CERN-TH-5304/89
WU B 89-6

**Current Algebra Parameters on the Lattice with a
Renormalisation Group Improved Wilson Action***

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Abstract

Using our renormalisation group improved Wilson fermion action in the quenched approximation and working on an $8^3 \times 16$ lattice with a lattice spacing of $a \approx 0.15 fm$, we determine: (a) the baryon decuplet and vector meson octet mass splittings, (b) the lattice renormalisations of the non-singlet axial and vector currents and (c) f_π and f_ρ down to rather small values of the quark mass. As a result, we gain an insight into the chiral behaviour and proximity to continuum of our improved action at this lattice spacing.

CERN-TH-5304/89
February 1989

* Work supported in part by Deutsche Forschungsgemeinschaft grant Schi 257/1-2.

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1. Introduction

Lattice formulations of quantum field theories are known to have less explicit symmetries than the regulated continuum theories that they are supposed to define as a limiting procedure. Any numerical simulation of a lattice theory is bound to work at finite lattice spacing and volume, and consequently a natural question that arises at the end is: How far away are the numerical results from the continuum physics, or how well has one succeeded in removing the lattice artifacts?

The majority of numerical lattice computations on the dynamics of hadrons have so far been concerned with the spectrum. In order to attain low energy hadron physics within this approach, one has to carry out two limits, each extremely non-trivial to perform numerically: (a) the limit of weak coupling/large lattice and (b) the chiral limit of vanishing quark mass. Not too much progress has been achieved over the years to meet these requirements in a satisfactory way. This holds in particular for the attempts to study full QCD by computer simulations [1]. That is the reason that the quenched approximation to QCD continues to serve as a laboratory to preview what might be going on in the complete theory.

The most obvious monitor on the distance to continuum physics of a lattice computation is the scaling behaviour of the resulting “physical” quantities. When it comes to restoration of the internal symmetries broken in a particular lattice formulation, it is however better to turn to the observables more sensitive to the symmetry in question.

The staggered fermion formulation [2] contains explicit flavour symmetry breaking. Suitable quantities to study the restoration of this symmetry are the masses of particles in a flavour multiplet and the topological charge. Numerical studies have shown that in quenched $SU(3)$ gauge theory with staggered fermions, it is necessary to have $\beta = 6/g^2 \geq 6.2$ in order to bring flavour symmetry violations under control [3,4].

In the Wilson prescription [5] one faces an explicit violation of the chiral symmetry, which is an essential phenomenological concept in our understanding of hadro- and flavour-dynamics. The testground to study the chiral structure of the Wilson lattice approach is PCAC and current algebra. The chiral aspects of Wilson fermions were investigated in perturbation theory by Karsten and Smit [6] a long time ago. Comforting enough, these authors found that the Wilson fermion scheme can be reconciled with all important aspects of current algebra: the dangerous contribution of the symmetry violating Wilson term to the divergence of the non-singlet axial

current can be absorbed into a redefinition of the current quark mass on one hand and a finite rescaling of the axial current on the other. The hard divergence in the singlet axial current can be traced back to the presence of the Wilson term and produces the well-known Adler-Bell-Jackiw triangle anomaly [7].

More recently, Bochicchio et al. [8] have extended these ideas and turned them into a practical lattice renormalisation prescription which respects the validity of current algebra on the lattice at every stage. They generalized, from perturbation theory to the non-perturbative situation, the decoupling of the undesirable symmetry violating terms. In particular they showed that the matrix elements of the renormalised chiral variation of the Wilson term vanish between on-shell hadron states. The renormalisation procedure is carried out by imposing the continuum chiral Ward identities to hold on the lattice. These Ward identities then offer the possibility to determine the renormalisation parameters of the currents in a non-perturbative manner.

These ideas have subsequently been tested in ref.[9] in quenched simulations at somewhat large quark masses. The determination of the renormalisation Z -factors of the non-singlet axial and vector currents and their direct couplings to pseudoscalar and vector mesons led to quite reasonable results.

Related is the question of the renormalisation of the quark mass and scalar density matrix elements between hadronic states. In terms of physics they offer to look both into the dynamics of mass splittings in the hadron multiplets and the hadronic σ -terms. In the case of the baryon octet, these quantities have been studied by Maiani et al. [10] with the standard Wilson action, and more recently by us [11] with a renormalisation group improved Wilson action [12]. Our improved action provided us with better signals and thus enabled us to proceed closer to the chiral limit than before [13,14,11]. As a result, we found indications that the traditional first order flavour $SU(3)$ breaking analysis might be inadequate. The conventional D parameter for the baryon octet mass splitting was found to be by no means mass independent in the range $m_q \leq m_s$, as assumed in first order flavour $SU(3)$ breaking.

In the present paper we intend to address two points:

- First we want to supplement our previous work on mass splittings within $SU(3)$ multiplets [11] by presenting some additional results for the baryon decuplet and the vector meson octet.
- Second we plan to determine the non-perturbative renormalisation of the axial and vector currents along the lines of ref.[8]. With our improved action we again expect to see more details of the limiting chiral behaviour than previously

possible.

Since we are dealing with non-perturbative, low energy phenomena, the actual improvement of any supposedly “improved” action cannot really be established by perturbation theory. Perturbative arguments have been of little help in the past in the construction of improved actions; they can at best be suggestive. The simple question (which is so complicated) to be answered is whether with a change of action one ends up closer to the scaling regime than without. The true test of an “improved” action such as ours can only be achieved *a posteriori* by a full fledged scaling study of physical quantities. This is prohibitively expensive in the present context. However, a partial answer to the question can be obtained by studying the departure of non-perturbative quantities like the Z -factors of the currents from their known continuum limits. The results for both the standard and “improved” Wilson formulations would then provide a handle for comparing the two actions.

The plan of the paper is as follows: In section 2 we define the quark bilinear operators and their lattice renormalisations underlying the present work. Section 3 contains new results concerning mass splittings in the baryon decuplet and vector meson octet states. The numerical results of the non-perturbative lattice renormalisation of the non-singlet axial and vector currents are presented in section 4. We use the methods developed originally in the context of the standard Wilson action. Justification for applying them to the improved action is relegated to an Appendix. The results for pseudoscalar and vector meson decay constants are discussed in section 5.

2. Composite Quark Operators and Their Renormalisation Constants

We list here the quark bilinears used in the subsequent calculations and discuss their renormalisations. For simplicity we use only strictly local operators throughout this work. We note in passing that relaxing the condition of strict locality would not allow us to construct an exactly conserved vector current as was possible in the case of the standard Wilson action. The local vector and axial currents to be considered are:

$$V_\mu^L(x) = \bar{u}(x)\gamma_\mu d(x) , \quad (1)$$

and

$$A_\mu^L(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x) , \quad A_\mu^{(3)L}(x) = \bar{q}(x)\frac{\tau^3}{2}\gamma_\mu\gamma_5 q(x) . \quad (2)$$

We denote lattice quantities with a superscript L , while the corresponding continuum variables carry no such index. We further define the pseudoscalar and scalar densities, with the Pauli matrices τ^c representing isospin:

$$P^{(c)L}(x) = \bar{q}(x)\frac{\tau^c}{2}\gamma_5 q(x) , \quad S^{(c)L}(x) = \bar{q}(x)\frac{\tau^c}{2} q(x) , \quad c = 1, 2, 3. \quad (3)$$

All these composite operators reduce to the continuum operators in the limit $a \rightarrow 0$. For non-zero a in the scaling region, however, they suffer finite multiplicative renormalisations relative to the continuum operators:

$$\begin{aligned} V_\mu(x) &= Z_V^L V_\mu^L(x), & A_\mu(x) &= Z_A^L A_\mu^L(x), \\ S^{(c)}(x) &= Z_S^L S^{(c)L}(x), & P^{(c)}(x) &= Z_P^L P^{(c)L}(x). \end{aligned} \quad (4)$$

These equalities are operator equalities. The Z-factors can be determined from a study of hadronic matrix elements of these operators, and they should be independent of the hadronic states considered once one is in the scaling region. As a matter of fact one can turn around the argument and use the dependence of the Z-factors on the hadronic states as a check of scaling.

The continuum currents $V_\mu(x)$ and $A_\mu(x)$ are not renormalised due to their (partial) conservation. Therefore the deviation from unity of Z_V^L and Z_A^L is a measure of how far off we are from continuum physics. With the standard Wilson action at $\beta = 6.0$, corresponding to $a \approx 0.10 fm$, the non-perturbative analysis for the lattice renormalisation of the local currents produced [9]:

$$Z_V^L = 0.57(2), \quad Z_A^L = 0.70(5). \quad (5)$$

In this case Z_V^L was calculated using the exactly conserved vector current and 2-point correlators. When Z_V^L was extracted from 3-point correlators measuring form factors of pion and proton, the result was [15] $Z_V^L = 0.73(3)$. The large variation indicates the impact of $O(a)$ lattice corrections and failure of the scaling assumption.

Both the pseudoscalar and scalar densities possess anomalous dimensions and get renormalised in the continuum theory. The operator $m\bar{q}q$ and the divergence of the axial current ($2mP$) are however renormalisation group invariants. A mass independent renormalisation scheme therefore yields $Z_P = Z_S = Z_m^{-1}$ in the continuum theory. The lattice regularisation, on the other hand, is not mass independent. Before making meaningful predictions, one has to make sure that the Z-factors become essentially independent of the quark mass for sufficiently small masses. Moreover, for any non-trivially renormalised operator in continuum perturbation theory, a matching of continuum and lattice results requires a convention for the matching scale. In absence of an explicitly defined matching scale, only combinations without anomalous dimensions such as Z_P/Z_S and $Z_m Z_P$ can be determined. With the standard Wilson action at $\beta = 6.0$ [10], $Z_P/Z_S = 0.68(1)$.

Our choice of the lattice quark mass in terms of the hopping parameter κ is (details of our action can be found in the Appendix)

$$m^L = \ln\left[1 + \frac{3}{2}\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right)\right], \quad m = Z_m^L m^L, \quad (6)$$

with the critical hopping parameter κ_c determined from spectrum calculations [13]. Since for our action an insertion of $\bar{q}q$ is equivalent to differentiation with respect to $\frac{3}{2\kappa}$, we have for sufficiently light quark masses the same relation as in the continuum

$$Z_m^L Z_S^L = 1. \quad (7)$$

3. Scalar Density Matrix Elements between Δ and ρ States

In a previous paper[11], we have presented results on the matrix elements of scalar quark densities sandwiched between proton states. Such matrix elements contain interesting physics, since they are related to the baryon octet mass splitting and to the pion-nucleon σ term. These computations can readily be extended to the scalar density matrix elements in other multiplets. We present here results on such matrix elements in the baryon decuplet and the vector meson octet. The analysis starts from the ratio of correlators:

$$R_q^{\mathcal{P}}(t) = \frac{\sum_{\vec{x}} \langle \mathcal{P}(\vec{x}, t) | \sum_z \bar{q}q(z) | \mathcal{P}(\vec{0}, 0) \rangle}{\sum_{\vec{x}} \langle \mathcal{P}(\vec{x}, t) | \mathcal{P}(\vec{0}, 0) \rangle}, \quad (8)$$

where $\mathcal{P}$ denotes an appropriate interpolating field for the particle in question, and q stands for a u or d quark operator. For large time separations t , these ratios are expected to linearly increase in t , the slope being related to the matrix element of interest:

$$\begin{aligned} R_q^{\Delta}(t) &\xrightarrow{t \text{ large}} c_{\Delta} + f \langle \Delta | \bar{q}q | \Delta \rangle t \\ R_q^{\rho}(t) &\xrightarrow{t \text{ large}} c_{\rho} + f \langle \rho | \bar{q}q | \rho \rangle t \cdot \tanh[M_{\pi}(t - L_t/2)] \end{aligned} \quad (9)$$

Here L_t stands for the lattice length in time direction, and the factor $f = \frac{\partial m^L}{\partial (3/2\kappa)}$ converts the lattice matrix elements to those of the operator $\frac{\partial}{\partial m^L}$.

The computations of the 2- and 3-point correlators have been carried out on the 47 configurations of an $8^3 \times 16$ lattice, as they were used by us before [13,11]: the effective gauge fields were derived using two $\sqrt{3}$ blocking steps from $24^3 \times 48$ quenched configurations produced with the simple plaquette action at $\beta = 6.3$. The quark propagators were calculated for five values of the hopping parameter κ : 0.1320, 0.1345, 0.13525, 0.1360, 0.1362. The spectrum analysis determined $\kappa_c = 0.13650(2)$, and fixed the lattice scale at $a^{-1} \approx 1.3 \text{ GeV}$ by matching the lattice ρ mass to its experimental value.

In figs. 1 and 2, we display our signals for the u -quark density in the Δ^{++} and the ρ^+ states respectively. The resulting matrix elements

$$G_{\Delta} = \langle \Delta^{++} | \bar{u}u | \Delta^{++} \rangle, \quad G_{\rho} = \langle \rho^+ | \bar{u}u | \rho^+ \rangle. \quad (10)$$

are shown in fig. 3a. All the statistical errors have been calculated with the jack-knife method.

With exact isospin symmetry and conventional parametrisation of $SU(3)$ flavour symmetry breaking [16]

$$H = H_{sym} - \frac{\Delta m}{\sqrt{3}} \cdot \bar{q} \lambda_8 q, \quad \Delta m = m_s^L - m_u^L, \quad (11)$$

we obtain

$$M_\Omega - M_\Delta = \Delta m G_\Delta, \quad M_{K^*} - M_\rho = \Delta m G_\rho. \quad (12)$$

The experimentally observed mass splittings are

$$M_\Omega - M_{\Xi^*} \approx 139 \text{ MeV}, \quad M_{\Xi^*} - M_{\Sigma^*} \approx 149 \text{ MeV}, \quad M_{\Sigma^*} - M_\Delta \approx 152 \text{ MeV}, \quad (13)$$

predicted to be equal in first order $SU(3)$ breaking analysis, and

$$M_{K^*} - M_\rho \approx 124 \text{ MeV}, \quad M_\phi - M_{K^*} \approx 125 \text{ MeV}, \quad (14)$$

again equal according to first order $SU(3)$ breaking.

Our previous spectrum calculation [13] determined

$$\begin{aligned} \pi, \rho, K \text{ masses} : \Delta m &\approx 0.069, \\ \pi, N, K \text{ masses} : \Delta m' &\approx 0.096. \end{aligned} \quad (15)$$

In fig. 3 we have converted our results for scalar density matrix elements to dimensionless ratios using $M_\rho = 0.576(10)$ [13]. The square of the ratio of the pseudoscalar meson mass (M_{PS}) to the vector meson mass (M_V) is used as the horizontal coordinate, and the experimentally observed mass splittings are included at the $SU(3)$ symmetric point:

$$M_{PS}^2/M_V^2 = 3(2M_K^2 + M_\pi^2)/(2M_{K^*}^2 + M_\rho^2) \approx 0.234. \quad (16)$$

Simple quark counting arguments (which should work for sufficiently heavy quarks) suggest a 3 : 1 ratio between G_Δ and G_ρ . Our data are roughly consistent with this expectation in the whole mass range. We could not obtain a reasonable signal at the smallest quark mass point, but an extrapolation of the higher mass lattice results gives a good agreement with the experimental numbers. The vector meson mass splitting is more or less correctly estimated, while the decuplet mass splitting is underestimated by about 20%. The latter situation is similar to the one in the case of the F -coupling in the baryon octet, and we have reproduced our older results [11] in fig. 3b. Were we to use $\Delta m'$ and M_N to set the scale for the baryons, our results would be in better agreement with the experimental numbers. The reason is

that in the present quenched calculations, the nucleon to rho mass ratio is affected by a “systematic error”; it comes out too large. While we cannot get rid of this uncertainty, our results point out that it is better to set the scale using M_ρ for mesonic observables, and using M_N for baryonic observables.

4. Non-perturbative Determination of the Renormalisation Constants

As mentioned already in the introduction, the renormalisation constants Z can be determined non-perturbatively by imposing current algebra identities on the lattice in the scaling region [8]. We begin with the PCAC relation for the isospin off-diagonal axial current:

$$\partial_\mu A_\mu(x) = 2m_u P(x), \quad P(x) = P^{(1)}(x) + iP^{(2)}(x). \quad (17)$$

It can be used to exploit the pion propagator

$$\langle \sum_{\vec{x}} (P^L(\vec{x}, t))^\dagger P^L(0) \rangle \xrightarrow{t, L_t - t \text{ large}} C_P^2 [\exp(-M_\pi t) + \exp(-M_\pi(L_t - t))], \quad (18)$$

and the 2-point function¹

$$\langle \sum_{\vec{x}} (A_0^L(\vec{x}, t))^\dagger P^L(0) \rangle \xrightarrow{t, L_t - t \text{ large}} \frac{C_P C_A}{2 \sinh(M_\pi/2)} [\exp(-M_\pi t) - \exp(-M_\pi(L_t - t))] \quad (19)$$

to build up the combination [9]

$$\rho = C_A/C_P = 2m^L Z_m^L Z_P^L / Z_A^L. \quad (20)$$

It is gratifying to convince ourselves by looking at fig. 4, that even at small quark masses we can base our investigation on very good signals for both $\langle PP \rangle$ and $\langle AP \rangle$ correlators. We first determined M_π and C_P from $\langle PP \rangle$, eq.(18), and then made a constrained fit for C_A to the $\langle AP \rangle$ correlator, eq.(19). We checked that a direct *tanh*-fit to the ratio of correlators left the results for ρ unchanged. The fits are also depicted in fig. 4, and we notice that the $\langle AP \rangle$ correlator exhibits asymptotic pion dominance earlier than $\langle PP \rangle$. This is a consequence of the Goldstone nature of the pion; the divergence of the axial current is more strongly coupled to the pion than to other pseudoscalar states.

According to eq.(20), in the chiral limit ρ is expected to vanish linearly in m^L , just like M_{PS}^2 . The data points in fig. 5 have very small statistical errors and follow a straight line with high precision:

$$\rho = 3b \left(\frac{1}{\kappa} - \frac{1}{\kappa_c} \right). \quad (21)$$

¹We use hyperbolic functions of hadron masses in our formulae to obtain the correct limiting behaviour for heavy quarks. Their inclusion in no way implies removal of all lattice artifacts.

The intercept of this linear fit provides an alternate estimate of κ_c :

$$\kappa_c = 0.13656(3) , \quad b = Z_m^L Z_P^L / Z_A^L = 0.76(3) . \quad (22)$$

The agreement of this value of κ_c with that where M_π vanishes is an important confirmation of the validity of current algebra manipulations.

Next we turn to the determination of Z_A^L and Z_V^L from the axial Ward identity [8]:

$$\langle \sum_x [2mP^{(3)}(x) - \partial_\mu A_\mu^{(3)}(x)] A_\nu(y) V_\rho^\dagger(0) \rangle = -\langle V_\nu(y) V_\rho^\dagger(0) \rangle + \langle A_\nu(y) A_\rho^\dagger(0) \rangle . \quad (23)$$

Using eq.(20) and performing a summation over $\vec{y}$ it can be rewritten as

$$\langle \sum_x P^{(3)L}(x) \sum_{\vec{y}} A_\mu^L(y) V_\mu^{L\dagger}(0) \rangle = -C_1 \langle \sum_{\vec{y}} V_\mu^L(y) V_\mu^{L\dagger}(0) \rangle + C_2 \langle \sum_{\vec{y}} A_\mu^L(y) A_\mu^{L\dagger}(0) \rangle \quad (24)$$

with

$$C_1 = \rho^{-1} Z_V^L (Z_A^L)^{-2} , \quad C_2 = \rho^{-1} (Z_V^L)^{-1} . \quad (25)$$

Computationally, the most difficult part of eq.(24) is the 3-point correlator $\langle PAV \rangle$. It is evaluated following the method of ref.[17] for pseudoscalar insertion at zero 4-momentum. Once the expectation values $\langle VV \rangle$, $\langle AA \rangle$ and $\langle PAV \rangle$ are determined, eq.(24) constitutes a system of two linear equations (for μ space-like and time-like), with the unknowns C_1 and C_2 . The two equations are rather complementary in nature. For $\mu = 0$ and large y_0 the $\langle AA \rangle$ correlator, dominated by the pion as the intermediate state, controls the r.h.s. of eq.(24). On the other hand, in the spatial components of this equation the r.h.s. is controlled by the $\langle VV \rangle$ term, this time due to the dominance of the rho as the intermediate state. This independence of the two equations helps stabilise the evaluation of C_1 and C_2 .

We first improved the signal by symmetrising the 3-point correlator under $y \leftrightarrow 0$, and then determined the coefficients C_1 and C_2 as a function of $t = y_0$. A typical sample of our results is given in fig. 6, for $\kappa = 0.13525$. Both C_1 and C_2 show a strong t -dependence below $t = 4$, and then level off for large t as anticipated. Upon careful observation, both C_1 and C_2 appear to exhibit a shoulder around $t = 4$ followed by a gentle decrease for larger t . The effect is in fact more significant for C_1 because of smaller statistical errors.

We suspected that this might be due to the finite time extent of the lattice. Indeed, finite size effects from insufficient time extent of the lattice are expected to be present in the correlators; they show up as t -independent terms carrying exponential suppression factors of the form $\exp(-const. \times L_t)$. Inspection of our correlators did reveal such terms in channels where they cannot be hidden underneath a light intermediate state signal: $\langle V_0 V_0 \rangle$, $\langle A_k A_k \rangle$ and $\langle SS \rangle$. In order to study the possible

implications for C_1 and C_2 , we periodically doubled some of our gauge configurations in the time direction, and repeated the calculation of C_1 and C_2 on these extended lattices. A comparison of our original and doubled lattice results, for an identical set of configurations and an intermediate value of κ , is made in fig. 7. Unfortunately, the problem is *not* cured and an oscillatory behaviour ($\approx 7\%$ in amplitude for C_1) survives². We subsequently extracted C_1 and C_2 from the data points at $t = 6, 7$, and included in the errors the variation between $t = 5$ and $t = 8$ as a systematic uncertainty.

Our final results for the renormalisation constants Z_A^L and Z_V^L are shown in fig. 8. $Z_A^L \approx 0.8$ is more or less independent of the quark mass, while Z_V^L exhibits a stronger mass dependence³ and can be extrapolated to ≈ 0.5 in the chiral limit. Combining eq.(22) with Z_A^L , we find $Z_m Z_P \approx 0.61$. It is reassuring to see that our improved action results at a lattice spacing $a \simeq 0.15 fm$ are not too different from the standard Wilson action values at $a \simeq 0.10 fm$. Z_A^L is a somewhat closer to the continuum limit of 1.0, while Z_V^L and $Z_m Z_P$ are a bit farther away.

We have also looked at the renormalisation of a 1-link vector current that obeys current conservation up to terms of $O(a^2)$:

$$\hat{V}_\mu^L(x) = \frac{1}{2} [\bar{u}(x)(\gamma_\mu + 1)U_\mu(x)d(x + \hat{\mu}) + \bar{u}(x + \hat{\mu})(\gamma_\mu - 1)U_\mu^\dagger(x)d(x)] , \quad (26)$$

$$\Delta_\mu \hat{V}_\mu^L(x) = O(a^2) , \quad \Delta_\mu f(x) = a^{-1}(f(x) - f(x - \hat{\mu})) . \quad (27)$$

In a Taylor series expansion, the leading difference between $\hat{V}_\mu^L$ and V_μ^L is the covariant derivative operator $a\bar{u}D_\mu d$. This piece, however, does not contribute in correlation functions containing the vector current at zero 3-momentum, and the difference becomes $O(a^2)$. From the ratio of the 2-point correlators $\langle \hat{V}V \rangle$ and $\langle VV \rangle$, we extracted Z_V^L/Z_V^L , and for all the values of masses found it to be 1.0 within 5%. The departure of Z_V from 1.0 is then more likely to be due to $O(g^2)$ renormalisation effects. For not entirely obvious reasons, the lattice renormalises the vector current more strongly than the axial current. In the case of the standard Wilson action, this observation has already been noted in ref.[15].

From the residue of the pion pole, we can determine the chiral condensate per quark flavour:

$$\chi(m^L) = -\frac{C_A^2(Z_A^L)^2}{4m^L \tanh(M_\pi/2)} \xrightarrow{m^L=0} \chi(0) = \langle 0|\bar{q}q|0 \rangle . \quad (28)$$

²An oscillatory behaviour of a similar nature has been noted in ref.[10] in the determination of Z_P/Z_S from the $\langle PPS \rangle$ Ward identity.

³It is hard to see any quark mass dependence in the results of ref.[9] since their signal starts deteriorating at $M_{PS}^2/M_V^2 \simeq 0.53$. We have gone down to $M_{PS}^2/M_V^2 \simeq 0.23$.

A linear fit to $\chi(m^L)$ is shown in fig. 5, and it produces $\chi(0) = -0.016(2)$. In terms of a renormalisation group invariant result, we have, $\langle 0|m_u^L \bar{q}q|0\rangle^{1/4} = -0.147(5)$. It differs by about 20% from the experimental number extracted using the Gell-Mann-Oakes-Renner formula: $-2^{-1/4}\sqrt{f_\pi M_\pi}/M_\rho \approx -0.125$.

5. pseudoscalar and Vector Meson Decay Constants

We adopt the conventions of ref.[18] for the normalisations of the decay constants. The charge pion decay constant, f_π , is defined through the PCAC relation

$$\langle 0|A_\mu(0)|\pi^-(p)\rangle = \sqrt{2}p_\mu f_\pi . \quad (29)$$

In our lattice laboratory, the way to get at the pseudoscalar decay constant is to combine the results for C_A , Z_A^L and M_{PS} :

$$f_{PS} = C_A Z_A^L \sqrt{\sinh(M_{PS})} (2 \sinh(M_{PS}/2))^{-2} . \quad (30)$$

The dimensionless quantity f_{PS}/M_V is plotted in fig. 9. In this representation, our lattice results appear to have a rather flat behaviour in the vicinity of the chiral point. That would be in accordance with the experimental numbers

$$f_\pi \approx 0.121 M_\rho , \quad f_K \approx 0.127 M_{K^*} , \quad (31)$$

except that our values are too high. This is of course related, through the Gell-Mann-Oakes-Renner formula, to the fact that our result for $\langle 0|\bar{q}q|0\rangle$ is too large.

We now proceed to the vector couplings in an ideally mixed $SU(3)$ nonet, e.g., we assume the quark content of the ϕ to be pure $\bar{s}s$. The charged vector meson decay constant is [18]

$$\langle 0|V_k(0)|V(\vec{p}=0)\rangle = \epsilon_k M_V^2 f_V^{-1}(m^L) . \quad (32)$$

The couplings of the physical vector mesons to the electromagnetic current are expressed in terms of f_V as

$$\begin{aligned} f_\rho^{-1} &= \frac{1}{\sqrt{2}} f_V^{-1}(m_u^L) , & f_\omega^{-1} &= \frac{1}{3\sqrt{2}} f_V^{-1}(m_u^L) , \\ f_\phi^{-1} &= -\frac{1}{3} f_V^{-1}(m_s^L) , & f_\psi^{-1} &= \frac{2}{3} f_V^{-1}(m_c^L) . \end{aligned} \quad (33)$$

Experimentally, the e^+e^- decay widths yield:

$$\begin{aligned} f_\rho^{-1} &= 0.200(9) , & f_\omega^{-1} &= 0.059(2) , \\ f_\phi^{-1} &= -0.078(3) , & f_\psi^{-1} &= 0.083(6) . \end{aligned} \quad (34)$$

On the lattice, we extract f_V^{-1} from the correlation function

$$\frac{1}{3} \sum_{k=1}^3 \langle \sum_{\vec{x}} (V_k^L(\vec{x}, t))^{\dagger} V_k^L(0) \rangle \xrightarrow{t, L_t-t \text{ large}} C_V^2 [\exp(-M_V t) + \exp(-M_V(L_t - t))] , \quad (35)$$

using the relation

$$f_V^{-1} = C_V Z_V^L \sqrt{2 \sinh(M_V)} (2 \sinh(M_V/2))^{-2} . \quad (36)$$

As can be seen from fig. 10, the data though showing reasonable mass dependence are substantially higher than the experimental values. This feature is not at all unexpected. In the vector dominance model, the electromagnetic pion form factor, F_{π} , tells us that [19]

$$F_{\pi}(s=0) = 1 = g_{\rho\pi\pi} \cdot f_{\rho}^{-1} , \quad (37)$$

neglecting the ρ -width. In the quenched approximation, we inhibit the decay $\rho \rightarrow \pi\pi$, suppressing the coupling $g_{\rho\pi\pi}$ and therefore enhancing f_{ρ}^{-1} . We expect significant improvement in the results of fig. 10, once dynamical quarks are incorporated in the Monte Carlo simulations.

6. Summary

We have been able to push the lattice analysis of chiral parameters farther into the small quark mass regime. New results presented for the mass splittings in the baryon decuplet and the vector meson octet are in general agreement with the experimental values.

The PCAC relation between the pion and the axial vector current is verified to a very good precision, down to $m_q \approx m_s/3$. Our lattice renormalisation constants at $a \approx 0.15 fm$ are comparable to the values found with the standard Wilson action at $a \approx 0.10 fm$. Though these Z -factors are renormalisation scheme dependent, they give an indication for the distance of our action from the continuum limit.

In the scaling region, the Z -factors must be the same no matter which correlation functions they are extracted from. Cross-checks, though difficult to carry out in practice, can give a feel for the scaling violations. We have not addressed this issue here, and are therefore unable to judge the amount of systematic errors in our results for Z -factors. Taking the numbers at face value, the pseudoscalar decay constant comes out some 30% too high when M_{ρ} is used to set the scale. The quark mass dependence of the vector meson decay constant f_V^{-1} is reproduced remarkably well by our data, but the overall magnitude is largely overestimated. As discussed above, our underestimate of f_V is in part due to the absence of dynamical quarks in our calculation.

As a general conclusion, the quenched approximation works as well as one could have hoped at present lattice spacing. All dimensional quantities remain affected by a systematic error of about 20%, which is due to the well-known mismatch of the mesonic and baryonic mass scales found in all quenched lattice results so far. For most quantities, the full QCD computations will have to answer questions on the $< 5\%$ level of accuracy to improve our understanding of the QCD dynamics. However, one may direct attention to observables sensitive to quenching, such as the vector meson decay constant.

Acknowledgements

We thank Phillipe de Forcrand for the configurations on which this work is based. The computations were performed on the CRAY X-MP/48 of the HLRZ in Jülich. We are grateful Prof. Hossfeld and the staff of the computer center for their kind support. We enjoyed numerous discussions with Guido Martinelli.

Appendix : The Renormalisation Group Improved Wilson Action

Our renormalisation group improved Wilson fermion action is described by the matrix:

$$\Delta(x', x) = \frac{3}{2} \left(\frac{1}{\kappa} - 8 \right) \delta_{x', x} + \sum_{i=1}^8 [-A_i c_i(x', x) + B_i \gamma_i s_i(x', x)] , \quad (\text{A.1})$$

with the fixed-point coefficients [12]

$$\begin{aligned} A_i &= \frac{1}{2\sqrt{2}} , \quad B_i = \frac{3}{8} , \quad \text{for } i \in \{1, \dots, 4\} ; \\ A_i &= \frac{1}{6(2 + \sqrt{2})} , \quad B_i = \frac{1}{24} , \quad \text{for } i \in \{5, \dots, 8\} . \end{aligned} \quad (\text{A.2})$$

Here the index x denotes the lattice sites and c, s are shorthand notations for discrete lattice derivatives:

$$\begin{aligned} c_i(x', x) &= U(x', x) (\delta_{x', x+e_i} + \delta_{x', x-e_i} - 2\delta_{x', x}) , \\ s_i(x', x) &= U(x', x) (\delta_{x', x+e_i} - \delta_{x', x-e_i}) . \end{aligned} \quad (\text{A.3})$$

Our choice of the basis vectors is:

$$e_1 = (1, 0, 0, 0) , \quad e_2 = (0, 1, 0, 0) , \quad e_3 = (0, 0, 1, 0) , \quad e_4 = (0, 0, 0, 1) ; \quad (\text{A.4})$$

$$e_5 = (1, 1, 1, 0) , \quad e_6 = (-1, 1, 0, 1) , \quad e_7 = (1, 0, -1, 1) , \quad e_8 = (0, -1, 1, 1) ; \quad (\text{A.5})$$

and the γ_i denote the Dirac matrices projected onto the relevant directions:

$$\gamma_i = (\gamma, e_i) , \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) , \quad \text{for } i \in \{1, \dots, 8\} ; \quad (\text{A.6})$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \quad , \quad \text{for } i, j \in \{1, \dots, 4\} \quad . \quad (\text{A.7})$$

The action, eq.(A.1), thus contains two types of couplings: nearest neighbours in direction of the four lattice axes ($i = 1, \dots, 4$), and 3-space diagonal neighbours a distance $\sqrt{3}$ away ($i = 5, \dots, 8$). The effective gauge fields connecting these neighbours entering the combinations c_i, s_i are defined as follows:

- For $i \in \{1, \dots, 4\}$, $U(x', x)$ is the link matrix along the straight line path connecting the sites x' and x .
- For $i \in \{5, \dots, 8\}$, $U(x', x)$ is the path ordered product of the link matrices, averaged over the six topologically equivalent 3-link paths connecting the sites x' and x , and then projected back on the $SU(3)$ group manifold.

The lattice dispersion relation following from the free field form of the action, eq.(A.1), is:

$$K(p) = \frac{3}{2} \left(\frac{1}{\kappa} - 8 \right) + \sum_{i=1}^8 [-2A_i + 4A_i \sin^2(\frac{1}{2}p \cdot e_i) + 2B_i \gamma_i \sin(p \cdot e_i)] \quad . \quad (\text{A.8})$$

Flavour doubling and anomalies do not depend on the detailed structure of the dispersion relation, but only on global properties such as the total number of poles of the propagator in the Brillouin zone [20]. The momentum dependent part of $K(p)$ has only one zero at $p = (0, 0, 0, 0)$, ensuring that in the continuum limit our action does represent a single Dirac fermion with correct anomalies.

With improved gauge actions extending over more than a single time slice, complex eigenvalues appear in the transfer matrix at finite lattice spacing [21]. The same is true for standard Wilson action when $r \neq 1$. Our action only couples the fermion fields on adjoining time slices, just like the standard Wilson action. Even if spurious eigenvalues were to arise in the transfer matrix, they can be bounded and hence become unimportant in the continuum limit. If present, complex eigenvalues make correlation functions oscillatory in time, and in our simulation we have not observed such an effect.

This sets the stage for applying the arguments of ref.[6,8] to our improved action. We have not evaluated the diagrams appearing in weak coupling perturbation theory, but compared to the standard Wilson fermion case, the only difference is expected to appear in the numerical Brillouin zone integrals. In practice, we must determine the renormalisation constants non-perturbatively; the perturbative values are of not much help except to give us an idea of the asymptotic limit.

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Figure Captions

1. The signal for the matrix element of the scalar density $\bar{u}u$ between Δ states, eq.(9), as a function of t for $\kappa = 0.1360$.
2. Same as fig.1, but for the matrix element between ρ states.
3. The mass splittings in units of the lattice ρ mass *vs.* M_{PS}^2/M_V^2 : (a) within the vector meson octet (triangles) and the baryon decuplet (circles), and (b) within the baryon octet. See text for determination of Δm and M_ρ .
4. The pion propagator (full circles), eq.(18), and the $\langle AP \rangle$ correlator (open circles), eq.(19), as a function of the time separation t for $\kappa = 0.1360$. The curves are the asymptotic *cosh* and *sinh* fits appropriate to pion dominance of the channels.
5. The quantity ρ , eq.(20), and the chiral condensate, eq.(28), as a function of the lattice quark mass.
6. The coefficients C_1 (full circles) and C_2 (open circles), as determined from eq.(24), *vs.* t for $\kappa = 0.13525$.
7. Effect of a periodic doubling of the lattice in the time direction on the coefficient functions C_1 and C_2 . The results were extracted from 5 configurations at $\kappa = 0.1320$. Full (open) symbols represent the data on the short (long) lattice, while circles (squares) stand for C_1 (C_2).
8. The lattice renormalisation constants for non-singlet vector (full triangles) and axial (full circles) currents.
9. The pseudoscalar decay constant, f_{PS} , normalised to the vector meson mass as a function of M_{PS}^2/M_V^2 . The filled symbols labeled with particle names represent experimental values.
10. The vector meson decay constant, f_V^{-1} , as a function of M_{PS}^2/M_V^2 . The filled symbols labeled with particle names represent experimental values.

Fig. 1

Fig. 2

Fig. 3a

Fig. 3b

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10