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On the Interpretation of Minimal Length in String Theories

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ABSTRACT

The existence of a minimal observable length in fundamental string theories has been suggested from recent investigations of the string amplitudes under various extreme conditions. Several observations about the interpretation of this phenomenon are presented. In particular, a new kind of dual indeterminacy relation is suggested as a possible basis of quantum string geometry.

encountering the boundaries of the parameter space. Because of the symmetries with respect to the exchange of external lines, the parameters describing the string propagation in different channels correspond to different choices of the time direction on the world sheet are connected by discrete transformations (called duality transformations) which preserve the form of the integrand apart from the exchange of the external lines. For instance, take the integrand of the 4-point Veneziano amplitude.

$$I(x) = x^{-\alpha(t)-1}(1-x)^{-\alpha(t)-1}. \quad (1)$$

The proper time parameters associated with the s and t channels are naturally chosen to be $t_1 = -\lambda \log x$ and $t_2 = -\lambda \log(1-x)$, respectively. The duality transformation is $x \rightarrow 1-x$. Evidently, one of the t_1 and t_2 satisfies the inequality

$$t_i \geq \lambda \log 2, \quad (2)$$

which can be interpreted as an indication of the minimal propagation length.

In this argument, the identification of the proper time in each channel is not uniquely determined. The point is that there is a direction of time on the world sheet in which the proper length measured along the chosen direction of time cannot become arbitrarily small, *provided* only natural sets of the choices of the proper times are allowed. Here, *natura* means that the boundaries (at 0 and ∞ by definition) of the range of a proper time should be only on the boundaries of the moduli space, and that the infinite value of the proper time for a particular choice of the time direction should correspond to the region where the pole singularity in the corresponding channel is generated. For instance, in the Veneziano case, the choice $t_1 = -\lambda \log 2x$, $t_2 = -\lambda \log 2(1-x)$ is unnatural according to this criterion. The definition of the interaction time in string field theory with a particular gauge is also unnatural. This seems reasonable if one remembers that the concept of interaction in string theory is a gauge artifact and that the crucial

symmetry and finiteness properties of the theory are not manifest in string field theories. In case of the Virasoro-Shapiro amplitude, an appropriate choice of the proper time variable corresponding to the s -channel is, say,

$$t_1 = -\lambda \log(|x||1 + |1-x|^{-1}|/(1+|x|+|1-x|^{-1})).$$

Of course, it will be nice if it is possible to formulate the existence of minimal length independently of the choice of the time coordinate on the world sheet. One of the purposes of the present paper is to suggest a direction towards a realization of this hope. Before turning to this subject, it is noted that our qualitative arguments conform to the behavior of the string amplitudes in the high-energy fixed-angle limit [1].

The essence of the exponential falloff of the string amplitudes in arbitrary string-loop order under the limit is that the integrals over the moduli parameters are dominated by saddle points which are not on the boundary of the moduli space. If one follows our criterion for the choice of the proper times of the string propagation, it is clear that the value of the proper times at the saddle points can never approach zero for any choice of the time direction. In contrast with this, the high-energy fixed momentum-transfer scattering [6] is dominated by the boundary of the moduli space at which at least one of the proper times of string propagation becomes infinite, corresponding to the exchange of soft gravitons.

2. Extremal length and a dual indeterminacy relation.

Our next task is to present the concept of the minimal proper length in a coordinate independent fashion. The above arguments lead us to ask whether it is possible to define a conformally invariant length on general Riemann surfaces. I propose that what mathematicians call extremal length is an appropriate notion here. Let us first define the extremal length following the mathematical literature [7].

Let Γ be a family of curves which are in a region Ω of a Riemann surface. Consider the whole family of the metrics ρ^2 on the Riemann surface. Denote the

It seems important to recognize that the string theory started a completely new and unified way of understanding the nature of particle interactions and symmetries, which is perhaps beyond the framework of quantum field theories (QFT). The departure of the string theory from QFT is firstly signified by the duality property of the scattering amplitudes and a natural realization of (ultraviolet) finiteness of the perturbation theory. It also manifests itself in drastically different behaviors of the high-energy fixed-angle limit [1] and the high-temperature limit [2] of the string amplitudes from those of QFT. These properties are qualitatively natural in view of intrinsic softness of the string and indicate that the true degrees of freedom of the string theory at short distance regime ($\sim$ Planck length) are vastly smaller than those in QFT. This also suggests that there exists a minimal length below which the spacetime structure has no observable meaning.

It must have been discussed in innumerable occasions that in any quantum theory of gravity, the Planck length puts a limitation [3] on the smallest length scale of arbitrary measurement. What is lacking in the past discussions is perhaps a knowledge about how this limitation is reflected in observable quantities. The properties mentioned above of the string amplitudes now provide us a concrete example. Thus it seems likely that giving a correct interpretation to the departure of the string theory from QFT is a crucial first step towards the nonperturbative and geometrical formulation of the string theory.

The purposes of this paper are to clarify why the conformally invariant string theory has a minimal length and to propose to interpret this as a consequence of a new kind of dual indeterminacy relation.* We have at hand a set of rules for constructing the amplitudes in a perturbative fashion. The perturbation series do not converge, owing to a rapid (factorial) increase of the volume of the moduli space for increasing genus [5]. Thus, as in QFT, the set of the perturbative rules cannot be taken as the definition of the string theory. Nevertheless, we

* See also previous discussions of the present author [4].

may expect that the key to the theory is buried in some disguise in the structure, especially, the symmetry structure of the perturbative rules. From this viewpoint, it seems appropriate to ask the meaning of the world-sheet conformal invariance, duality and modular invariance. Let us start from that. The first part of the paper is meant for motivation to the remaining discussions.

1. Modular invariance, duality, and minimal length.

The perturbative string amplitudes are expressed in the form of integral representations whose integration parameters are conformal invariants (i. e., moduli) characterizing the world sheet. Consequently, any spacetime properties of the amplitudes should be expressible in terms of the conformal invariants. For instance, the modulus $|\tau|$ of the familiar Teichmüller parameter τ of the $g = 1$ compact Riemann surface (torus) is interpreted as the $\lambda^{-2}(\text{proper time})^2$ (or the Schwinger parameter) for the self-closed propagation of a closed string, where λ is a constant with the dimension of length. It is a common sense that if the modular invariance is valid the range of τ can be restricted in the fundamental region $\mathcal{F} = \{\tau : |\tau| \geq 1, -1/2 \leq \text{Re}\tau \leq 1/2\}$.

This property, which is one of the crucial prerequisites of the string theory, is equivalent to the fact that for an arbitrary value of the Teichmüller parameter there is always a choice of the direction of the string propagation along a closed curve such that the proper time of the propagation can never be smaller than a finite minimal value which is proportional to λ . It is also clear from the open-closed string duality that the same property concerning the existence of the minimal proper length holds for open string one-loop amplitudes, although in this case there is no modular invariance.

The duality property of the string-tree amplitudes can also be interpreted as being intimately related to the existence of minimal proper length. The crux of the duality is the following. The positions of the singularities of the integrand of the amplitudes corresponding to the propagation of the string in different channels are connected to each other by paths in the parameter space without

flat spacetime. Consider the transition amplitude corresponding to a rectangle with the sides a and b , with a Dirichlet boundary condition

$$x^\mu(0, \xi_2) = x^\mu(a, \xi_2) = \delta^{\mu 2} B \xi_2 / b,$$

$$x^\mu(\xi_1, 0) = x^\mu(\xi_1, b) = \delta^{\mu 1} A \xi_1 / a,$$

where the range of the coordinates (ξ_1, ξ_2) is in the rectangle with the metric $\rho = 1$. Then, the path integral $\int [dx] \exp -\frac{1}{2\alpha'} \int d^2 \xi \sqrt{g} g^{ab} \partial_a x \partial_b x^*$ contains the factor

$$\exp -\frac{1}{\lambda^2} \left[A^2 \left(\frac{b}{a} \right) + B^2 \left(\frac{a}{b} \right) \right].$$

Thus, the indeterminacies of A and B are given as

$$(\Delta A)^2 \sim \lambda^2 \Lambda(\Gamma_a^*), (\Delta B)^2 \sim \lambda^2 \Lambda(\Gamma_b^*). \quad (10)$$

It is noted that (10) together with the boundary condition is consistent with the Virasoro constraints $\langle \dot{x}^2 \rangle = \langle \dot{x}'^2 \rangle = 0$ at the edges of the rectangle.

The relations (8) and (7) express the existence of the minimal length. (8) says that there is always a choice of the direction of propagation such that the proper time is not smaller than a finite minimum value. (7) suggests that this should rather be interpreted as a consequence of a general dual indeterminacy relation between the proper time of propagation and the proper extendedness of the propagating state.

$$\Delta A \Delta B \sim \lambda^2. \quad (11)$$

If α shrinks to a point, (10) shows that $\Delta A \rightarrow 0$ and $\Delta B \rightarrow \infty$. Does this mean the extendedness of the string can be neglected if it propagates for infinitely long time? This would seem strange in view of the fact that the naive

* Such a path integral with Dirichlet boundary condition, in general, contains anomaly induced by the edge effect under conformal transformation. However, the net effect of the anomaly should always cancel after many of these integrals are convoluted to give physical amplitudes.

extendedness of a free on-mass shell state of a string is apparently nonzero. It should be stressed however that there is in fact no observable measure of the string extendedness in spacetime other than the S -matrix in string theories. If the string could be probed by a point particle, the extension would be logarithmically infinite. Instead I suggest the following interpretation. In string theories, the physical objects associated with a point on a Riemann surface are the vertex operators (especially, the massless vertex operators since the massive states are unstable) which represent asymptotic scattering states with infinite propagation. That these massless vertex operators are elegantly represented by the ordinary background local fields, such as the spacetime metric, antisymmetric field and scalar dilaton, in the Polyakov action is one of the great mysteries of fundamental strings. Because of this property, the string graviton couples with string through its spacetime energy-momentum tensor (Equivalence Principle) and the string theory reduces to General Relativity at low energies. (9) may be taken to signify this important correspondence between string theory and the local field theory. It is noted that (11) is based on the proper length and hence in principle the dual indeterminacy relation applies to any background spacetime if it is compatible with conformal invariance.

In general scattering experiments it is difficult to clearly separate the effects of the propagation and the extendedness. Then the length scale-probed in the scattering experiments should be an average of ΔA and ΔB ,

$$\Delta X \sim \Delta A + \Delta B. \quad (12)$$

If ΔB being regarded as the propagation length is replaced by the inverse of the energy scale $\Delta B \sim \hbar/E$, (12) reduces to the form suggested from the investigations of the high-energy limit of the string amplitude [8] [9]. Note that λ is related to the Newton constant G_N and the dimensionless coupling constant g by

$$\lambda^2 = \frac{G_N \hbar}{g^2}. \quad (13)$$

length of a curve γ and the area of Ω with respect to a given metric as

$$L(\gamma, \rho) = \int_{\gamma} \rho |dz|, \quad (3)$$

$$A(\Omega, \rho) = \int_{\Omega} \rho^2 dz d\bar{z}. \quad (4)$$

Then, the extremal length of Γ , which is a conformal invariant, is defined as

$$\Lambda(\Gamma) = \sup_{\rho} \frac{L(\Gamma, \rho)^2}{A(\Omega, \rho)} \quad (5)$$

where $L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho)$. In the left hand side of (5), the subscript Ω is dropped because it actually depends only on Γ . As a simplest example, let us take Ω to be an arbitrary quadrilateral, two of whose different pairs of opposite sides are (α, α') and (β, β') . There are two natural choices of Γ : $\Gamma_{\alpha} = \{\gamma \text{ connecting } \alpha \text{ and } \alpha'\}$ and $\Gamma_{\beta} = \{\gamma \text{ connecting } \beta \text{ and } \beta'\}$. The extremal distances between α and α' , and β and β' are respectively

$$\Lambda(\Gamma_{\alpha}) = \frac{b}{a}, \quad \Lambda(\Gamma_{\beta}) = \frac{a}{b}, \quad (6)$$

where a and b are the lengths (with metric $\rho = 1$) of the sides α and β , respectively, of the rectangle which is conformally equivalent to the quadrilateral. (6) is natural as the length on a Riemann surface, because for a straight strip domain (or ring domain) with constant width (or circumference), it coincides with the Euclidean length of the domain, corresponding to the Schwinger parameter of the string propagator. It should be stressed that the notion of the extremal length can be defined for an arbitrary set of curves on a Riemann surface. For other interesting examples, see [7].

There is also the concept of the conjugate extremal length. If α and α' are two sets in the boundary of an arbitrary region Ω and $\Gamma_{\alpha} = \{\gamma \text{ connecting } \alpha$

and $\alpha'\}$, let $\Gamma_{\alpha}^* = \{\gamma \text{ separating } \alpha \text{ and } \alpha'\}$. Then, $\Lambda(\Gamma_{\alpha}^*)$ is called the conjugate extremal length of α and α' . In the example of the quadrilateral, $\Gamma_{\alpha}^* = \Gamma_{\beta}$. Hence,

$$\Lambda(\Gamma_{\alpha})\Lambda(\Gamma_{\alpha}^*) = 1. \quad (7)$$

It is known that (7) is generally valid if Γ is sufficiently regular. Other important general properties of the extremal length are the followings: (i) If every $\gamma \in \Gamma$ contains a $\gamma_1 \in \Gamma_1$ and a $\gamma_2 \in \Gamma_2$,

$$\Lambda(\Gamma) \geq \Lambda(\Gamma_1) + \Lambda(\Gamma_2);$$

(ii) If every $\gamma_1 \in \Gamma_1$ and every $\gamma_2 \in \Gamma_2$ contain a $\gamma \in \Gamma$,

$$\Lambda(\Gamma)^{-1} \geq \Lambda(\Gamma_1)^{-1} + \Lambda(\Gamma_2)^{-1},$$

where Γ_1 and Γ_2 consist of arcs in mutually disjoint regions.

From these properties it follows that for any pair of one-dimensional domains α and α' , either $\Lambda(\Gamma_{\alpha})$ or $\Lambda(\Gamma_{\alpha}^*)$ satisfies the inequality

$$\Lambda(\Gamma_{\alpha}) \geq 1, \text{ or } \Lambda(\Gamma_{\alpha}^*) \geq 1. \quad (8)$$

Furthermore, if α (or α') shrinks to a point,

$$\Lambda(\Gamma_{\alpha}) \rightarrow \infty, \quad \Lambda(\Gamma_{\alpha}^*) \rightarrow 0. \quad (9)$$

Namely, the distance from a point to an arbitrary domain on a Riemann surface is always infinite and the corresponding conjugate extremal length is zero.

If we regard the extremal length $\Lambda(\Gamma_{\alpha})$ as the (proptime)² of the propagation of α to α' , the conjugate extremal length should be interpreted as the (proper extendedness)² of the propagating state. That this is a reasonable interpretation can be seen by a simple observation on the Polyakov path integral in

The first term of (12) which is then proportional to the Schwarzschild radius may be related with the gravitational instability.

3. Possible nature of quantum string geometry.

Are our observations relevant to possible nonperturbative and geometric formulation of the string theory? I will finally suggest a simple reason why (11) might be important. The presently familiar view on the string theory is either that the string field theory is the classical theory whose ordinary quantization yields the quantum string theory, or that the set of 2 dimensional conformal field theories is the space of classical solutions to the quantum string theory. Two views may be combined by identifying the space of the classical solutions of the string field equation with the space of 2 dimensional CFTs.

Let us remember that in the unit $c = 1$ the natural constants of the string theory are only the $\hbar$ and the λ . The dimensionless coupling constant g^2 is determined by the ground-state expectation value of the dilaton field. Instead of λ , we can adopt the Newton constant using the relation (13). If the g^2 is assumed to be finite in the classical limit, (13) shows that the string extendedness is a quantum effect, as has been evident from the time when the Planck length was related to the Regge slope parameter. From this viewpoint, the existence of *classical* string field theory is unnatural since it already contains the length parameter. The string extension and the string loop effect should come out simultaneously. In this situation, the λ may be more fundamental than the Planck constant. Namely, we can take G_N and λ as the independent physical constants. Then, the relation such as (11) seems strikingly suggestive. Some sort of spacetime quantization might be the correct direction. The world-sheet conformal invariance might be understood as a secondary property, appearing in perturbation theory in order to realize the dual indeterminacy relation. To find a suitable mathematical and conceptual framework for doing this is a great challenge.

REFERENCES

- [1] D. J. Gross and P. E. Mende, Nucl. Phys. B303(1988)407.
- [2] J. Atick and E. Witten, preprint IASSNS-HEP-88/14(1988).
- [3] See, e. g., G. 'tHooft, in Symposium on anomalies, geometry, topology (Eds. W. A. Bardeen and A. R. White, World Scientific, Singapore, 1985), p. 33 and references therein.
- [4] T. Yoneya, in *Wandering in the fields* (Eds. K. Kawarabayashi and A. Ukawa, World Scientific, Singapore, 1987), p. 419; also in *Quantum string theory* (Eds. N. Kawamoto and T. Kugo, Springer, Berlin, 1988), p. 23.
- [5] D. J. Gross and V. Periwal, Phys. Rev. Lett. 60 (1988)2105.
- [6] M. Soldate, Phys. Lett. 186B(1987)321.
- [7] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. 197B(1987)81.
- [7] See, e. g., M. Otsuka, *Dirichlet problem, extremal length and prime ends* (Van Nostrand, New York, 1970);
L. V. Ahlfors, *Conformal invariants* (McGraw-Hill, New York, 1973).
- [8] D. J. Gross, preprint PUPT-1108(1988).
- [9] D. Amati, M. Ciafaloni and G. Veneziano, preprint CERN-TH 5209/88 (1988).