-modes in the Ramond sector, we choose to put all γ_0 operators to the left of all β_0 operators.

The composite operators $L_n^{(k)}$ and $J_q^{(K)}$ which generate the symmetries of the system are then defined in terms of the same expressions as before, understood now to be normal ordered. This leads to normal ordering subtraction constants in the total contributions:

$$L_n = L_n^{(X)} + L_n^{(B)} - a_+ \delta_{n,0}, \quad (4.57)$$

$$J_q = J_q^{(X)} + J_q^{(B)}, \quad (4.58)$$

with

$$L_n^{(X)} = L_n^{(\alpha)} + L_n^{(\lambda)}, \quad (4.59a)$$

$$L_n^{(B)} = L_n^{(c)} + L_n^{(\gamma)}, \quad (4.59b)$$

Similarly, the BRST charge is now given by:

$$Q_B = \sum_n c_{-n} \left[L_n^{(X)} + \frac{1}{2} L_n^{(B)} - a_+ \delta_{n,0} \right] + \sum_q \frac{1}{2\sqrt{2}} \gamma_{-q} \left[J_q^{(X)} + \frac{1}{2} J_q^{(B)} \right]. \quad (4.60)$$

In these equations, a_+ (resp. a_-) is the subtraction constant in the Ramond (resp. Neveu-Schwarz) sector. Moreover, (4.60) is strictly valid only for the open string. For the closed string, these definitions apply to the right-moving sector, with similar quantities defined for the left-moving sector in terms of the left-moving modes. In addition, Q_B is then given by a sum of two quantities as in (4.60), one for the right-moving sector, and the other for the left-moving sector (see (4.54)). In what follows, we shall not make the appropriate distinctions anymore, which should be obvious.

Let us note that following from (4.60), we have (see the Appendix):

$$(b_n, Q_B) = L_n, \quad [\beta_q, Q_B] = -\frac{i}{2\sqrt{2}} J_q. \quad (4.61)$$

Gauge invariance of the quantised system can be studied in the superconformal gauge by considering the superconformal algebra of the total generators L_n and J_q , or by checking nilpotency of the BRST charge (see the Appendix).

In the Ramond sector, we have $(J_q^{(K)} = F_n^{(K)}, K=X, B)$:

$$(F_n^{(B)}, F_m^{(B)}) = 2L_{n+m}^{(B)} - 5n^2 \delta_{n+m,0}.$$

$$[L_n^{(B)}, F_m^{(B)}] = (\frac{1}{2}n - m) F_{n+m}^{(B)}.$$

$$[L_n^{(\gamma)}, L_m^{(\gamma)}] = (n - m) L_{n+m}^{(\gamma)} + \frac{1}{12} [11n^3 - 2n] \delta_{n+m,0}. \quad (4.62)$$

Completing with our previous results for $F_n^{(X)}$, $L_n^{(a)}$, $L_n^{(\lambda)}$ and $L_n^{(c)}$, we then obtain for the total generators $(F_n = F_n^{(X)} + F_n^{(B)})$:

$$(F_n, F_m) = 2L_{n+m} + [\frac{1}{2}(D - 10)n^2 + 2a_-] \delta_{n+m,0}.$$

$$[L_n, F_m] = (\frac{1}{2}n - m) F_{n+m}.$$

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{1}{12} [\frac{3}{2}(D - 10)n^3 + 24a_-n] \delta_{n+m,0}. \quad (4.63)$$

Hence, the superconformal anomalies in the Ramond sector vanish only for:

$$D = 10, \quad a_- = 0. \quad (4.64)$$

These are precisely the conditions (4.25) following from the no-ghost theorem in the old covariant approach (and in the light-cone gauge, from Lorentz covariance [1]).

It may also be shown that the BRST charge is nilpotent precisely for the same conditions only. Actually, as in (3.90) and (3.91) for the bosonic string, the Ramond superconformal algebra with vanishing central extensions follows from (4.61) if Q_B is nilpotent.

In the Neveu-Schwarz sector, we have $(J_q^{(K)} = G_r^{(K)}, K=X, B)$:

$$(G_r^{(B)}, G_s^{(B)}) = 2L_{r+s}^{(B)} + [-5r^2 + \frac{1}{4}] \delta_{r+s,0}.$$

$$[L_n^{(B)}, G_r^{(B)}] = (\frac{1}{2}n - r) G_{n+r}^{(B)}.$$

$$[L_n^{(\gamma)}, L_m^{(\gamma)}] = (n - m) L_{n+m}^{(\gamma)} + \frac{1}{12} [11n^3 + n] \delta_{n+m,0}. \quad (4.65)$$

Completing with our previous results for $G_r^{(X)}$, $L_n^{(a)}$, $L_n^{(\lambda)}$ and $L_n^{(c)}$, we obtain for the total generators $(G_r = G_r^{(X)} + G_r^{(B)})$:

$$(G_r, G_s) = 2L_{r+s} + [\frac{1}{2}(D - 10)r^2 + \frac{1}{4}(1 + 8a_- - \frac{1}{2}D)] \delta_{r+s,0}.$$

$$[L_n, G_r] = (\frac{1}{2}n - r) G_{n+r}.$$

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{1}{12} [\frac{3}{2}(D - 10)n^3 + 3(1 + 8a_- - \frac{1}{2}D)n] \delta_{n+m,0}. \quad (4.66)$$

Hence, the superconformal anomalies in the Neveu-Schwarz sector only vanish for:

$$D = 10, \quad a_- = \frac{1}{2}. \quad (4.67)$$

These are precisely the conditions (4.25) following from the no-ghost theorem in the old covariant approach (and in the light-cone gauge, from Lorentz covariance [1]).

It may also be shown that Q_B is nilpotent precisely for the same conditions only. Indeed, in that case the Neveu-Schwarz superconformal algebra with no central extensions can then be obtained from (4.61).

Finally, let us simply say here that physical states in the BRST quantised spinning string are defined by the BRST invariant condition

$$Q_B |\psi\rangle = 0, \quad (4.68)$$

and an additional condition restricting their ghost and superghost charges (much like in (3.92) for the bosonic string). Here again, non-trivial cohomology classes defined by (4.68) correspond to strictly positive norm (transverse) physical states in the old covariant quantisation [39], whereas the trivial cohomology class corresponds to zero-norm states. The additional constraint on the ghost number of these states is more subtle, as it is related to the problem of picture changing in the superghost sector. We do not consider this question here, as we simply wanted to present the BRST quantisation of spinning string theories, and discuss how the critical values (4.64) and (4.67) follow in such an approach, so that the same spectrum of physical states is seen to be obtained as in the old covariant approach.

4.4. Comments

The same comments as for the bosonic string can be made here. Without going into the details of the Batalin-Fradkin-Vilkovisky formalism, we derived the BRST quantisation of the spinning string in the superconformal gauge, by taking advantage of the global $N=1$ world-sheet supersymmetry preserved by this gauge fixing. We derived the conditions under which the gauge symmetries of the system, namely super-reparametrisation and super-Weyl invariance on the world-sheet, can be preserved at the quantum level. In the superconformal gauge, these conditions reduce to conditions for the cancellation of the superconformal anomalies in the $N=1$ superconformal algebra.

When these conditions are satisfied, the present quantisation method of the theory is equivalent to Polyakov's approach using path-integral methods [40]. In that case, super-reparametrisation invariance is explicitly

preserved in the quantisation procedure, but super-Weyl invariance is broken through quantum anomalies which precisely cancel under the same conditions as derived above.

In the same way that the (b,c) system is universal for the quantisation of any two-dimensional reparametrisation and Weyl invariant system in the conformal gauge, the (b,c) and (β, γ) ghost and superghost systems are universal for the quantisation of any two-dimensional super-reparametrisation and super-Weyl invariant system in the superconformal gauge.

Therefore, the results of the analysis are to be interpreted as follows. The first constraint, namely $D=10$, actually restricts the matter sector. The total contributions of these fields to the central charges of the supersymmetry and Virasoro algebras should compensate the corresponding contributions from the ghost sectors. Since these two central charges are related by supersymmetry, it is enough to do the counting in terms of the Virasoro central charge, say. Since the (b,c) and (β, γ) ghost systems contribute with $c=-26$ and $c=+11$ respectively, the total contribution of the matter sector should be $c-(-26+11)=+15$. The second constraint involving the normal ordering subtraction constants, puts a restriction on the lowest mass eigenvalue in the spectrum of these theories.

For example, if all fields x^μ and their superpartners λ^μ correspond to space-time coordinates, the first condition fixes the dimension of space-time to be $D=10$, since the corresponding central charge is $c=(1+1/2)D-3D/2=15$. The second condition changes with the Ramond or Neveu-Schwarz sector, and determines the corresponding physical mass spectra of the theory.

But clearly, some of the matter fields may be regarded as internal degrees of freedom,

transforming under the superconformal algebra (in the superconformal gauge). This is the fundamental remark which is at the basis of the construction of all lower dimensional string theories.

5. CONCLUSION

In these notes, an introduction to BRST quantisation of bosonic and spinning strings was presented, following Hamiltonian methods developed for constrained systems by Batalin, Fradkin and Vilkovisky (for references and a review, see Ref.[8]). As the technical aspects of the discussion only become more involved but are not really different when going from the scalar particle (for references, see Refs.[9,10]) to the bosonic string [9] and then to the spinning string, we first considered in some detail the corresponding analysis for the point particle, and compared the results with the well known canonical quantisation approach.

Turning then our attention to the bosonic string, we showed, through a rather detailed discussion, how world-sheet reparametrisation invariance reduces to conformal invariance in the conformal gauge, and we also determined the conditions necessary to preserve these gauge symmetries in the quantised theory [30].

Rather than applying the full BFV formalism to the spinning string, which is not the purpose of these notes, the results for the bosonic string were extended to the spinning string, by taking advantage of the global $N=1$ world-sheet supersymmetry of the theory in the superconformal gauge [4-6]. The corresponding results for gauge invariance in BRST quantisation were then also derived.

Both for bosonic and spinning strings, the results of the old covariant approach were briefly reviewed [1,2], and shown to also follow from BRST quantisation.

Clearly, these notes are only meant to be a starting point for further study of the subject. For example, we did not address aspects related to superghost bosonisation and the construction of vertex operators [4,5,41], nor to the calculation of string amplitudes.

The main lesson to be learned however, is that string theories in the (super)conformal gauge are nothing but (super)conformal theories, for which the central extensions of the corresponding conformal algebras need to take well determined critical values in order to cancel the corresponding contributions from the ghost sector following from gauge fixing. This is one of the most important aspects in the construction of all four-dimensional string theories, as discussed by Bert Schellekens in his lectures [11].

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APPENDIX

In this Appendix, we show how the conformal algebras given in the main text can be obtained using methods of conformal field theory [4-6,42]. In such an approach, those algebras follow from the singular terms in the operator product expansion (OPE) of local operators on the world-sheet. To illustrate the method, let us consider an operator $T(z)$,

function of the complex variable z , having the mode expansion:

$$T(z) = \sum_n L_n z^{-n-2}, \quad (A.1)$$

with

$$L_n = \oint \frac{dz}{2i\pi} z^{n+1} T(z), \quad (A.2)$$

where the contour integration is taken around the origin.

Corresponding to the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \left[\frac{1}{12} c n(n^2-1) + b n \right] \delta_{n+m,0}, \quad (A.3)$$

we have the OPE:

$$T(x) T(y) = \frac{c/2}{(x-y)^4} + \frac{by^2}{(x-y)^2} + \frac{-by^3}{(x-y)} + \frac{2}{(x-y)^2} T(y) + \frac{1}{(x-y)} \partial_y T(y). \quad (A.4)$$

In this relation, the left-hand side is understood to be the radial ordered product of the two operators, namely:

$$T(x) T(y) = R(T(x) T(y)) = \begin{cases} T(x) T(y) & \text{if } |x| > |y| \\ T(y) T(x) & \text{if } |y| > |x| \end{cases} \quad (A.5)$$

In the right-hand side, only the singular terms as $(x \rightarrow y)$ are displayed, although an infinite number of positive powers of $(x-y)$ also appears. This notational convention will be used throughout.

The equivalence between (A.3) and (A.4) can easily be established by contour integrations (see Ref.[1] for more details).

The calculation of the conformal algebras which appear in the quantisation of bosonic and spinning strings thus reduces to obtaining the OPE of specific world-sheet operators. Let us then introduce the following field operators, in the notations used in the main text:

$$X^\mu(z) = q^\mu - i\alpha_0^\mu \ln z + i \sum_n \frac{1}{n} \alpha_n^\mu z^{-n}, \quad (A.6)$$

$$\psi^\mu(z) = \sum_q \alpha_q^\mu z^{-q-1/2}, \quad (A.7)$$

$$c(z) = \sum_n c_n z^{-n+1}, \quad b(z) = \sum_n b_n z^{-n-2}, \quad (A.8)$$

$$\gamma(z) = \sum_q \gamma_q z^{-q+1/2}, \quad \beta(z) = \sum_q \beta_q z^{-q-3/2}. \quad (A.9)$$

Correspondingly, we define the composite operators:

$$T^{(\alpha)}(z) = \frac{1}{2} (i\partial_z X^\mu(z))^2, \quad (A.10)$$

$$T^{(\lambda)}(z) = \frac{1}{2} \psi^\mu(z) \partial_z \psi_\mu(z), \quad (A.11)$$

$$T^{(c)}(z) = [2\partial_z c(z) b(z) + c(z) \partial_z b(z)], \quad (A.12)$$

$$T^{(\gamma)}(z) = i \left[\frac{3}{2} \partial_z \gamma(z) \beta(z) + \frac{1}{2} \gamma(z) \partial_z \beta(z) \right], \quad (A.13)$$

and

$$J^{(X)}(z) = \frac{1}{2} : (i\partial_z X^\mu(z)) \psi_\mu(z) : , \quad (A.14)$$

$$J^{(B)}(z) = -i\sqrt{2} : \left[\frac{3}{2} \partial_z c(z) \beta(z) + c(z) \partial_z \beta(z) - \frac{1}{4} i \gamma(z) b(z) \right] : , \quad (A.15)$$

They have the following mode expansions:

$$T^{(k)}(z) = \sum_n L_n^{(k)} z^{-n-2}, \quad k=\alpha, \lambda, c, \gamma, \quad (A.16)$$

$$J^{(K)}(z) = \sum_q \frac{1}{2} J_q^{(K)} z^{-q-3/2}, \quad K=X, B, \quad (A.17)$$

where $L_n^{(k)}$ ($k=\alpha, \lambda, c, \gamma$) and $J_q^{(K)}$ ($K=X, B$) are the quantities given in the main text.

The OPE of these operators can easily be obtained using Wick's theorem and the correlation functions of the fundamental fields. Using the notation $\langle \phi(x) \chi(y) \rangle$ for the Wick contraction of the R-ordered product $R(\phi(x) \chi(y))$, we have:

$$\langle (i\partial_x X^\mu(x)) (i\partial_y X^\nu(y)) \rangle = \eta^{\mu\nu} \frac{1}{(x-y)^2}, \quad (A.18)$$

$$\langle c(x) b(y) \rangle = -\frac{1}{2} x y^{-2} \frac{x-y}{x-y} , \quad (A.19)$$

Ramond sector:

$$\langle \psi^\mu(x) \psi^\nu(y) \rangle = -\frac{1}{2} \eta^{\mu\nu} (xy)^{-1/2} \frac{x-y}{x-y} , \quad (A.20)$$

$$\langle \chi(x) \beta(y) \rangle = -i \frac{x^{1/2} y^{-1/2}}{x-y} , \quad (A.21)$$

Neveu-Schwarz sector:

$$\langle \psi^\mu(x) \psi^\nu(y) \rangle = \eta^{\mu\nu} \frac{1}{x-y} , \quad (A.22)$$

$$\langle \chi(x) \beta(y) \rangle = -i \frac{x y^{-1}}{x-y} , \quad (A.23)$$

The OPE of $T^{(k)}(z)$ ($k=\alpha, \lambda, c, \gamma$) with itself can then easily be obtained. In the parametrisation of (A.4), one finds:

$$T^{(\alpha)}(z) : \quad c = D , \quad b = 0 , \quad (A.24)$$

$$T^{(c)}(z) : \quad c = -26 , \quad b = -2 , \quad (A.25)$$

Ramond sector:

$$T^{(\lambda)}(z) : \quad c = \frac{1}{2} D , \quad b = \frac{1}{8} D , \quad (A.26)$$

$$T^{(\gamma)}(z) : \quad c = +11 , \quad b = \frac{3}{4} , \quad (A.27)$$

Neveu-Schwarz sector:

$$T^{(\lambda)}(z) : \quad c = \frac{1}{2} D , \quad b = 0 , \quad (A.28)$$

$$T^{(\gamma)}(z) : \quad c = +11 , \quad b = 1 , \quad (A.29)$$

For the OPE of $J^{(K)}(z)$ ($K=X, B$) with itself, we have:

$$J^{(K)}(x) J^{(K)}(y) = \frac{\hat{c}/4}{(x-y)^3} + \frac{\hat{a}y^{-2}}{(x-y)} + \frac{1}{2} \frac{1}{(x-y)} T^{(K)}(y) , \quad K=X, B , \quad (A.30)$$

where:

$$T^{(X)}(z) = T^{(\alpha)}(z) + T^{(\lambda)}(z) , \quad (A.31a)$$

$$T^{(B)}(z) = T^{(c)}(z) + T^{(\gamma)}(z) , \quad (A.31b)$$

In the Ramond sector, one obtains:

$$J^{(X)}(z) : \quad \hat{c} = D , \quad \hat{a} = \frac{1}{32} D , \quad (A.32)$$

$$J^{(B)}(z) : \quad \hat{c} = -10 , \quad \hat{a} = -\frac{5}{16} , \quad (A.33)$$

and in the Neveu-Schwarz sector:

$$J^{(X)}(z) : \quad \hat{c} = D , \quad \hat{a} = 0 , \quad (A.34)$$

$$J^{(B)}(z) : \quad \hat{c} = -10 , \quad \hat{a} = -\frac{1}{4} , \quad (A.35)$$

Finally, we also have:

$$T^{(K)}(x) J^{(K)}(y) = \frac{3/2}{(x-y)^2} J^{(K)}(y) + \frac{1}{(x-y)} \partial_y J^{(K)}(y) , \quad K=X, B , \quad (A.36)$$

irrespective of the Ramond or Neveu-Schwarz sector.

From these results, all conformal and superconformal algebras considered in the main text are easily derived. Indeed, corresponding to (A.4) we have the Virasoro algebra (A.3), and similarly from (A.30) and (A.36), one obtains:

$$(J_{q_1}^{(K)}, J_{q_2}^{(K)}) = 2 L_{q_1+q_2}^{(K)} + \left[\frac{1}{2} \hat{c} (q_1^2 - \frac{1}{4}) + 4\hat{a} \right] \delta_{q_1+q_2,0} , \quad K=X, B , \quad (A.37)$$

$$[L_n^{(K)}, J_q^{(K)}] = (\frac{1}{2} n - q) J_{q+n}^{(K)} , \quad K=X, B , \quad (A.38)$$

Using the same techniques, nilpotency of the BRST charge may easily be checked. For the bosonic string, defining:

$$J_B^{(\text{bosonic})}(z) = : c(z) [T^{(\alpha)}(z) + \frac{1}{2} T^{(c)}(z) - az^{-2}] : , \quad (A.39)$$

we simply have:

$$Q_B^{(\text{bosonic})} = \oint \frac{dz}{2i\pi} J_B^{(\text{bosonic})}(z) , \quad (A.40)$$

In this form, it is quite straightforward to obtain the result (3.89).

Similarly, for the spinning string, we have:

$$Q_B^{(\text{spinning})} = \oint \frac{dz}{2i\pi} j_B^{(\text{spinning})}(z) \quad , \quad (A.41)$$

with

$$j_B^{(\text{spinning})}(z) = c(z)[T^{(X)}(z) + \frac{1}{2}T^{(B)}(z) - a_{\frac{1}{2}}z^{-2}] + \frac{1}{\sqrt{2}}\gamma(z)[j^{(X)}(z) + \frac{1}{2}j^{(B)}(z)] \quad , \quad (A.42)$$

which again permits the use of conformal field theory techniques in checking nilpotency of the BRST charge.

Moreover, from these expressions for the BRST charge, it becomes very easy to derive the results in (3.84) and (4.61).

The OPE (A.36) is actually very general. A local operator $\phi(z)$ is said to be a primary field of conformal weight h [42] if its OPE with the stress-energy tensor $T(z)$ is:

$$T(x)\phi(y) = \frac{h}{(x-y)^2}\phi(y) + \frac{1}{(x-y)}\partial_y\phi(y) \quad , \quad (A.43)$$

which is also equivalent to:

$$[L_n, \phi(z)] = z^{n+1}\partial_z\phi(z) + h(n+1)z^n\phi(z) \quad , \quad (A.44)$$

Let us assume that $\phi(z)$ obeys a boundary condition of the type

$$\phi(e^{2i\pi}z) = e^{-2i\pi p/q}\phi(z) \quad , \quad (A.45)$$

with p and q being co-prime integers, and that h is a rational number. We then have the mode expansion

$$\phi(z) = \sum_s \phi_s z^{-s-h} \quad , \quad (A.46a)$$

where the summation is done over all rational numbers of the form $s = n + (p/q - h_r)$, i.e. $s \in \mathbb{Z} + (p/q - h_r)$ (h_r is h minus its integer part, i.e. the rational part of the conformal weight h). The modes ϕ_s are given by:

$$\phi_s = \oint \frac{dz}{2i\pi} z^{s+h-1}\phi(z) \quad , \quad (A.46b)$$

The OPE (A.43) is then also equivalent to the commutation relations:

$$[L_n, \phi_s] = [(h-1)n - s]\phi_{s+n} \quad , \quad (A.47)$$

Therefore, (A.36) and (A.38) show that $j^{(K)}(z)$ ($K=X,B$) is a primary field of weight $h=3/2$. Actually, most of the fields introduced here are primary fields. One can indeed verify that $(i\partial_z x^\mu(z))$, $\psi^\mu(z)$, $c(z)$, $b(z)$, $\gamma(z)$ and $\beta(z)$ are all primary fields, having respectively the conformal weight $h=1$, $h=1/2$, $h=-1$, $h=2$, $h=-1/2$ and $h=3/2$. Moreover, for the fields $(i\partial_z x^\mu(z))$, $c(z)$ and $b(z)$ we have $p/q=0$, and for the fields $\psi^\mu(z)$, $\gamma(z)$ and $\beta(z)$ we have $p/q=1/2$ in the Ramond sector and $p/q=0$ in the Neveu-Schwarz sector.

As said above, $j^{(K)}(z)$ is a primary field of weight $h=3/2$, with $p/q=1/2$ (resp. $p/q=0$) in the Ramond (resp. Neveu-Schwarz) sector. Also, note that $T(z)$ in (A.4) is a primary field with $h=2$ and $p/q=0$ when the central extensions c and b vanish identically.

One may generalise the (b,c) and (β, γ) ghost systems to a system of commuting or anticommuting b and c fields of arbitrary rational conformal weight $h=j$ and $h=1-j$ respectively. Let us define the fields:

$$c(z) = \sum_{s_+} c_{s_+} z^{-s_+-(1-j)} \quad , \quad (A.48)$$

$$b(z) = \sum_s b_s z^{-s-j} \quad , \quad (A.49)$$

where the summation is performed over all rational numbers

$$s_+ = n + \alpha \quad , \quad s_- = n - \alpha \quad , \quad (A.50)$$

with α being a rational number such that $-1 < \alpha < 1$. This is the most general situation such that the results below make sense. Clearly, the cases $\alpha=0$ and $\alpha=\pm 1/2$ are particular, and correspond to Ramond or Neveu-Schwarz boundary conditions respectively. The fields

$c(z)$ and $b(z)$ satisfy boundary conditions of the type (A.45), with p/q being given by the rational part of $\pm(\alpha - j_r)$.

The modes c_{s_+} and b_{s_-} satisfy either the commutation relations

$$[c_{s_+}, b_{s_-}] = i \delta_{s_+, s_-, 0} \quad (A.51)$$

or the anticommutation relations

$$\{c_{s_+}, b_{s_-}\} = \delta_{s_+, s_-, 0} \quad (A.52)$$

One then introduces the local operator

$$T^{(j)}(z) = \eta : [j \partial_z c(z) b(z) + (j-1) c(z) \partial_z b(z)] : \quad (A.53)$$

$$= \sum_n L_n^{(j)} z^{-n-2} \quad (A.54)$$

with:

$$L_n^{(j)} = \eta \sum_{s_-} : [s_- - (j-1)n] c_{s_-} b_{n+s_-} : \quad (A.55)$$

where $\eta=i$ (resp. $\eta=1$) for commuting (resp. anticommuting) fields $c(z)$ and $b(z)$.

One can then show that $c(z)$ and $b(z)$ satisfy the OPE (A.43) with $h=j$ and $h=(1-j)$ respectively, and that the OPE (A.4) is obtained with the following central extensions:

commuting fields:

$$c = 2[1 + 6j(j-1)], \quad b = (j-\alpha)(j+\alpha-1) \quad (A.56)$$

anticommuting fields:

$$c = -2[1 + 6j(j-1)], \quad b = -(j-\alpha)(j+\alpha-1) \quad (A.57)$$

Of course, these general results lead to the same values as obtained before for the (b,c) and (β, γ) ghost systems, with $j=2$ and $j=3/2$ respectively, and actually also for the matter fermions λ^μ with $j=1/2$, since the corresponding (b,c) system then represents two such fermions.

Finally, let us remark here that it is not an accident that the fields introduced above allow for straightforward calculations of the operator algebras of string theories. Indeed, as was shown at length in the main text, in the (super)conformal gauge, these theories are

simply conformal field theories. Then, by appropriate changes of variables from (σ, τ) to (σ_+, σ_-) and to z and $\bar{z}$, it is possible to rewrite entirely the formulation of these theories in terms of the primary fields above only. This is left as a useful exercise.

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