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On the Explicit Construction of the Superstring Loop Measure in the Super-Schottky-Reggeon Formalism

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ABSTRACT

A technique for performing string perturbation theory in flat space time is reviewed with particular emphasis on the features arising as a consequence of the Ramond string complications. Details on results obtained so far are presented.

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1 Introduction

Despite a massive effort on the part of a large number of groups and people in order to work out in detail (super-) string perturbation theory in flat space-time background, and despite the fact that the completion of such a programme would only represent a somewhat modest step towards a full understanding of string dynamics, many issues remain unclear (for a review, see [1]). Whereas the situation for the bosonic string is rather well understood, recent effort has concentrated on the Neveu-Schwarz-Ramond superstring. Techniques involved include direct considerations of the path integral [2,3,4] [5,6], an improved version of the dual resonance model oscillator formalism with ghosts included [7,8,9,10,11,12,13], considerations of algebraic geometry [14,15], exploitations of bosonization-techniques [16,17], relating determinants to θ -functions [18,19,20], a new operator formalism [21,22], a group theoretic approach [23,24,25,26] and techniques pertaining to special cases [27,28,29] (see also [1] for many more references).

Here we shall concentrate on an approach which attempts to unify the path integral and the operator (oscillator) approach. It is rather well developed for the bosonic string [30,29,31] and worked out in broad outline for the Neveu-Schwarz-Ramond superstring [32,33,34]. For the bosonic string it is closely related to the pure oscillator approach [8,9] as discussed in [35,36]. The Riemann surface structure of multiloop amplitudes emerges in a Schottky representation [37] from the process of "sewing" reggeons [38], a process which is seen as a way of gradually performing the path integral (see also [5]). However, emphasis is put on using a manifestly world-sheet supersymmetric formulation building on ideas of [39], the spirit in this respect being similar to [22,23]. Thus one looks for a super-Schottky description of super Riemann surfaces [40,41] which, however, turns out to differ from the one of [42].

The present approach consists in evaluating the Polyakov path integral in the BRST-formalism (in the Weyl-invariant situation of the critical dimension) with specified boundary conditions. Those are carefully chosen not to violate the complex structure (we think of closed strings) by choosing an external coherent state basis, thereby also making a translation to an oscillator language immediate. Thus, external currents are chosen to provide a classical (super-) string field, (super-) analytic on the world sheet with singularities encoding the coherent external state parameters [30,32,33,43,31]. Holomorphic [14,15] and superholomorphic factorization [44,4,45,22] are manifestly maintained.

The technique for building the measure on moduli space (counting tree diagram Koba-Nielsen (KN) points on an equal footing with loop moduli) consists in constructing (super-) Beltrami differentials which are distributions concentrated around certain closed contours and built as (anti-

holomorphic) derivatives of certain quasi-(super-) conformal transformations: ones that are (super-) conformal Killing vector fields, but discontinuous across the contour. Following [46] these find a natural place as ghost "zero-modes". They only couple to specific anti-ghost-modes which turn into the Schottky form of holomorphic quadratic differentials on the Riemann surface (super-holomorphic $3/2$ - differentials on the super-Riemann surface).

The result of working out loop amplitudes in such an approach consists in explicit expressions for the integrand on (super-) moduli space, in very particular choices of coordinates depending on the order in which 3-vertices are joined together to form a tree, and on how the branches of the tree are sewn to form loops (cf. also [38]). In this way moduli space gets covered with patches which are neighbourhoods of fully pinched surface-configurations. One may hope that such explicit expressions might be useful for better understanding the rather complicated structure of super moduli space.

In this approach there is a nearly obvious relation between sewing, factorization and the path integral [30]: On a punctured Riemann sphere corresponding to a particular string amplitude (with punctures corresponding to infinite external state "tubes", conformally mapped to points) draw a contour separating the surface in two. The full path integral may be thought of as (i) path integrating over fields living on only one or the other of the two parts with particular boundary conditions along the contour, and (ii) path integrating over the boundary conditions. These in turn refer to Laurent-expansions of the field on the contour with powers of opposite sign for the two halves of the surface. The first of the above steps builds amplitudes corresponding to the two halves, each with one external "reggeon" in a coherent state corresponding to the boundary condition: one reggeon is described by the positive power part of the Laurent-expansion on the contour, the other by the negative powers (in suitable coordinates). The second step is sewing - or in reverse - factorization. In sewing the two amplitudes together, a mapping must be provided to relate the coordinates on the two individual surfaces. This mapping plays the role of a propagator and introduces $(1|0)$ or $(1|1)$ (even|odd) additional moduli for a Neveu-Schwarz (NS) or Ramond (R) sewing. Building the Riemann surface by sewing is therefore a way of gradually doing the path integral (for a recent discussion, see [5]).

In sec. 2 we describe the comparatively simple situation of tree diagrams. In flat space-time the action is gaussian, and the result of the path integral is just given by the classical action evaluated at the classical field configuration with the specified boundary conditions (singularities). To write down the classical field in terms of singularities is trivial for the bosonic and the NS-string. For the R-string states, complications arise due

to the square-root branch cuts [16,17,39]. Once the classical field is known the action is easily found by contour integrations.

In sect. 3 we briefly describe the treatment of BRST-invariant physical external states.

In sect. 4 the 3 NS-string vertex and the RR-NS vertex are treated with special emphasis on the differences: the 3 NS vertex depends on a $(0|1)$ modulus, the RR-NS vertex depends on no moduli.

In sect. 5 the NS- and R-sewings and their associated quasi-superconformal ghosts are described.

In sect. 6 we give some indications on how to perform loop calculations in the pure (non-modular invariant) NS-problem, and some general results are written down [32], obtained also in [9].

Sect. 7 describes how to treat the non-zero-mode part of the sewing for a completely general vacuum amplitude with g_{NS} NS-loops and g_R R-loops.

Finally in sect. 8 we venture to provide some details and techniques on how to perform the important ghost zero-mode-integrations providing the measure on moduli space. The present state of the art can handle a single R-loop together with an arbitrary number of NS-loops.

In sect. 9 we summarize the results.

2 Tree Diagrams and Notations

We consider a closed superstring with action [32,17,40]

$$S(X, B, C) = \int dZ d\bar{Z} \left\{ \frac{1}{2} D X^\mu(Z, \bar{Z}) \bar{D} X^\mu(Z, \bar{Z}) + B(Z, \bar{Z}) \bar{D} C(Z, \bar{Z}) + \bar{B}(Z, \bar{Z}) D \bar{C}(Z, \bar{Z}) \right\} \quad (2.1)$$

We shall be concentrating on the (super-) holomorphic part only, and hence we shall not even specify the meaning of $\bar{D}$ and $\bar{Z}$, that meaning being different for the heterotic and the type II superstring. $Z = (z, \theta)$, $dZ = dz d\theta$, $D = \partial/\partial\theta + \theta\partial/\partial z$ with $D^2 = \partial/\partial z$ and θ is the grassmannian coordinate. The superfields for matter, ghosts and antighosts, which we shall often denote as the triple $\Phi = (X^\mu, B, C)$, have components

$$\begin{aligned} X^\mu(Z) &= x^\mu(z) + \theta\psi^\mu(z) \\ B(Z) &= \beta(z) + \theta b(z) \\ C(Z) &= c(z) + \theta\gamma(z), \end{aligned} \quad (2.2)$$

and have superconformal dimensions 2, 3/2 and -1 respectively.

As explained above the result of performing the path integral with prescribed boundary conditions, i.e. singularities in $\Phi(Z)$, may be written as

$$W_N = \int d^{D_N} \mu \exp\{+S(\Phi_{Cl})\} (2\pi)^d \delta^d(\sum p_i)$$

$$\prod_{i=1}^N \exp\{-S(\Phi_i)\}, \quad (d = 10). \quad (2.3)$$

Here $d^{D_N} \mu$ is the measure on (tree-diagram-) moduli space which (see below) has $D_N = (N_B + 2N_F - 3 | N_B + N_F - 2)$ (bosonic|fermionic) dimensions. N_B is the number of external bosons (Neveu-Schwarz or NS-string states) and $2N_F$ the number of external fermions (Ramond or R-string states). The factors $\exp\{-S(\Phi_i)\}$ are (divergent) external state normalizations removing some self contraction infinities in $S(\Phi_{Cl})$, (and some otherwise explicit Weyl-breaking dependence, cf. [31] and [30]).

Φ_{Cl} is expressed in terms of individual "reggeon" fields Φ_i , each specifying the singularities of a particular string state.

This is very simple for the bosonic string and the pure NS-string, which we consider first. In fact by sewing (sec. 5) we shall build

$$\Phi_{Cl}(Z) = \sum_{i=1}^N \Phi^{(i)}(Z), \quad \Phi^{(i)}(Z) = V_i \Phi_i(Z). \quad (2.4)$$

Here a general "reggeon" field in the standard configuration (see below) is transported by a superconformal transformation V_i which in this NS-case is just an $OSp(2|1)$ transformation. For a superconformal field of superconformal dimension, h , and a superconformal map, V [40]

$$\begin{aligned} V : Z \rightarrow V(Z) &= (V_B(Z), V_F(Z)) \quad , \quad DV_B(Z) = V_F(Z) DV_F(Z) \\ V^{-1} \Phi_h(Z) &= (DV_F(Z))^{2h} \Phi_h(V(Z)). \end{aligned} \quad (2.5)$$

One parametrization of an $OSp(2|1)$ transformation is in terms of super-space points $U = (u, \mu)$, $V = (v, \nu)$ and a complex parameter, K

$$\begin{aligned} V(Z) &= (\tilde{z}, \tilde{\theta}) \\ \tilde{z} &= K^{-1} \frac{Z - U}{Z - V} (U - V), \quad U - V \equiv u - v - \mu\nu \text{ etc.} \\ \tilde{\theta} &= K^{-1/2} (Z - V)^{-1} \{(\theta - \mu)(U - V) - (\mu - \nu)(Z - U)\} \end{aligned} \quad (2.6)$$

If the i 'th external state is a coherent state with bosonic parameters $\{p_i, a_n^i, \beta_r^i, \gamma_r^i\}$ and fermionic parameters $\{\psi_r^i, b_n^i, c_n^i\}$, then the reggeon fields in the "standard" configuration take the form

$$\begin{aligned} X_i(Z) &= -ip_i \log z + i \sum_{n=1}^{\infty} \frac{a_n^i}{n} z^{-n} + i\theta \sum_{r=1/2}^{\infty} \psi_r^i z^{-r-1/2} \\ B_i(Z) &= \sum_{r=-1/2}^{\infty} \beta_r^i z^{-r-3/2} + \theta \sum_{n=-1}^{\infty} b_n^i z^{-n-2} \\ C_i(Z) &= \sum_{n=2}^{\infty} c_n^i z^{-n+1} + \theta \sum_{r=3/2}^{\infty} \gamma_r^i z^{-r+1/2}. \end{aligned} \quad (2.7)$$

Thus $V_i(0)$ is the KN-point Z_i . Since $\bar{D}$ on these vanish except at singularities, the classical action evaluates, by Stoke's theorem, to

$$S_{Cl}(\Phi_{Cl}) = \sum_{i,j} [\frac{1}{2}(X^{(i)}|X^{(j)}) + (B^{(i)}|C^{(j)})] \quad (2.8)$$

where the bilinear forms

$$\begin{aligned} (X^{(i)}|X^{(j)}) &= \oint_{Z_j} dZ D X^{(i)}(Z) X^{(j)}(Z) \\ (B^{(i)}|C^{(j)}) &= \oint_{Z_j} B^{(i)}(Z) C^{(j)}(Z) \end{aligned} \quad (2.9)$$

are symmetric and $OSp(2|1)$ invariant (technical complications arise for "zero-modes" labelled by $p, b_n, n = \pm 1, 0, \beta_r, r = \pm 1/2$, see refs.[30,32] and below). The super-contour integrals are evaluated by the super-Cauchy rule:

$$\oint_Z dZ' \frac{\theta' - \theta}{Z' - Z} F(Z') = F(Z) \quad (2.10)$$

(a factor $2\pi i$ is absorbed in dZ) and identities obtained by acting with D 's on that.

The measure is obtained by the process of sewing and has a general form like ($N_B = N$ here)

$$\begin{aligned} d^{(N-3|N-2)}\mu &= \prod_{i=1}^{N-3} d \log K_i \prod_{i=1}^{N-2} d\vartheta_i \\ &\cdot \int \prod_{j=1}^{N-3} d\check{c}_{s,i} \prod_{i=1}^{N-2} d\check{\gamma}_{\vartheta,i} \prod_{n=-1}^1 d\check{c}_n \prod_{r=-1/2}^{1/2} d\check{\gamma}_r \\ &\cdot \exp(B_{Cl} | \sum \check{C}_{s,i} + \check{C}_0). \end{aligned} \quad (2.11)$$

K_i and ϑ_i are the tree diagram bosonic and fermionic moduli, $\check{C}_{s,i}$ are the quasi (super-) conformal fields, expressed in terms of the modes

$$\{\check{c}_{s,i}, \check{\gamma}_{\vartheta,i}, \check{c}_n, \check{\gamma}_r\}, \quad (2.12)$$

discontinuous across the "sewing contours" $\mathcal{C}_{s,i}$ so that

$$(B_{Cl}|\check{C}_{s,i}) = \oint_{\mathcal{C}_{s,i}} dZ B_{Cl}(Z) \check{C}_{s,i}(Z) \quad (2.13)$$

whereas the superconformal Killing vector field (SCKVF) – describing the ghost zero-modes – is given by

$$\check{C}_0(Z) \equiv [\check{c}_{-1}z^2 + \check{c}_0z + \check{c}_1 + \theta(\check{\gamma}_{-1/2}z + \check{\gamma}_{1/2})]\vartheta(\mathcal{C}_0) \quad (2.14)$$

with $\vartheta(\mathcal{C}_0)$ vanishing outside the contour $\mathcal{C}_0$ which encloses all reggeon singularities (or, equivalently a single spurious singularity, see sect. 8 below).

We use the *check* accent to denote the quasi superconformal and zero-mode degrees of freedom.

The general situation when Ramond string states are present requires more care due to the presence of square root branch points at arbitrary points in superspace. We introduce the following definitions:

1. We say $\Phi \in \mathcal{R}$ iff there exists a Laurent type expansions (*for the entire classical field*)

$$\begin{aligned} X(Z) &= q - ip \log z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} + i\theta \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1/2} \\ B(Z) &= \sum_{n \in \mathbb{Z}} \beta_n z^{-n-3/2} + \theta \sum_{n \in \mathbb{Z}} b_n z^{-n-2} \\ C(Z) &= \sum_{n \in \mathbb{Z}} c_n z^{-n+1} + \theta \sum_{n \in \mathbb{Z}} \gamma_n z^{-n+1/2}. \end{aligned} \quad (2.15)$$

2. For any superconformal transformation, V (i.e. $DV_B = V_F DV_F$)

$$\mathcal{R}[V] \equiv \{\Phi | \exists \tilde{\Phi} \in \mathcal{R} : \Phi = V\tilde{\Phi}\}. \quad (2.16)$$

3. For any number of maps V_i , we define

$$\mathcal{R}[V_1, V_2, \dots] = \bigcap_i \mathcal{R}[V_i]. \quad (2.17)$$

Thus the classical field belonging to a situation describing N_B external NS-strings and $2N_F$ external R-strings will have N_F branch cuts and belong to a space $\mathcal{R}[V_1, V_2, \dots, V_{N_F}]$. We shall see that contrary to the NS-case, these V_i 's in general do *not* belong to $OSp(2|1)$.

It is crucial to notice that the symmetry, or equivalently, the set of SCKVF's in this case gets increased. Indeed, whereas in the pure NS-problem we only had invariance under

$$\Phi_{Cl} \rightarrow V\Phi_{Cl}, \quad V \in OSp(2|1) \quad (2.18)$$

(all other superconformal transformations would introduce new singularities, thereby violating the boundary conditions) corresponding to (3|2) degrees of freedom ($osp(2|1)$ is generated by $(L_{-1}, L_0, L_1 | G_{-1/2}, G_{1/2})$), we now have the following larger invariance:

Define the "super-dilation" (generated by L_0 and G_0)

$$d_{[K, \kappa]} : Z = (z, \theta) \rightarrow d_{[K, \kappa]}(Z) = (K(z + \theta\kappa\sqrt{z}), \sqrt{K}(\theta + \kappa\sqrt{z})). \quad (2.19)$$

The action (and the amplitude) is now invariant under

$$\begin{aligned} \Phi &\rightarrow (V_i \circ d_{[K, \kappa]} \circ V_i^{-1})\Phi \in \mathcal{R}[\tilde{V}_1, \dots, \tilde{V}_{N_F}] \neq \mathcal{R}[V_1, \dots, V_{N_F}] \\ \tilde{V}_j &\equiv V_i \circ d_{[K, \kappa]} \circ V_i^{-1} \circ V_j. \end{aligned} \quad (2.20)$$

Thus we have $(3|2 + N_F)$ invariances resulting in the well-known $(N_B + 2N_F - 3|N_B + N_F - 2)$ moduli of the general tree-diagram.

Similarly we have more SCKVF modes. In fact for each Ramond-cut there is an additional contribution of

$$\Delta_i \check{C}_0(Z) = \theta \check{\gamma}_0^R \sqrt{z} \vartheta(\mathcal{C}_0), \quad i = 1, \dots, N_F \quad (2.21)$$

each of these being transported by their respective V_i as a field with superconformal dimension -1 , and $\check{\gamma}_0^R$ denoting the associated d.o.f.

3 External (Physical) States

Begin with the ghost part and use the notation of ref.[17]. We have the vacua $|q, q' \rangle$ where q and q' are ghost numbers of (b, c) and (β, γ) respectively. In the general notation (a hat denotes a quantum oscillator operator as distinct from a c-number or Grassmann-number)

$$\hat{b}_n |q \rangle = 0, \quad n > \epsilon q - \lambda; \quad \hat{c}_n |q \rangle = 0, \quad n \geq -\epsilon q + \lambda \quad (3.1)$$

where $\epsilon = 1, \lambda = 2$ for the (b, c) and $\epsilon = -1, \lambda = 3/2$ for the (β, γ) .

NS-coherent states are built from the vacuum $|0, 0 \rangle$ whereas R-coherent states are built from $|0, 1/2 \rangle$.

A useful observation is [39,32,22]

$$\begin{aligned} |q + \epsilon \rangle &= \delta(\hat{c}_{-\epsilon q + \lambda - 1}) |q \rangle \\ |q - \epsilon \rangle &= \delta(\hat{b}_{\epsilon q - \lambda}) |q \rangle \\ \delta(\hat{c}_n) &\equiv \int db_{-n} \exp(b_{-n} \hat{c}_n) \\ (&= \hat{c}_n \text{ for } \hat{c}_n \text{ fermionic).} \end{aligned} \quad (3.2)$$

For physical strings we may take ghosts to be in the $|1, -1 \rangle$ and $|1, -1/2 \rangle$ vacua respectively for NS- and R- strings [17].

Our formalism directly gives amplitudes in coherent state bases. For an arbitrary state, however, they are easily obtained by either (i) replacing coherent state parameters by oscillators and letting the ensuing operator act on the desired states (with suitable vacua to the left, this is the most common formalism, cf. [8,9,7,23,12] etc.); or (ii) equivalently we may fold by coherent state wave functions of the desired states, amounting to acting on our coherent state amplitude with certain differential operators, obtained by replacements like $\hat{a}_{-n}^i \rightarrow \partial/\partial a_n^i$ etc. in the constructions of the desired external states.

For Ramond cuts the fermionic zero-mode ψ_0^μ is identified by (however, cf. also [34])

$$\langle \Theta_1 | \hat{\psi}_0^\mu | \Theta_2 \rangle = \langle \Theta_1 | \Theta_2 \rangle \psi_0^\mu \quad (3.3)$$

where the R-vacua $\langle \Theta_1 |, | \Theta_2 \rangle$ describe the corresponding pair of spinors. The operators $\hat{\psi}_0^\mu$ satisfy the Clifford algebra of the gamma matrices. In expressing the spinor states in a coherent state basis corresponding to $\hat{\psi}^{[1]} = \hat{\psi}^i + i\hat{\psi}^{i+5}$ (and the hermitean conjugate) one convinces oneself that the Grassmann parameters ψ_0 translate into antisymmetrized products of gamma matrices, and the spinor vacua to suitable spinors u_1, u_2 [39,32,34].

4 The 3 String Vertices

4.1 The 3-NS string vertex

See refs. [7,8,32,9,11,12]. By $OSp(2|1)$ invariance we may arrange that

$$Z_1 = (0, 0), \quad Z_2 = (\infty, 0), \quad Z_3 = (1, \vartheta) \quad (4.1)$$

For simplicity we chose

$$V_1(Z) = Z, \quad V_2(z, \theta) = (1/z, i\theta/z), \quad V_3(Z) = (1 - z, \vartheta - \theta). \quad (4.2)$$

But unlike for the 3-reggeon bosonic vertex, a fermionic modulus, ϑ remains. Our main point here is the fact that this gives rise to an associated superconformal ghost field describing the change in that modulus, while keeping the KN points at their form eq.(4.1). There are two possible forms of $\delta\theta$ that may be employed. We think in terms of the conformal dimension $-1/2$ field $\gamma(z)/\sqrt{dz}$:

$$1/\sqrt{dz} \quad \text{and} \quad z/\sqrt{dz} \quad (4.3)$$

related by $z \rightarrow 1/z$ and vanishing at ∞ and 0 respectively. Thus the first may be used with C_ϑ an anticlockwise circle of radius < 1 and vanishing inside (case 1), and the second with a clockwise circle of radius > 1 and vanishing outside (case 2).

At this point, consider in general the form of a (ghost-) displacement field: Put $\tilde{Z} = (z + \delta z, \theta + \delta\theta)$ and get [46] $C(Z) \sim \tilde{Z} - Z = \delta z + \theta\delta\theta$. However, both δz and $\delta\theta$ may depend in general on both z and θ . Requiring $\tilde{Z}$ to be a superconformal map gives

$$D(\delta z) = \delta\theta + \theta D(\delta\theta) \quad (4.4)$$

Putting $\delta z(z, \theta) = \delta z_B + \theta\delta z_F$ one finds $\theta\delta z_F = \theta\delta\theta$. Hence ¹

$$C(Z) = \delta z_B + 2\theta\delta\theta \quad (4.5)$$

So we take

$$\check{C}_\vartheta(Z) = 2\theta\check{\gamma}_\vartheta\vartheta(C_\vartheta) \begin{cases} 1 & (\text{case 1}) \\ z & (\text{case 2}) \end{cases} \quad (4.6)$$

When integration over the SCKVF and this ghost mode, $\check{\gamma}_\vartheta$, is performed there arise 3 bosonic and 3 fermionic antighost delta functions. This structure is generic.

¹I am very grateful to K.O. Roland for discussions on this factor of 2.

4.2 The RR-NS 3-string vertex

See [23,12,32,22]. There are no moduli in this case because we may use the extra invariance eq.(2.20) to put $Z_3 = (1, 0)$. Thus, compared to the 3-NS vertex, this extra SCKVF mode results in the "measure factor" being replaced by the "constant"

$$d^0\mu[RR - NS] = \int d\check{c}_{-1}d\check{c}_0d\check{c}_1d\check{\gamma}_{-1/2}d\check{\gamma}_{1/2}d\check{\gamma}_R \exp\{(B_{Cl}|\check{C}_{0,R})\} \quad (4.7)$$

with

$$\check{C}_{0,R}(Z) = \check{C}_0(Z) + \theta\check{\gamma}_R\sqrt{z}\vartheta(C_0) \quad (4.8)$$

$\check{\gamma}_R$ being the d.o.f. associated with the extra invariance. This form again gives rise to 3 fermionic and 3 bosonic delta functions in the anti-ghost modes, when integrations over the $\check{C}_{0,R}$ modes have been performed. But the origin of the 3'd bosonic delta function is quite different from the NS-case.

The square root cut causes some subtleties. Let us choose the Ramond reggeons to be at $Z_1 = (0, 0)$ and $Z_2 = (\infty, 0)$. The contribution to ψ_{Cl}^μ from those is easily parametrized as

$$\psi_R^{\mu(1+2)}(z) = i \sum_{n=-\infty}^{\infty} \psi_n^\mu z^{-n-1/2} \quad (4.9)$$

where the terms with $n < 0$ refer to reggeon 2 at $z = \infty$ (the modes ψ_{-n} differ from the coherent state parameters of that reggeon by a phase found by considering the superconformal map $z \rightarrow 1/z$, $\theta \rightarrow i\theta/z$).

For reggeon 3 at $z = 1$ ($\theta = 0$) we would use the following form in a 3-NS vertex:

$$\psi_{NS}^{\mu(3)}(z) = i \sum_{r=1/2}^{\infty} \psi_r^{\mu(3)}(1-z)^{-r-1/2}. \quad (4.10)$$

However, although this field has the required singularity near $z = 1$ it clearly does not belong to the proper space $\mathcal{R}$ (sect. 2). Thus we introduce a projection operator, projecting on to that space, with the requirement that the square root cut introduced by this projection should not modify the modes of reggeons 1 and 2. In the present case of a single R-cut, it is easy to provide an explicit form of this projection:

$$\begin{aligned} \psi_R^{\mu(3)}(z) &\equiv \mathcal{P}_{\mathcal{R}}(\psi_{NS}^{\mu(3)}(z)) \\ &= \oint dz' \left[\frac{z+z'}{2\sqrt{zz'(z-z')}} \right] \psi_{NS}^{\mu(3)}(z'). \end{aligned} \quad (4.11)$$

One easily checks that this is indeed a projection (cf. [34] for a slightly modified treatment). In eq.(4.11) a careful treatment has been provided for the ψ_0^μ zero-mode. This form is dictated by the requirement of it yielding

the correct Ramond propagator which therefore appears in the integration kernel [32].

For the ghost and anti-ghost fields, similar projections have to be considered, but we leave them for the more explicit discussion in sect. 8.

For illustration we give the result of performing the evaluation of the action and the amplitude for the case where the ghosts belong to the physical vacuum (sec. 3). The wave function factors in eq.(2.3) do not know about the above projection to the $\mathcal{R}$ spaces and hence the cancellation of infinite self-contractions in this case leaves a finite contribution [32]. This is a new feature. A simple point splitting technique gives the following oscillator result:

$$\begin{aligned}
W[RR - NS] = & \\
= & (2\pi)^d \delta^d(\sum p_i) \prod_{i=1}^3 (V_i'(0))^{p_i^2/2} \exp\left\{\sum_{n>0} \frac{1}{2} \frac{p_i a_n^i}{n!} \partial_z^n \log V_i'(z) \Big|_{z=0}\right\} \\
& \cdot \exp\left\{\frac{1}{2} \sum_{i \neq j} \sum_{n,m \geq 0} \frac{a_n^i a_m^j}{n! m!} \partial_z^n \partial_y^m \log(V_i(z) - V_j(y)) \Big|_{z=y=0}\right\} \\
& + \frac{1}{4} \sum_{r,s \geq 1/2} \frac{\psi_r^3}{(r-1/2)!} \frac{\psi_s^3}{(s-1/2)!} \partial_y^{(r-1/2)} \partial_x^{(s-1/2)} \\
& \left[\frac{1}{2} \left(\sqrt{\frac{x+1}{y+1}} + \sqrt{\frac{y+1}{x+1}} - 2 \right) \right] \Big|_{y=x=0} \\
& - \sum_{r,n \geq 0} \frac{\psi_n}{n!} \frac{\psi_r^3}{(r-1/2)!} \partial_z^n \partial_y^{(r-1/2)} \frac{z+y+1}{2\sqrt{y+1}(z-y-1)} \Big|_{z=y=0} \\
& + \sum_{r \geq 1/2, n < 0} \frac{\psi_n}{(-n)!} \frac{\psi_r^3}{(r-1/2)!} \partial_z^{(-n)} \partial_y^{(r-1/2)} \frac{zy+z+1}{2\sqrt{y+1}(1-zy-z)} \Big|_{z=y=0} \\
& + \sum_{n>0} \psi_{-n} \psi_n \} \tag{4.12}
\end{aligned}$$

Here we have had in mind the choices $V_1(z) = z$, $V_2(z) = 1/z$, $V_3(z) = 1 - z$. This expression[32] satisfies the overlap conditions [23] and agrees with [12] when allowance is made for the different (and simpler) choice of V_i 's employed here.

5 Sewing

Consider sewing two tree diagrams. One diagram is expressed in terms of classical fields $(\Phi_A + \Phi_\alpha)(Z^I)$, where Φ_A denotes the contribution from any number of external string states (singularities) *not* to be sewn, and Φ_α denotes the contribution from the reggeon to be sewn. Z^I is the world sheet super variable for that diagram. Similarly the other diagram is described by $(\Phi_B + \Phi_\beta)(Z^{II})$.

Sewing Φ_α and Φ_β to form a new diagram involves the following:

1. Variables Z^I and Z^{II} must be identified by a "propagator" mapping. This involves some ambiguity, part of which is associated with the overall invariances of the resulting diagram, and part of which should be viewed as giving rise to different coordinations of the same thing. To fix matters, choose KN-points Z_A^I and Z_B^{II} arbitrarily among the KN-variables of the A - and B - sets, and choose coordinates such that

$$Z_\alpha^I = (0, 0), \quad Z_\beta^{II} = (\infty, 0), \quad Z_A^I = (\infty, 0), \quad Z_B^{II} = (0, 0). \quad (5.1)$$

Then the possible "relative mappings" for the propagators are, respectively for NS and R-sewings:

$$\begin{aligned} NS: P = d_{[K]} &: z \rightarrow Kz, \quad \theta \rightarrow \sqrt{K}\theta \\ R: P = d_{[K, \kappa]} &: z \rightarrow K(z + \theta\kappa\sqrt{z}), \quad \theta \rightarrow \sqrt{K}(\theta + \kappa\sqrt{z}). \end{aligned} \quad (5.2)$$

The phase choice of $\sqrt{K}$ will become a spin structure choice in the loop problem (sects. 6,7,8).

2. Now sewing is the rule of gaussian integration:

$$\begin{aligned} \int \mathcal{D}(\Phi_\alpha, \Phi_\beta) &\cdot \exp\{+S[\Phi_A + \Phi_\alpha]\} \exp\{+S[\Phi_B + \Phi_\beta]\} \\ &\cdot \exp\{-S[\Phi_\alpha + P\Phi_\beta]\} \\ &= \exp\{+S[\Phi_A + P\Phi_B]\}. \end{aligned} \quad (5.3)$$

In the integrand the first two factors are the amplitudes to be sewn, the third is the "sewing"- i.e. path integral weight. The result is the full, sewn amplitude. In the process the V_i^B 's of the B-field have been changed to PV_i^B . This is how the final V_i 's of the full amplitude are gradually being built up by successive sewings from the "trivial" V_i 's of the 3-vertices.

3. When Ramond-cuts are present in Φ_A and/or Φ_B we must first write

$$\Phi_A + \Phi_\alpha = \mathcal{P}_R(\Phi_A) + \tilde{\Phi}_\alpha \quad (5.4)$$

(and similarly for the other amplitude) where the projection operator projects from the sort of Ramond-space (sect. 2) relevant for Φ_A , Φ_α , $\Phi_A + \Phi_\alpha$, to the one relevant for $\Phi_A + P\Phi_B$. The crucial observation is that this implies a shift in Φ_α , the path integral variable, compatible with the singularities of that field; i.e. the extra cuts that have to appear after the projection, may be described by the type of Laurent expansion that defines Φ_α , i.e. the allowed location of the singularities of Φ_α . [32].

4. Corresponding to the NS- and R-propagators eq.(5.2), we must construct quasi-superconformal ghost fields generating the shifts $K \rightarrow K + dK$ and $\kappa \rightarrow \kappa + d\kappa$ in order to calculate the measure. The result is found (cf. eq.(4.5)) to be [32]

$$\begin{aligned} NS: \check{C}_K(Z) &= \check{c}_K \cdot z \vartheta(\mathcal{C}_K) \\ R: \check{C}_{K,\kappa}(Z) &= [\check{c}_K \cdot z + \check{\gamma}_\kappa(-\kappa z + 2\theta\sqrt{z})] \vartheta(\mathcal{C}_{K,\kappa}) \end{aligned} \quad (5.5)$$

where the contours $\mathcal{C}_K$ and $\mathcal{C}_{K,\kappa}$ include all $P\Phi_B$ -singularities and exclude all Φ_A -singularities, i.e. run around the tube connecting the two diagrams to be sewn. The associated measure factor, say for R-sewing, is then

$$(d\mu)_{K,\kappa} = d \log K d\kappa \int d\check{c}_K d\check{\gamma}_\kappa \exp\left\{ \oint_{\mathcal{C}_{K,\kappa}} dZ B_{CI}(Z) \check{C}_{K,\kappa}(Z) \right\} \quad (5.6)$$

which is easy to evaluate to a product of a fermionic and a bosonic delta function in B_{CI} -field modes.

5. The sewing is a path integral entirely analogous to the one in the tree diagram, sect. 2. For the same reason then we get δ -functions arising from zero-mode integrations: momentum conservation in the sewing, and δ -functions in the lowest antighost sewing modes. We shall mostly imagine that these δ -functions have been used to integrate out antighost modes b_n, β_n with $|n| \leq 1$ in one of the reggeons. In the case of tree diagrams, the same modes of the other sewing reggeon may be integrated out and we are left with the requisite antighost δ -functions for the entire diagram. In the case of loops (sect. 8) these last integrations provide what we refer to as the "measure factor" in loop calculations.

We see that adding a 3 vertex always gives rise to 1 more bosonic and 1 more fermionic modulus. But their origin is quite different for NS- and R-sewing. This means that the requirement of duality works out in a rather non-trivial way. Thus, we may consider the well-known example of the 2R-2NS amplitude. It may be obtained either by NS-sewing or by R-sewing, the two procedures giving rise to very different coordinate descriptions, indeed two very different choices for the (1|1) moduli. It is gratifying to check [32] by explicit computation that the two calculations give rise to the same (well-known) physical amplitude.²

6 Sewing Loops from a General NS-Tree

We first consider this simple but non modular invariant case. Out of a total of 2^{2g} spin structures (a concept strictly speaking only making sense

²K.O. Roland has recently revised the calculation and found a missing factor of 2 in both cases in [32].

near pinching limits, or, when the super moduli vanish) this corresponds to the subset of 2^g spin structures (around b-cycles) arising from the different possible phase choices of $\sqrt{K_i}$ in the fermionic part of the sewing propagator (cf. eq.(6.2), eq.(5.2)). The remaining spin structures (around a-cycles) require sewings of any combination of R- and NS-reggeons (cf. sect. 7,8).

The calculational techniques represent a very straightforward generalization of the ones developed for the bosonic string [30]. Indeed, the results obtained for the most general N -string g -loop amplitude [32] were also obtained by a simple supersymmetrization of the corresponding bosonic result [9].

We decompose matter, ghost and antighost superfields for an NS-tree amplitude with $2g + N$ external lines as

$$\begin{aligned}\Phi_{tot}^{Cl} &= \sum_{i=1}^g (\Phi_{\alpha_i} + \Phi_{\beta_i}) + \Phi_e \\ \Phi_e &= \sum_{k=1}^N \Phi^{(k)} = \sum_{k=1}^N V_k \Phi_k\end{aligned}\tag{6.1}$$

An advantage in the formalism is that the detailed form of Φ_e expressed in terms of external fields is of no concern. We may even regard the total Φ_e as describing one single reggeon.

Reggeons Φ_{α_i} and Φ_{β_i} are to be sewn together to form g loops. This is done using the g "propagator" $OSp(2|1)$ transformations P_i , with fixed points Z_{α_i} and Z_{β_i} and with multipliers K_i . These concepts are defined by

$$P_i = \Gamma_i^{-1} \circ d_{K_i} \circ \Gamma_i\tag{6.2}$$

where Γ_i is of the form eq.(2.6) mapping Z_{β_i} and Z_{α_i} to $(0, 0)$ and $(\infty, 0)$, and d_{K_i} is a dilation with multiplier K_i .

The loop amplitude is then constructed as

$$\begin{aligned}&\int d\check{C} \int \mathcal{D}[\Phi_{\alpha_i}, P_i^{-1}\Phi_{\beta_i}] \exp[+S(\Phi_{tot}^{Cl})] \exp[(B_{tot}^{Cl}|\check{C})] \\ &\cdot \prod_{i=1}^g \exp[-S(\Phi_{\alpha_i} + P_i^{-1}\Phi_{\beta_i})]\end{aligned}\tag{6.3}$$

where $\check{C}$ is the sum of all the quasi-superconformal ghost modes introduced by the various sewings. In sect. 8 we shall treat in some detail the evaluation of $\int d\check{C} \exp[(B_{tot}^{Cl}|\check{C})]$.

The integral over loop modes is gaussian. It is convenient to distinguish between "zero-modes" (momentum modes and ψ_0^μ modes for the matter fields; ghost/antighost "zero-modes", see sect. 8) and non-zero-modes. The integral over momentum modes gives rise to a (super-) period matrix and is the only part of the result that breaks manifest (super-) holomorphic factorization [14,15,1,30,32]. The integral over non-zero-modes (performed first)

involves the $OSp(2|1)$ -mapping valued $2g \times 2g$ matrix A , easily obtained from eq.(6.3) acting between the bilinear forms eq.(2.8)

$$A = \begin{pmatrix} \Pi - I & \Pi - P^{-1} \\ \Pi - P & \Pi - I \end{pmatrix} \quad (6.4)$$

where

$$\begin{aligned} \Pi_{ij} &\equiv 1 & , & & I_{ij} &\equiv \delta_{ij} \\ P_{ij} &\equiv P_i \delta_{ij} & , & & P_{ij}^{-1} &\equiv P_i^{-1} \delta_{ij} \\ i, j &= 1, \dots, g \end{aligned} \quad (6.5)$$

Expressing eq.(6.3) in terms of A it is seen to act between a row and a column of the fields (for the zero-modes, a modification occurs [30]) $\{|\Phi_{\alpha_i}\rangle, |\Phi_{\beta_i}\rangle\}$. Standard techniques [37,30,32,8,9] allow to evaluate $\det(A)$ in terms of the spectrum of the elements, γ , of the the (super-) Schottky group, $\mathcal{S}$, generated by the propagators, $\{P_i, i = 1, \dots, g\}$. This spectrum is almost the same whether A acts between dim 0 matter super fields or between dimension $3/2$ and $-1/2$ anti-ghost and ghost super fields. However, the lowest eigenvalue is absent in the ghost case (see the expression eq.(6.13)). The analogous feature for the bosonic string has been discussed [30,8] and found [29] to be crucial for consistency with general results from algebraic geometry [27], modular invariance in particular.

In addition to producing the determinants, the gaussian integration results in evaluating the action on some shifted, critical fields, expressed in terms of A^{-1} . This last operator has the effect of converting the original tree diagram external field Φ_e into an "automorphized" form, making it well defined on the ensuing (super-) Riemann surface (automorphic with respect to the Schottky group in the covering space).

If we restrict ourselves to external string states with ghosts in the physical vacua of sect. 3, the final result may be written as [32]

$$\begin{aligned} W_{N,g} &= (2\pi)^d \delta^d(\sum p_i) (\det 2\hat{T})^{-d/2} \\ &\cdot \prod_{(\gamma)} ' \prod_{n=1}^{\infty} \left[\frac{1 - K_{\gamma}^{n-1/2}}{1 - K_{\gamma}^n} \right]^d \cdot \prod_{(\gamma)} ' \prod_{n=2}^{\infty} \left[\frac{1 - K_{\gamma}^{n-1/2}}{1 - K_{\gamma}^n} \right]^{-2} \\ &\cdot \exp\left\{ \frac{1}{2} \int DX_e^{Aut} \overline{D}X_e^{Aut} \right\} \\ &\cdot \exp\left\{ \frac{1}{4} \sum_{i,j} \left(\int [\omega_i \overline{D}X_e^{Aut} + DX_e^{Aut} \overline{\omega}_i] \right. \right. \\ &\quad \left. \left. (2\pi\hat{T})_{ij}^{-1} \int [\omega_j \overline{D}X_e^{Aut} + DX_e^{Aut} \overline{\omega}_j] \right) \right\} \\ &\cdot d\mu_g d\mu_N \\ d &\equiv 10. \end{aligned} \quad (6.6)$$

Here

$$X_e^{Aut}(Z) = \sum_{\gamma \in \mathcal{S}} \gamma X_e^{Aut} \quad (6.7)$$

is the automorphized external matter field. The ω_i 's are the superholomorphic half-forms (cf. eq.(2.5) for notation)

$$\omega_i(Z) = \sum_{\gamma}^{(i)} \left[\frac{\theta - \gamma_F(Z_{\alpha_i})}{Z - \gamma(Z_{\alpha_i})} - \frac{\theta - \gamma_F(Z_{\beta_i})}{Z - \gamma(Z_{\beta_i})} \right] dZ \quad (6.8)$$

($\sum^{(i)}$ omits elements $\gamma \in \mathcal{S}$ with a rightmost P_i^n , $n \neq 0$; similarly for $^{(i)}\sum$),

$$\hat{T}_{ij} \equiv \frac{1}{2\pi i} (T_{ij} - \bar{T}_{ij}) \quad (6.9)$$

$$\begin{aligned} T_{ij} = \int_Z^{P_i(Z)} \omega_j &= \frac{1}{2\pi i} (\delta_{ij} \log K_i - \\ &- \sum_{\gamma}^{(i)} \log \left[\frac{Z_{\beta_i} - \gamma(Z_{\alpha_j})}{Z_{\beta_i} - \gamma(Z_{\beta_j})} \cdot \frac{Z_{\alpha_i} - \gamma(Z_{\beta_j})}{Z_{\alpha_i} - \gamma(Z_{\alpha_j})} \right]) \end{aligned} \quad (6.10)$$

(The prime means the identity is absent for $i = j$; $\bar{T}$ is a similar expression for anti-holomorphic modes; in general $\bar{T}_{ij} \neq T_{ij}^*$). In eq.(6.6) we have treated DX_e^{Aut} etc. as forms, including a factor of dZ . The primed products over (γ) denote products over cyclic classes of primary elements [37,30].

The measure factors may be expressed in various ways, for example as

$$\begin{aligned} d\mu_g &= \prod_{i=1}^g \frac{dK_i}{K_i^{3/2}} \left(\frac{1 - K_i^{1/2}}{1 - K_i} \right)^{-2} \prod_{i=1}^g \frac{1}{Z_{\alpha_i} - Z_{\beta_i}} \prod_{i=2}^{g-1} dZ_{\alpha_i} dZ_{\beta_i} \cdot dZ_{\alpha_g} d\theta_{\alpha_1} \\ d\mu_N &= \prod_{k=1}^N \frac{dZ_k}{DV_{k,F}(0)} \end{aligned} \quad (6.11)$$

where we used $OSp(2|1)$ invariance to fix $Z_{\beta_g} = (0, 0)$, $Z_{\beta_1} = (\infty, 0)$, $Z_{\alpha_1} = (1, \theta_{\alpha_1})$. The calculation of $d\mu_g$ will be treated in some detail in sect. 8. A manifest $OSp(2|1)$ generalization is easily written down [9].

Using contour integration techniques

$$\begin{aligned} \int d^2 Z \bar{D} X^{(k)}(Z) D X^{(l)}(Z) &= - \oint_{Z_l} dZ X^{(k)}(Z) D X^{(l)}(Z) \\ D X^{(l)}(Z) &= - \oint_{Z_l} dZ' \frac{\theta' - \theta}{Z' - Z} D X^{(l)}(Z') \\ X^{(k)}(Z) &= + \oint_{Z_k} dZ'' \log[Z'' - Z] D X^{(k)}(Z'') \end{aligned} \quad (6.12)$$

the two exponential factors in eq.(6.6) may be expressed in terms of oscillator modes of the external (coherent) states. In the calculation one must

remember the peculiar properties of momentum modes under conformal transformations[30,32]. The result is [32]

$$\begin{aligned}
& \prod_{k=1}^N \exp \left\{ \sum_{m=0}^{\infty} \frac{p_k A_m^{(k)}}{[m]!} D_{Z'}^{2m} \log DV_{k,F}(Z') \right\} \Big|_{Z'=0} \\
& \cdot \exp \left\{ \frac{1}{2} \sum_{\gamma \in \mathcal{S}}' \sum_{n,m=0}^{\infty} \sum_{k,l=1}^N \frac{A_n^{(k)}}{[n]!} D_Z^{2n} \frac{A_m^{(l)}}{[m]!} D_{Z'}^{2m} \log[V_k(Z) - \gamma V_l(Z')] \Big|_{Z=Z'=0} \right\} \\
& \cdot \exp \left\{ \frac{1}{4} \sum_{n,m=0}^{\infty} \sum_{k,l=1}^N \frac{A_n^{(k)}}{[n]!} D_Z^{2n} \frac{A_m^{(l)}}{[m]!} D_{Z'}^{2m} \right. \\
& \left. \Omega_i(V_k(Z))(2\pi\hat{T})_{ij}^{-1} \Omega_j(V_l(Z')) \Big|_{Z=Z'=0} \right\}, \tag{6.13}
\end{aligned}$$

where $\sum'_{\gamma \in \mathcal{S}}$ means that the identity element is absent from the sum whenever $k = l$; $[n]$ denotes the integer part of n , and the sum over n, m extends over all positive half odd integers and over all non-negative integers, and we use the notation [9,32]

$$A_n^{(k)} = \begin{cases} a_n^{(k)} & n \text{ integer} \\ \psi_n^{(k)} & n \text{ half odd integer} \end{cases} \tag{6.14}$$

and $a_0^{(k)} \equiv p_k$ for notational convenience. The first Abelian superintegrals, Ω_i , satisfy $D\Omega_i = \omega_i$ and may be given by

$$\Omega_i(Z) = \sum_{\gamma \in \mathcal{S}}^{(i)} \log \frac{Z - \gamma(Z_{\alpha_i})}{Z - \gamma(Z_{\beta_i})}. \tag{6.15}$$

The convergence of the exponent of eq.(6.13) rests on momentum conservation: $\sum a_0^{(k)} = 0$. The expression may then further be related [32] in a very simple way to the NS-super-prime form [9]. Notice that the term $\gamma = I$ represents a convenient oscillator expression for the tree diagram (cf. [7,23,47]).

7 The Non-Zero Mode Part of the Most General Vacuum Diagram

To calculate the most general vacuum diagram built by sewing g_R loops of R-reggeons and g_{NS} loops of NS-reggeons, it is convenient to distinguish between zero-modes and non-zero-modes.

The zero-modes may further be separated into (i) the momentum modes giving rise to some generalized form of period matrix (a contribution breaking (super-) holomorphic factorization), (ii) the fermionic matter zero-mode ψ_0^μ , and (iii) the quasi superconformal ghost modes and the (super-) Teichmüller anti-ghost modes to which they couple. This last part is of particular

interest since it is here that the measure on (super-) moduli space resides. It will be treated in some detail in the next section.

In this section we treat the (infinitely many) non-zero-modes in the most general case. It is interesting that apparently a very simple structure emerges, even though the functional dependence on (moduli|supermoduli) is quite complicated.

For simplicity we restrict ourselves to coordinates corresponding to building the $g = g_R + g_{NS}$ loop vacuum amplitude from a g -point NS-tree amplitude by attaching g_{NS} 3-NS-vertices and g_R RR-NS vertices to the legs, thereby using NS-sewings only, cf. fig.1. This has the simplifying consequence that the corresponding tree diagram is built entirely using $OSp(2|1)$ transformations, V_i . The loop amplitude is then constructed using the propagator transformations (cf. eq.(5.2)) P_i^{NS} , P_j^R characterized as follows:

1. For the g_{NS} NS-loops, $P_i^{NS} \in OSp(2|1)$, $i = 1, \dots, g_{NS}$ with fixed points $Z_{\alpha_i}^{NS}$, $Z_{\beta_i}^{NS}$ and multipliers K_i^{NS} .
2. The tree diagram classical fields belong to the space (cf. sect. 2), $\mathcal{R}[V_1^R, \dots, V_{g_R}^R]$ with a V_i^R for each R-cut, i.e. for each *pair* of R-reggeons to be sewn into a loop. Then

$$P_j^R = V_j^R \circ d_{[K_j^R, \kappa_j]} \circ (V_j^R)^{-1}, \quad j = 1, \dots, g_R. \quad (7.1)$$

In the coordination employed here the V_i^R 's are $OSp(2|1)$ -maps constructed in the process of building the tree amplitude (cf. sect. 8 for examples), but the "super-dilations", $d_{[K, \kappa]}$ and hence the R-propagators, P_i^R are not.

The super-Schottky group as usual consists of all products of all powers of the generators, $\{P_i^{NS}, P_i^R\}$. It should be noticed that therefore we are led to consider a somewhat incomplete uniformization of the super-Riemann surface: Our uniformizing space is *not* the super Riemann sphere, but a double cover of that, glued together at the Ramond-branch points in a rather complicated way, as described by the space $\mathcal{R}[V_1^R, \dots, V_{g_R}^R]$. Thus our treatment differs from that of [42] and others in which the super-Schottky group is taken to be a subgroup of $OSp(2|1)$. Our point is that such a subgroup does not seem to naturally arise from the sewing picture.

Formally, however, most of the analysis in the previous section goes through, and the integration over non-zero-modes gives rise to a determinant of the matrix A , and again that may be expressed in terms of the spectrum of elements in this new super-Schottky group. However, whereas this spectrum was trivial to write down for an $OSp(2|1)$ map γ in terms of the multipliers – it is given as $\{(K_\gamma^n | K_\gamma^{n-1/2}), n \in \mathbb{N}\}$, corresponding to

(bosonic|fermionic) eigenfields – it is not immediately obvious how to write it down in the general case when $\gamma \notin OSp(2|1)$.

Nevertheless, we now argue that a very similar structure emerges also here:

When γ is conjugate to a Ramond propagator, P_j^R the spectrum is trivially obtained as in the 1-loop problem:

$$\{((K_j^R)^n | (K_j^R)^n), n \text{ integer}\}. \quad (7.2)$$

When $\gamma \notin OSp(2|1)$ we may still introduce the concept of fixed points ($\gamma(Z_{F.P.}) = Z_{F.P.}$) and convince ourselves that generically there are two (since this is the case when all odd super-moduli vanish). Further, when γ is not conjugate to any Ramond propagator, one easily sees that the fixed points are regular points of the map. Now observe that when $U = (u, \mu)$ is a regular fixed point of any superconformal map γ , then

$$\gamma'_B(U) + \mu \gamma'_F(U) = (D\gamma_F(U))^2 \quad (7.3)$$

and that is the multiplier for $\gamma \in OSp(2|1)$. Sufficiently close to the fixed point, U , γ "looks like" an $OSp(2|1)$ transformation with fixed point U and multiplier given by

$$K_\gamma^{1/2} \equiv D\gamma_F(U). \quad (7.4)$$

The phase here corresponds to a spin structure choice, see below. Eigenfunctions must vanish or be singular at U . Take the one with a principal (cf. [41]) zero at U (we think of scalar fields for simplicity). Clearly then the eigenvalue must be K_γ^{-1} (if U is attractive). Denote the eigenfunction by $\Gamma_B^{-1}(Z)$ and find $\Gamma_F^{-1}(Z)$ such that the two together constitute a superconformal map:

$$D\Gamma_B^{-1}(Z) = \Gamma_F^{-1}(Z) D\Gamma_F^{-1}(Z). \quad (7.5)$$

One verifies that $\Gamma_F^{-1}(Z)$ is then an eigenfield with eigenvalue $K_\gamma^{-1/2}$ and indeed that

$$\gamma \equiv \Gamma \circ d_{K_\gamma} \circ \Gamma^{-1} \quad (7.6)$$

from which the spectrum is trivially written down as

$$\{(K_\gamma^n | K_\gamma^{n-1/2}), n \text{ integer}\}. \quad (7.7)$$

The different phase choices of $K_\gamma^{1/2}$ follow uniquely by eq.(7.4) from the spin structure choice of the phases of the $(K_i^{NS})^{1/2}$, $(K_i^R)^{1/2}$.

It is now obvious how to write down the non-zero-mode contribution to the most general vacuum amplitude. It is given by

$$\begin{aligned} & \prod_{i=1}^{gR} \prod_{n=2}^{\infty} \left[\frac{1 \pm (K_i^R)^n}{1 - (K_i^R)^n} \right]^{d-2} \cdot \left[\frac{1 \pm K_i^R}{1 - K_i^R} \right]^d \\ & \cdot \prod_{(\gamma)}^{\sim} \prod_{n=2}^{\infty} \left[\frac{1 - K_\gamma^{n-1/2}}{1 - K_\gamma^n} \right]^{d-2} \cdot \left[\frac{1 - K_\gamma^{1/2}}{1 - K_\gamma} \right]^d. \end{aligned} \quad (7.8)$$

Here $\prod_{(\gamma)}'$ is only over those cyclic classes of primary elements for which that primary element is *not* a Ramond-generator. The contribution from those form the first line in eq.(7.8).

It should be pointed out that the functional dependence of K_γ on the (moduli|supermoduli) is completely specified by eq.(7.4) but will in general be fairly complicated. One would expect the integration over supermoduli to lead to a tractable result, but it might not be simple. Also notice that the analysis was performed without providing an explicit form of the eigenfunctions. This was fortunate, since these must be very complicated (involving infinite accumulations of cuts for example).

8 The Measure Factor: Ghost Zero Modes

We treat here in detail the situation depicted in figs. 1,2,3: The case $g_R = 1$, g_{NS} arbitrary, $g = g_{NS} + g_R = g_{NS} + 1$. This case too does not provide a modular invariant result, however, the techniques developed are hoped to pave the way towards the completely general situation. The principal lesson is that most of the serious complications due to the presence of the Ramond-cut, can be circumvented by manipulations of contour integrals.

8.1 Parametrization of the tree

We first build the $g + 2$ string tree in fig. 1. The dashed line connecting Z_{α_g} and Z_{β_g} indicates that these two strings are R-ones. All the sewing contours, $\mathcal{C}_{A_i} = -\mathcal{C}_{\beta_{i-1}}$ (see below) refer to NS-sewings. This represents the the simplest choice in our formalism (for discussions of other possibilities in the bosonic case, cf.[35]). In fig.2 we have attached 3-NS-vertices to the $g - 2$ strings $Z_{\alpha_2}, \dots, Z_{\alpha_{g-1}}$, and the loop diagram is completed by sewing $Z_{\alpha_i}, Z_{\beta_i}$ together, $i = 1, \dots, g = g_{NS} + 1$. In fig. 3 the KN-points and the various sewing contours are indicated in the (super-) Z -plane.

For the purpose of the calculation we shall find it convenient to use also the mapped variable, $W = (w, \phi)$:

$$\begin{aligned} z &= \frac{z_{\alpha_g} w}{w - (1 - z_{\alpha_g})} \quad ; \quad \theta = i \frac{\sqrt{z_{\alpha_g}(1 - z_{\alpha_g})}}{w - (1 - z_{\alpha_g})} \phi \\ w &= \frac{1 - z_{\alpha_g}}{z - z_{\alpha_g}} z \quad ; \quad \phi = -i \theta \frac{\sqrt{z_{\alpha_g}(1 - z_{\alpha_g})}}{z - z_{\alpha_g}} \\ D_\theta \phi &= -i \frac{\sqrt{z_{\alpha_g}(1 - z_{\alpha_g})}}{z - z_{\alpha_g}} \quad ; \quad D_\phi \theta = i \frac{\sqrt{z_{\alpha_g}(1 - z_{\alpha_g})}}{w - (1 - z_{\alpha_g})}. \end{aligned} \quad (8.1)$$

Koba-Nielsen points $Z_{\alpha_i}, Z_{\beta_i}$ are mapped to KN-points $W_{\alpha_i}, W_{\beta_i}$.

In the Z -plane the tree diagram is built by using for the sewings always simple dilations with multiplier A_i . Thus at the level of fig. 1 we have used

$OSp(2|1)$ invariance as well as superdilation invariance for the R-cut (sect. 2) to put

$$\begin{aligned} \theta_{\alpha_g} &= 0 \quad ; \quad z_{\beta_g} = 0, \quad \theta_{\beta_g} = 0 \\ z_{\alpha_1} &= 1, \quad \theta_{\alpha_1} \equiv \vartheta_{\alpha_1} \quad ; \quad z_{\beta_1} = \infty, \quad \theta_{\beta_1} = 0. \end{aligned} \quad (8.2)$$

Further

$$\begin{aligned} z_{\alpha_i} &= A_i \cdot A_{i-1} \cdot \dots \cdot A_2 \quad , \quad i = 2, \dots, g \\ \theta_{\alpha_i} &= \sqrt{z_{\alpha_i}} \vartheta_{\alpha_i} \quad , \quad i = 1, \dots, g-1 \end{aligned} \quad (8.3)$$

where ϑ_{α_i} , $i = 1, \dots, g-1$ are the odd moduli of the 3-NS-vertices. Our moduli at this point are the $\{A_i\}$ and the $\{\vartheta_{\alpha_i}\}$, *not* the KN-points.

Associated with these moduli are the quasi-superconformal ghost fields

$$\begin{aligned} \check{C}_{A_i}(Z) &= z \check{c}_{A_i} \vartheta(C_{A_i}) \quad , \quad i = 2, \dots, g \\ \check{C}_{\vartheta_{\alpha_i}}(Z) &= 2\sqrt{z_{\alpha_i}} \theta \check{\gamma}_{\vartheta_{\alpha_i}} \vartheta(C_{\vartheta_{\alpha_i}}) \quad , \quad i = 1, \dots, g-1 \end{aligned} \quad (8.4)$$

where we have taken $C_{\vartheta_{\alpha_i}} = -C_{A_{i+1}}$ as indicated ($\vartheta(C)$ vanishes *outside* C , $\vartheta(-C)$ vanishes *inside* C) corresponding to the choice, case 1, in eq.(4.6). The i 'th 3-NS-vertex has been subjected to a total dilation by z_{α_i} , whence the factor $\sqrt{z_{\alpha_i}}$ in $\check{C}_{\vartheta_{\alpha_i}}$.

Next we attach the $g-2$ 3-NS-vertices of fig. 2. We choose to sew using propagators with fixed points Z_{α_i} and $(0, 0)$ and multipliers H_i . Figuring out the various associated conformal transformations (using eq.(2.6)) gives

$$\begin{aligned} z_{\beta_i} &= z_{\alpha_i} / [1 - H_i - \sqrt{H_i} \vartheta_{\beta_i} \vartheta_{\alpha_i}] \quad , \quad i = 2, \dots, g-1 \\ \theta_{\beta_i} &= \sqrt{z_{\alpha_i}} (\sqrt{H_i} \vartheta_{\beta_i} + \vartheta_{\alpha_i}) / (1 - H_i) \quad , \quad i = 2, \dots, g-1 \end{aligned} \quad (8.5)$$

where again ϑ_{β_i} are the basic odd moduli of the new 3-NS-vertices.

The associated superconformal ghost fields are found to be (using case 2 in eq.(4.6))

$$\begin{aligned} \check{C}_{H_i}(Z) &= \check{c}_{H_i} \cdot (Z - Z_{\alpha_i}) z / z_{\alpha_i} \vartheta(C_{H_i}) \\ \check{C}_{\vartheta_{\beta_i}}(Z) &= \check{\gamma}_{\vartheta_{\beta_i}} \cdot 2(Z - Z_{\alpha_i}) z_{\alpha_i}^{-3/2} H_i^{-1/2} (\theta z_{\alpha_i} - \theta_{\alpha_i} z) \vartheta(C_{H_i}) \end{aligned} \quad (8.6)$$

with $C_{\vartheta_{\beta_i}} = C_{H_i}$.

For each A_i , H_i , ϑ_{α_i} and ϑ_{β_i} we get the following differential factors associated with the sewings

$$\begin{aligned} d \log A_i \quad , \quad i &= 2, \dots, g \\ d \log H_i \quad , \quad i &= 2, \dots, g-1 \\ d \vartheta_{\alpha_i} \quad , \quad i &= 1, \dots, g-1 \\ d \vartheta_{\beta_i} \quad , \quad i &= 2, \dots, g-1. \end{aligned} \quad (8.7)$$

Thus for the tree we have $(2g-3|2g-3)$ moduli in accordance with the general rule $(N_B+2N_F-3|N_B+N_F-2)$ (sect. 2) for $N_B = 2(g-1)$, $N_F = 1$.

It is now straightforward to transform all these expressions to the W -variable, using eq.(8.1). In particular we find

$$\begin{aligned} W_{\alpha_1} &= (1, -i\vartheta_{\alpha_1} \sqrt{\frac{z_{\alpha_g}}{1-z_{\alpha_g}}}) \quad , \quad W_{\beta_1} = (1 - z_{\alpha_g}, 0) \\ W_{\beta_g} &= (0, 0) \quad , \quad W_{\alpha_g} = (\infty, 0). \end{aligned} \quad (8.8)$$

Thus, in the W -variable, all (tree diagram) fields belong to the space $\mathcal{R}$ (sect. 2). Therefore it will be more convenient to use transformations to that variable in the calculations. We shall not give all the explicit expressions which are, however, easy to work out.

To finish the treatment of the tree diagram we write down the SCKV zero-mode field relevant to the $\mathcal{R}$ -space

$$\check{C}_0(W) = [\check{c}_{-1}w^2 + \check{c}_0w + \check{c}_1 + \phi(\check{\gamma}_{-1/2}w + \check{\gamma}_{1/2} + \check{\gamma}_0\sqrt{w})]\vartheta(C_0). \quad (8.9)$$

Here we have introduced the contour C_0 running around all the physical singularities, or, equivalently, around the spurious singularity, Z_0 (or W_0 in the W -plane). See further below.

8.2 The antighost sewing fields

When we introduce the antighost sewing modes, labelled for each NS-reggeon by $\{b_n^i, \beta_r^i, n = \pm 1, 0, r = \pm 1/2\}$ $i = 1, \dots, g_{NS}$, and for the R-reggeon by $\{b_n^R, \beta_n^R, n = \pm 1, 0\}$, we meet with two different kinds of subtleties.

First, because of the (super-) Riemann-Roch theorem [48,41] there is no superconformal dim 3/2 super-field on the (super-) Riemann sphere (with or without Ramond cuts) that has only those modes, i.e. only such singularities at one single KN-point. Inevitably, if such singular behaviour is imposed, the corresponding B -field will be singular elsewhere. As in the case of the bosonic string [30] and in the pure NS-case [32], we deal with this problem by allowing all "individual" B-reggeon fields to have the necessary spurious singularity at some common point, Z_0 (W_0 in the W plane). The δ -functions in B arising from the $\check{C}$ -zero-modes eq.(8.9) will then precisely make sure that the total singularity here is vanishing. Obviously, for consistency the final result should be independent of Z_0 .

The second subtlety concerns the fact that all fields must be projected to the space $\mathcal{R}$ (in the W variable) corresponding to the present situation in which we have a single R-cut.

It turns out to be very convenient to parametrize the problem in terms of the "full" unprojected sewing fields

$$B_\alpha + P^{-1}B_\beta \text{ or } PB_\alpha + B_\beta \quad (8.10)$$

in which we imagine that the ghost zero-modes in the sewing of that loop have already been made use of to ensure that the fields eq.(8.10) have no spurious singularity, but instead are singular both at Z_α and Z_β but with one single set $\{b_n, \beta_r\}$ of degrees of freedom. We first write down the expressions relevant before projection to $\mathcal{R}$. Afterwards we come back to that projection as well as to the question of the form of the "individual" fields B_α and B_β , then with some spurious singularity.

Thus for the Ramond B -field we simply have

$$\begin{aligned}
(B_{\alpha_g}^R + P_R^{-1} B_{\beta_g}^R)(W) &= \sum_{n=-1}^1 \{ \beta_n^R w^{-n-3/2} + \phi b_n^R w^{-n-2} \} \\
(P_R B_{\alpha_g}^R + B_{\beta_g}^R)(W) &= \sum_{n=-1}^1 \{ \beta_n^R(P_R) w^{-n-3/2} + \phi b_n^R(P_R) w^{-n-2} \} \\
\beta_n^R(P_R) &\equiv K_R^n (\beta_n^R - \kappa_R b_n^R) \\
b_n^R(P_R) &\equiv K_R^n (b_n^R - \kappa_R n \beta_n^R)
\end{aligned} \tag{8.11}$$

and these fields already belong to $\mathcal{R}$.

For the NS-fields $B_{\alpha_i}, B_{\beta_i}, i = 1, \dots, g_{NS}$ it is similarly trivial to write down expressions if $W_{\alpha_i} = (0, 0)$ and $W_{\beta_i} = (\infty, 0)$. By the superprojective transformation eq.(2.6) we then get in the general case

$$\begin{aligned}
(B_{\alpha_i} + P_i^{-1} B_{\beta_i})(Z) &= \sum_{r=-1/2}^{1/2} \beta_r^i (Z - Z_{\alpha_i})^{-r-3/2} [(Z - Z_{\beta_i}) / (Z_{\alpha_i} - Z_{\beta_i})]^{r-3/2} \\
&+ \{ (\theta - \theta_{\alpha_i}) - (Z - Z_{\alpha_i}) \frac{(\theta_{\alpha_i} - \theta_{\beta_i})}{(Z_{\alpha_i} - Z_{\beta_i})} \} \\
&\cdot \sum_{n=-1}^1 b_n^i (Z - Z_{\alpha_i})^{-n-2} [(Z - Z_{\beta_i}) / (Z_{\alpha_i} - Z_{\beta_i})]^{n-2} \\
i &= 1, \dots, g_{NS} = g - 1
\end{aligned} \tag{8.12}$$

and the field $(P_i B_{\alpha_i} + B_{\beta_i})(Z)$ looks the same with $\beta_r^i \rightarrow K_i^r \beta_r^i, b_n^i \rightarrow K_i^n b_n^i$. Here we have the convenient situation, that in the W -plane these NS-fields look exactly the same with $Z \rightarrow W$ throughout (after a trivial change of variables with (super-) Jacobian 1).

From the generic fields $(B_\alpha + P^{-1} B_\beta)$ and $(P B_\alpha + B_\beta)$ given above in eq.(8.11) and eq.(8.12), we project onto $\mathcal{R}$ and construct "individual" reggeon fields $(B_\alpha)_{W_0}^R, (B_\beta)_{W_0}^R$ with a spurious singularity at W_0 as follows

$$\begin{aligned}
(B_\alpha)_{W_0}^R(W) &= \oint_{W_\alpha} dW' p(W, W'; W_0) (B_\alpha + P^{-1} B_\beta)(W') \\
(B_\beta)_{W_0}^R(W) &= \oint_{W_\beta} dW' p(W, W'; W_0) (P B_\alpha + B_\beta)(W') \\
p(W, W'; W_0) &= \left(\frac{w' - w_0}{w - w_0} \right)^3 \frac{1}{w - w'} \{ -\phi + \sqrt{\frac{w}{w'}} \phi' \}.
\end{aligned} \tag{8.13}$$

Here we have chosen a $W_0 = (w_0, \phi_0) = (w_0, 0)$. One easily verifies that (i) these fields lie in $\mathcal{R}$, (ii) $(B_\alpha)_{W_0}^R$ ($(B_\beta)_{W_0}^R$) have the required singularities near W_α (W_β) but are regular at W_β (W_α) and have a spurious singularity at W_0 .

As we shall see below, explicit evaluation of these projections can be completely circumvented.

8.3 The loop sewing overlaps associated with $\mathcal{C}_i$

Associated with each loop sewing is a contour, $\mathcal{C}_i$, $i = 1, \dots, g$, and, in the covering space, the contours $\mathcal{C}'_i$, images of $\mathcal{C}_i$ under P_i . There are quasi-superconformal fields $\check{C}_i(Z)$ discontinuous across $\mathcal{C}_i$, as well as the factors

$$d\kappa_R d\log K_R \prod_{i=1}^{g_{NS}} d\log K_i \quad (8.14)$$

($K_R \equiv K_g$). It is easy to work out the overlaps $(B_{\mathcal{C}_i}|\check{C}_i)$ for two reasons: (i) since $\check{C}_i$ vanishes outside $\mathcal{C}_i$ we need only worry about B_{α_i} in the overlap, in fact, any field with the singularities of B_{α_i} (in the neighbourhood of Z_{α_i}) will do. Thus the projection onto $\mathcal{R}$ becomes irrelevant and we may in fact use directly the fields $(B_{\alpha_i} + P^{-1}B_{\beta_i})$; (ii) this overlap is independent of which variable is used for its evaluation, thus we may use variables for which $Z_{\alpha_i} = (0, 0)$ and $Z_{\beta_i} = (\infty, 0)$, so that in fact the $\check{C}_i$ are given by eq.(5.5) directly. The integrals over $\check{c}_i$ ($\equiv \check{c}_{K_i}$), $i = 1, \dots, g_{NS}$ will produce δ -functions in the modes b_0^i with a factor 1, and integrating over $\check{c}_R$ and $\check{\gamma}_{\kappa_R}$ will give δ -functions in b_0^R and β_0^R and a factor 1/2. Thus in the sequel we may forget all those modes.

8.4 The $\mathcal{R}$ -projection

Consider the generic overlap

$$\begin{aligned} ((B_\alpha)_{W_0}^R|\check{C}) &= \oint_{\mathcal{C}} dW (B_\alpha)_{W_0}^R \check{C}(W) \\ &= \oint_{\mathcal{C}} dW \oint_{Z_\alpha} dW' p(W', W; W_0) B_\alpha(W') \check{C}(W) \\ &\equiv \oint_{Z_\alpha} dW' B_\alpha(W') \check{C}_{W_0}^R(W') \\ \check{C}_{W_0}^R(W') &\equiv \oint_{\mathcal{C}} dW \check{C}(W) p(W, W'; W_0) \end{aligned} \quad (8.15)$$

where $B_\alpha(W)$ denotes any field with the correct singularities near $Z = Z_\alpha$ (in particular we might use $(B_\alpha + P^{-1}B_\beta)$). Now $p(W, W'; W_0)$ is singular at $W = W'$ and $W = W_0$ and has a cut from $(0, 0)$ to $(\infty, 0)$ eq.(8.13). However, often we shall meet cases where $\mathcal{C}$ does not enclose W_0 (see fig. 3) and where the cut complication drops out (see below). Then we simply have

$\check{C}_{W_0}^R(W) = \check{C}(W)$, and the unpleasant complications of the $\mathcal{R}$ projection drop out – this is another way of justifying the treatment of the previous subsection. In fact the only exception will be in the overlaps with $\check{C}_0$.

Indeed, considering fig. 3 we notice that the $\mathcal{R}$ projection will be irrelevant for the contours $\mathcal{C}_{H_i}$ and $\mathcal{C}_{\vartheta_{\alpha_i}}$. Further, for the contour $\mathcal{C}_{A_i}$, $i = 2, \dots, g_{NS}$ we first deform that to a union of contours

$$\mathcal{C}_{A_i} \sim \mathcal{C}_{H_i} \cup \mathcal{C}_{H_{i+1}} \cup \dots \cup \mathcal{C}_{H_{g_{NS}}} \cup \mathcal{C}_{A_g}. \quad (8.16)$$

But if we imagine evaluating the overlap associated with $\mathcal{C}_{A_g}$ first, integrating over $\check{C}_{A_g}$ to provide a δ -function in B -fields, then similarly treating $\mathcal{C}_{H_{g-1}}$, $\mathcal{C}_{A_{g-1}}$ etc., we see that in the above decomposition of $\mathcal{C}_{A_i}$, only $\mathcal{C}_{H_i}$ will provide a non-vanishing contribution. This is because the functional form of all the $\check{C}_{A_i}$'s are the same. A similar argument applies to the $\mathcal{C}_{\vartheta_{\alpha_i}}$ (thinking of the order of evaluation as being reversed).

8.5 The remaining $(B|\check{C})$ overlaps

Using the above with $\check{C}_0$ given by eq.(8.9) we evaluate

$$\begin{aligned} (\check{C}_0)_{W_0}^R(W) &= \check{c}_{-1}w^2 + \check{c}_0w + \check{c}_1 + \phi\check{\gamma}_0^R(w) \\ \check{\gamma}_0^R(w) &\equiv \sum_{n=-1}^1 \check{\gamma}_n^R w^{-n+1/2} \\ \check{\gamma}_{-1}^R &\equiv \frac{3}{8}w_0^{-1/2}\check{\gamma}_{-1/2} - \frac{1}{8}w_0^{-3/2}\check{\gamma}_{1/2} \\ \check{\gamma}_0^R &\equiv \frac{3}{4}w_0^{1/2}\check{\gamma}_{-1/2} + \frac{3}{4}w_0^{-1/2}\check{\gamma}_{1/2} + \check{\gamma}_0 \\ \check{\gamma}_1^R &\equiv -\frac{1}{8}w_0^{3/2}\check{\gamma}_{-1/2} + \frac{3}{8}w_0^{1/2}\check{\gamma}_{1/2}. \end{aligned} \quad (8.17)$$

This shows that we may change variables

$$(\check{\gamma}_{-1/2}, \check{\gamma}_0, \check{\gamma}_{1/2}) \rightarrow (\check{\gamma}_{-1}^R, \check{\gamma}_0^R, \check{\gamma}_{+1}^R) \quad (8.18)$$

with Jacobian $= 1/8$. This we shall do in the following, thereby explicitly achieving independence of the position, W_0 , for the spurious singularity, and otherwise significantly simplifying the computation.

It is now a completely straightforward exercise in super-contour integration and super-determinant evaluation to complete the calculation. After all b_0 and β_0 modes have been gotten rid of, the calculation proceeds as follows:

The Ramond antighost d.o.f.'s $b_{\pm 1}^R$, $\beta_{\pm 1}^R$ couple to $\check{C}_{A_g}$ and $\check{C}_0$. From

$$(B_{\alpha_g}^R|\check{C}_{A_g}) = -b_1^R\check{c}_{A_g}(1 - z_{\alpha_g})^{-1} \quad (8.19)$$

and

$$((B_{\alpha_g}^R + P_R^{-1}B_{\beta_g}^R)|\check{C}_{A_g}) \equiv 0 \quad (8.20)$$

we immediately write down

$$(B_{\alpha_g}^R + B_{\beta_g}^R | \check{C}_{A_g}) = [(K_R - 1)b_1^R + K_R \kappa_R \beta_1^R] \check{c}_{A_g} (1 - z_{\alpha_g})^{-1}. \quad (8.21)$$

This technique is used throughout. Integrating over $b_{\pm 1}^R$, $\beta_{\pm 1}^R$ (it is convenient to perform a linear change of variables) one finds a factor of 1 together with δ -functions in $\check{c}_1$, $\check{\gamma}_{\pm 1}^R$. These may subsequently be ignored. Thus the contribution from the Ramond-loop is simply

$$4 \frac{(1 - K_R)^2}{(1 - K_R)^2} \frac{dK_R}{K_R} \frac{dA_g}{A_g} d\kappa_R \frac{1}{(1 - z_{\alpha_g})}. \quad (8.22)$$

The factor $1 = (1 - K_R)^2 / (1 - K_R)^2$ is to emphasize the difference to the NS-loop contributions below.

The remaining antighost d.o.f.'s from the i 'th NS loop, $b_{\pm 1}^i$, $\beta_{\pm 1/2}^i$ are taken care of by considering the overlaps along $C_{\vartheta_{\alpha_i}}$, $C_{\vartheta_{\beta_i}}$, C_{H_i} , C_{A_i} – all deformed to coincide. The result of integrating over $\{b_{\pm 1}^i, \beta_{\pm 1/2}^i, \check{c}_{A_i}, \check{c}_{H_i}, \check{\gamma}_{\vartheta_{\alpha_i}}, \check{\gamma}_{\vartheta_{\beta_i}}\}$ is

$$4^{-1} \frac{(1 - K_i)^2}{(1 - K_i^{1/2})^2} \frac{dK_i}{K_i^{3/2}} \frac{dA_i}{A_i} \frac{dH_i}{H_i^{3/2}} d\vartheta_{\alpha_i} d\vartheta_{\beta_i}, \quad i = 2, \dots, g_{NS} \quad (8.23)$$

(using $z_{\beta_i} / (Z_{\alpha_i} - Z_{\beta_i}) = H_i^{-1}$, cf. eq.(8.5)).

By our particular coordinate choice, NS loop no. 1 (the last in the calculation) is special. It is the only one overlapping with the remaining $\check{C}_0$ modes, thereby feeling the presence of the R-cut. The contribution turns out to be

$$8 \frac{\sqrt{1 - z_{\alpha_g}}}{2 - z_{\alpha_g}} \frac{dK_1}{K_1^{3/2}} \frac{(1 - K_1)^2}{(1 - K_1^{1/2})^2} \frac{1}{\sqrt{z_{\alpha_g}}} d\vartheta_{\alpha_1}. \quad (8.24)$$

Collecting all factors we finally find for the ghost zero-mode contribution in these coordinates to the $g_R = 1$ part of the vacuum amplitude:

$$\begin{aligned} & 32 \cdot 4^{-g_{NS}} \frac{\sqrt{1 - z_{\alpha_g}}}{2 - z_{\alpha_g}} \frac{dK_R}{K_R} d\kappa_R \\ & \cdot \prod_{i=1}^{g_{NS}} \frac{(1 - K_i)^2}{(1 - K_i^{1/2})^2} \frac{dK_i}{K_i^{3/2}} \prod_{i=2}^g \frac{dA_i}{A_i^{3/2}} \prod_{i=2}^{g_{NS}} \frac{dH_i}{H_i^{3/2}} \\ & \cdot d\vartheta_{\alpha_1} \prod_{i=2}^{g_{NS}} d\vartheta_{\alpha_i} d\vartheta_{\beta_i} \\ & z_{\alpha_g} \equiv \prod_{j=2}^g A_j \end{aligned} \quad (8.25)$$

This calculation works in a coordinate patch around the pinching limit $A_i, H_i, K_i \rightarrow 0$. One observes the expected singular behaviour in that

limit. The factors on the first line are associated with the Ramond loop. Apart from those this calculation agrees with the pure NS-result sect. 6. The result there for $d\mu_g$ eq.(6.11) is essentially that, after the (super-) Jacobian for the transformation to (super-) KN variables has been worked out and inserted. In the present case too, one might similarly change to super-KN variables, however, it would be less trivial to express the final answer in a way that manifestly expresses the symmetries of the problem: $OSp(2|1)$ and super-dilation invariance (cf. sect. 2). In contrast, the (moduli|supermoduli) employed here in eq.(8.25) are all manifestly invariant under these symmetries.

9 Conclusion

A method [32] based on sewing reggeons, for building superstring loop densities on supermoduli-space (in a super-Schottky parametrization) has been explained and investigated. It generalizes the one [30] developed and checked [29] for the bosonic string. The most general superloop amplitudes have not yet been dealt with in all detail. However, the treatment is completed for (i) the most general N -string g loop problem involving NS-strings only (sect. 6 eq.(6.6), eq.(6.11), eq.(6.13)); (ii) the non-zero-mode part of the most general vacuum amplitude with an arbitrary number of Ramond-loops, g_R , and NS-loops, g_{NS} ; (iii) the ghost zero-mode (i.e. measure) part for the case $g_R = 1$, g_{NS} arbitrary. Ramond strings introduce essential complications, however, experience gained so far indicates that they remain tractable or can be circumvented.

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Figure Captions

Figure 1: A particular construction of the tree with 2 R-strings (connected by a dashed line) at KN-points Z_{α_g} and Z_{β_g} , and $g_{NS} + 1 = g$ NS-strings. Sewing-contours $\mathcal{C}_{A_i}$ and $\mathcal{C}_{\vartheta_{\alpha_i}}$ are indicated.

Figure 2: The tree fig. 1, after $g - 2$ additional 3-NS-vertices have been attached to the branches of fig. 1 by NS-sewings along contours $\mathcal{C}_{H_i}$ ($= \mathcal{C}_{\vartheta_{\beta_i}}$). The loop amplitude – a tree with tadpoles – is obtained by sewing strings α_i and β_i together, along sewing contours $\mathcal{C}_i$ (to be identified with $\mathcal{C}'_i$).

Figure 3: The Z -plane picture corresponding to fig. 2. KN-points and sewing contours are indicated. The Ramond-cut connecting Z_{α_g} , Z_{β_g} is shown. So is the spurious singularity at Z_0 .

Fig. 1

Fig. 2

Fig. 3