CERN-TH.5299/89

MULTICRITICAL POINTS IN TWO DIMENSIONS,
THE RENORMALIZATION GROUP AND THE ϵ EXPANSION

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ABSTRACT

We calculate, using the ϵ expansion, the critical exponents for the two-dimensional potential ϕ^{2n} . It is found that the results at lowest perturbative order are not in agreement with what would be expected from the proposed correspondence between such Landau-Ginsburg Hamiltonians and minimal conformal field theories.

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CERN-TH.5299/89

February 1989

One of the most beautiful developments in recent years has been the application of the renormalization group to the theory of critical phenomena [1]. An important technique used within this approach is the ϵ expansion [2] which has had considerable success in predicting critical exponents in three dimensions starting from the perturbation theory in four dimensions (i.e., $\epsilon = 4-d$). The field theoretic model is based upon a Landau-Ginsburg Hamiltonian with quartic interaction. By making a double expansion in g and ϵ , one controls the infra-red divergences which are logarithmic in $d = 4$ and worse than logarithmic in $d < 4$. This technique enables one to compute not only the fixed point, but also anomalous dimensions and correlation functions in the critical theory.

The situation described above applies to systems in three dimensions which have a second-order phase transition with two phases. A similar approach has been tried for multicritical phenomena in two dimensions. It has been proposed [3-5] that multicritical behaviour is described by the Hamiltonian

$$\mathcal{H} = \int d^2x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{(2n)!} \phi^{2n} \right\} \quad (1)$$

which corresponds to a system in which n phases can coexist at the multicritical point.

There are a number of features which make this assumption very natural; however, it is desirable to have some concrete demonstration of this identification. The first step to establishing this would be to locate the infra-red stable fixed point g^* at which the above Hamiltonian describes a conformally invariant [i.e., $\beta(g) = 0$] system. This point in coupling space is presumably outside the range of perturbation theory in two dimensions about $g = 0$. A second step would be to compute the anomalous dimension of ϕ^m and other operators at the fixed point. Further calculations could evaluate the operator product expansion and Green functions at the critical point.

Zamolodchikov [5] has proposed an explicit identification of some of the primary fields of the minimal model with central charge

$$C = 1 - \frac{6}{(n+1)(n+2)}$$

with operators in the above Landau-Ginsburg model. In particular, he has suggested that

$$:\phi^m: \approx \begin{cases} \phi_{m+1, m+1} & \text{for } m=0, 1, \dots, n-1 \\ \phi_{m-n+1, m-n+2} & \text{for } m=n, \dots, 2n-2 \end{cases} \quad (2)$$

Since we know [6] the anomalous dimensions of all the primary fields, we should find the anomalous dimension of ϕ^m to be

$$\gamma_{\phi^m} = \begin{cases} \frac{(m+1)^2 - 1}{2(n+1)(n+2)} & m=0, 1, \dots, n-1 \\ \frac{(m+3)^2 - 1}{2(n+1)(n+2)} & m=n, \dots, 2n-2. \end{cases} \quad (3)$$

The ϵ -expansion would seem to be ideally suited for establishing the connections given above. One starts [3,4] with the Hamiltonian Eq. (1) in its critical dimension $d_c = 2 + 2/(n-1)$, where the coupling g is dimensionless and the infra-red divergences in, for example, the susceptibility just become logarithmic. In this dimension, the two fixed points coincide at $g = 0$. However, below d_c for $\epsilon = d - d_c$ the β function has the generic form $\beta(g) \sim -\epsilon g + g^2$ and the infra-red stable point of interest is at $g^* \sim \epsilon \sim 1/n$ for large n .

In two dimensions, all couplings in the potential have naive dimension 2 in perturbation theory around $g = 0$. However, in the ϵ -expansion the naive dimensions (i.e, those at zeroth order in g and 2 in perturbation theory) are those occurring in $d = d_c$. The coupling for ϕ^m has naive value $d_c - (m/2)(d_c - 2)$ and so one naturally finds that ϕ^m is a relevant operator for $m < 2n$, a marginal operator for $m = 2n$ and an irrelevant operator for $m > 2n$. If one includes all even powers of ϕ up to ϕ^{2n} in the Hamiltonian, there will be n relevant operators (including ϕ itself) corresponding to n -critical behaviour. In particular, it might seem that the anomalous dimension of ϕ is for n large of order

$$\gamma_{\phi} \sim g_*^2 \sim \epsilon^2 \sim \frac{1}{n^2} \quad (4)$$

in agreement with the identification of Eq. (3).

Another advantage of the ϵ expansion is the infra-red behaviour of the propagators. In two dimensions exactly, the propagator associated to the $g = 0$ perturbation expansion is logarithmic in x -space. As such, in perturbation expansion about $g = 0$, one encounters off-shell infra-red divergences which are divergences in the Feynman graphs as the loop momenta go to zero, regardless of the external momenta of the graphs. This is to be contrasted with the on-shell infra-red

divergences in QCD, say, which occur only for particular values of the external momenta. Such off-shell divergences pose the problem of how one is to supplement the Hamiltonian of Eq. (1) by a recipe that avoids these difficulties. The ϵ expansion about $d = d_c$ is one such recipe, since the propagator behaves naively as a power, i.e., $r^{-(d_c-2)}$. We note that the propagator in the $(n+1)^{\text{th}}$ minimal model also behaves as a power due to the non-zero anomalous dimension of ϕ . Despite the above, we will find that at least within the framework of the usual method employed for the ϵ -expansion we are unable to establish the connection between the Hamiltonian of Eq. (1) and the $(n+1)^{\text{th}}$ member of the minimal series. In particular, we will concentrate on the n large limit and show essentially that the correct combinatoric and crossing factors destroy relations such as those given in Eq. (4).

In the Seventies, a number of groups [3,4] used the ϵ -expansion starting in $d = d_c$ and computed the β function and anomalous dimension of ϕ at lowest and next-to-lowest order for the action of Eq. (1) in d dimensions. In particular, they found [7] the correction to the critical exponent γ_ϕ at the tricritical point (i.e., in ϕ^6 theory) to be +0.001. In fact, it is thought to be +0.075 = +3/2(n+1)(n+2) for $n = 3$ as a result of Eq. (3). Thus the result computed in the ϵ -expansion is 75 times too small. This is to be contrasted with the result for the three-dimensional Ising model, which is computed from the ϵ -expansion to be $\epsilon^2(54)^{-1} = (54)^{-1}$ when the estimated numerical value from high and low temperature expansions is 0.04. Thus the ϵ -expansion result, although low, is only out by a factor of 2^{-1} . The β function in $d = d_c$ was computed to all orders as $n \rightarrow \infty$ in a paper by Lipatov [8]. We will now repeat some of these calculations in outline in order to isolate the problems of interest here and discuss what modifications are possible.

We start from the d -dimensional Hamiltonian

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{(2n)!} \phi^{2n} \right\} \quad (5)$$

Evaluation of the multiloop Feynman diagrams is most easily performed in x space. The propagator is given by

$$\Delta(x) = \frac{1}{(2\pi)^d} \int d^d p \frac{e^{ip \cdot x}}{p^2} = \frac{a}{|x|^{d-2}} \quad (6)$$

where

$$a \equiv \frac{1}{(d-2)\Omega}, \quad \Omega \equiv \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \quad (7)$$

The lowest-order loop correction to the potential comes from the graph of Fig. 1 and is of the form

$$d_1 c_1 I_1 a^n g^2 \quad (8)$$

where d_1 and c_1 are the combinatoric and crossing factors. They are given by

$$d_1 = \frac{1}{2} \frac{(2n)!}{n! n!} \quad \text{and} \quad c_1 = \frac{1}{n!} \quad (9)$$

I_1 denotes the integral

$$\begin{aligned} I_1 &= \int \frac{d^d x}{|x|^{(d-2)n}} e^{i p \cdot x} = \pi^{d/2} 2^{(d-(d-2)n)} p^{((d-2)n-d)} \frac{\Gamma(\frac{d}{2} - \frac{(d-2)n}{2})}{\Gamma(\frac{(d-2)n}{2})} \\ &= \frac{2\pi^{d/2}}{\epsilon(n-1)} + O(\epsilon^0) \equiv I_1^{\text{div}} + O(\epsilon^0) \end{aligned} \quad (10)$$

The β function is then given to this order by

$$\beta = -\epsilon(n-1)g + c_1 d_1 a^n g^2 I_1^{\text{div}} [(n-1)\epsilon] \quad (11)$$

Hence one finds the fixed point to be

$$g_* = [c_1 d_1 a^n I_1^{\text{div}}]^{-1} \quad (12)$$

The lowest-order correction to ϕ^2 is given by the graph in Fig. 2, and it evaluates to

$$g^2 c_2 I_2 a^{2n-1} p^2 \quad (13)$$

where c_2 is the combinatoric factor and is given by

$$c_2 = \frac{1}{(2n-1)!} \quad (14)$$

and

$$\begin{aligned} \frac{I_2}{p^2} &= \frac{1}{p^2} \int \frac{d^d x}{|x|^{(d-2)(2n-1)}} e^{i p \cdot x} = \pi^{d/2} 2^{(d-(d-2)(2n-1))} \frac{\Gamma(\frac{d}{2} - (\frac{d-2}{2})(2n-1))}{\Gamma((\frac{d-2}{2})(2n-1))} \\ &\cdot p^{(d-2)(2n-1)-(d-2)} = -\frac{\pi^{d/2}}{4 \epsilon (n-1)} + O(\epsilon^0) \equiv I_2^{\text{div}} + O(\epsilon^0) \end{aligned} \quad (15)$$

The anomalous dimension of ϕ is then given by

$$\gamma_{\phi} = \frac{1}{2} g^2 c_2 a^{2n-1} [2(n-1)\epsilon] I_2^{\text{div}} \quad (16)$$

which, at the critical point, becomes

$$\begin{aligned} \gamma_{\phi} &= \frac{1}{a} \cdot (n-1)\epsilon \cdot \frac{c_2}{(c_1 d_1)^2} \cdot \frac{I_2^{\text{div}}}{(I_1^{\text{div}})^2} \\ &= \frac{1}{a d_c} \cdot \frac{\Gamma(\frac{d_c}{2})}{2 \pi^{d_c/2}} \cdot \epsilon^2 2^n (n-1)^2 \frac{(n!)^6}{(2n!)^3} \end{aligned} \quad (17)$$

in agreement with Ref. [4]. We recall that if the identification of the model with the minimal series is correct and the above calculation is valid, the result should be given by γ_{ϕ} of Eq. (3), namely $(3/2)(n+1)^{-1}(n+2)^{-1}$. It is particularly instructive to compare the results in the large n limit when $\epsilon \sim 2/n$.

Using the Stirling formula, we have

$$\gamma_{\phi} \sim 2 \epsilon^2 n^2 n^{3/2} \pi^{3/2} e^{-6n \ln 2} \quad (18)$$

It is now clear what the problem is; the combinatoric factors dominate the behaviour of γ_{ϕ} for large n , as the integral I and the propagator factor a can at best provide a power dependence on n .

One can also explore the anomalous dimensions of the composite fields to see if they fare better. For example, for a ϕ^2 insertion, one has the quantum correction

$$c_3 g^2 a^{2n} \int \frac{dx_1 dx_2 e^{i p \cdot x_1} e^{i q \cdot x_2}}{|x_1|^{(d-2)(2n-2)} |x_2|^{d-2} |x_1 - x_2|^{d-2}} \quad (19)$$

where the combinatoric factor $c_3 = [(2n-2)!]^{-1}$. To exhibit the result, we compare the anomalous dimension of ϕ^2 to that of ϕ in the large n limit. We have in essence

$$\frac{\gamma_{\phi^2}}{\gamma_{\phi}} \sim a \frac{c_3}{c_2} \sim n^2 \quad (20)$$

also in disagreement with Eq. (3). We note that g^* cancels out of the ratio and so the disagreement is insensitive to the precise value of the fixed point.

As another example, let us consider the operator insertion ϕ^n . There is a contribution of γ_{ϕ^n} from a linear order in g . The graph is shown in Fig. 3 for ϕ^n . The graph is

$$g c_4 a^n I_1 \quad (21)$$

where $c_4 = [n!]^{-1}$.

The corresponding anomalous dimension of ϕ^n is $\gamma_{\phi^n} = +g c_4 a^n I_1^{\text{Div}}$ which, at the fixed point, becomes

$$\gamma_{\phi^n} = \frac{c_4}{c_1 d_1} = 2 \frac{(n!)^2}{(2n)!} \approx 2 \sqrt{n\pi} e^{-n \ln 2} \quad (22)$$

again in disagreement with the large n result of Eq. (3). There are similar contributions to the anomalous dimensions of ϕ^m when $m > n$, since these insertions also have their leading contribution at order g rather than g^2 . The one feature that is in agreement with Eq. (3) is that the anomalous dimensions of ϕ^m exhibit a changed behaviour at $m = n-1$. This, however, is a property of a ϕ^{2n} potential rather than a feature of the ϵ -expansion.

There are now two possibilities. It is possible that the identification of Eq. (1) with the minimal series is not correct. This could happen in two ways; either it is simply not true, or it is true but the field identifications proposed in Ref. [5] are not valid. Clearly, if the ϵ -expansion were valid as computed above, we would have located an infra-red stable fixed point, and so we must be dealing with some conformal theory. The other possibility is that the ϵ -expansion does not work. It could be that this particular expansion is not relevant, or it is possible that the effect we are seeking is further out in the perturbation theory or some non-perturbative effect calculable within this framework.

We now examine the latter possibility by computing the next term in the β function. As explained in Ref. [8], the dominant contribution for large n at this order is the triangle-shaped graph with three vertices, with $2n/3$ internal lines exchanged between the vertices. This graph evaluates to

$$d_5 c_5 g^3 a^{2n} I_3 \quad (23)$$

where

$$d_5 = \frac{2n!}{3! n_1! n_2! n_3!}, \quad c_5 = \frac{1}{n_1! n_2! n_3!} \quad (24)$$

with $n_1 = n_2 = n_3 = 2n/3$ and

$$I_3 = \frac{\int dx_1 dx_2 e^{i(p \cdot x_1 + q \cdot x_2)}}{|x_1|^{(d-2)n_1} |x_2|^{(d-2)n_2} |x_1 - x_2|^{(d-2)n_3}} \quad (25)$$

The resulting β function to this order is of the form

$$\beta = -\epsilon g + g^2 h e^n a^n e^{n \ln 2} + g^3 a^{2n} e^{2n} k e^{(-2n \ln 2 + 4n \ln 3)} \quad (26)$$

for large n , where h and k are polynomial in n . Defining $g = \tilde{g} e^{n \ln n - n} a^{-n} e^{-2n \ln 2}$, the condition for β vanishing is

$$0 = \epsilon + h \tilde{g} + h \tilde{g}^2 e^{2n \ln 9/8} \quad (27)$$

Solving this equation, we find that the g^3 correction to the β function dominates the g^2 term and that $g \sim \sqrt{\epsilon(8/9)^{n/k}}$. This clearly suggests that the results obtained earlier are not the leading contributions and that higher order or non-perturbative effects are important.

Lipatov [8] has computed the β function to all orders in g in the large n limit in dimension $d_c = (2n)(n-1)^{-1}$, and proposed a generalization which agrees with $g = 0$ perturbation theory. The β function in d dimensions for n large is of the form

$$\beta(g) = -\epsilon g^{(n-1)} + \psi(g) + O(\epsilon g^2) + \dots \quad (28)$$

where $\psi(g)$ is the β function in dimension d_c computed in Ref. [8] [see Eqs. (57) and (58)]. If the β function in d_c dimensions $\psi(g)$ has a zero at g_0 , then we may

write in its vicinity

$$\beta(g) = -\epsilon g^{(n-1)} + a(g-g_0) + b(g-g_0)^2 + \dots \quad (29)$$

which has a fixed point for

$$g = g_0 + \frac{g_0 (n-1) \epsilon}{(a - \epsilon(n-1))} + \dots \quad (30)$$

It will be an infra-red stable fixed point if $a - \epsilon(n-1) > 0$.

Let us now briefly discuss some attempts to start from a different action than that of Eq. (1). One possibility would be to consider more than one scalar field. For example, we could take two fields with a potential of the form $\phi_1^n \phi_2^n$ or n scalar fields and $\phi_1^2 \phi_2^2, \dots, \phi_n^2$, or many other choices. We have, however, been unable to find a choice which avoids the problems mentioned before. Namely, in (17) we cannot find an example for which the crossing and combinatoric factors $c_2(c_{1d})^{-2}$ give a power behaviour in n .

An alternative approach is to change the integrals, that is, the factor $I_2^D a^{-1} (I_1^D)^{-2}$. One way to do this would be to modify the kinetic term to $\phi(\partial^2)^{1+\beta} \phi$. Since one is anyway dealing with a phenomenological Hamiltonian, this is perhaps acceptable. It is straightforward to recompute the above factor; however, it becomes readily apparent that there is no natural value of β which can cancel the factor $e^{-6n \ln 2}$. Another possibility, which we have not investigated, would be to use a non-linear sigma model kinetic term.

By using a Landau-Ginsburg Hamiltonian in four dimensions together with the ϵ -expansion, one can predict properties of two-phase critical phenomena in three dimensions with some precision. The analogous results starting from a Landau-Ginsburg action in $2+(2/n-1)$ dimensions and predicting multicritical phenomena in two dimensions are in poor agreement with known results. Indeed, in the limit in which the number of phases n goes to infinity, one should find a power behaviour for anomalous dimensions, but instead one finds an exponential decay. We discussed the extent to which this failure may be a result of the ϵ -expansion or the Landau-Ginsburg Hamiltonian. It must be admitted that precisely in two dimensions one finds a number of special features, one of which is the infinite dimensional conformal group. Another feature of two-dimensional field theory is the occurrence of off-shell infra-red divergences in $g = 0$ perturbation theory. Although these are avoided in perturbation theory about a non-trivial fixed point, nevertheless, in the ϵ -expansion, the propagator, although not logarithmic, contains a factor $1/(d-2)$. To lowest order in the ϵ -expansion this is 1.

There exist a number of calculations [9] which use renormalization group flows from one minimal model to another. However, these make no direct use of the Landau-Ginsburg Hamiltonian. It remains to find a perturbative scheme for the phenomenological Landau-Ginsburg Hamiltonian, and also a reliable perturbative scheme near the non-trivial fixed points of this Hamiltonian. Given such a scheme, one could then test whether or not this Hamiltonian and associated scheme describe multicritical points of the minimal models.

The situation is different for $N = 2$ supersymmetric theories. The ϵ -expansion can be carried out for supersymmetric theories, and in the $N = 2$ case, we have calculated various anomalous dimension, operator product expansions and the value of the central charge, c , at the fixed point. In contrast to the $N = 0$ case discussed above, the results are in complete agreement with those derived from the corresponding $N = 2$ superconformal (minimal) model. Thus, in the $N = 2$ case, we can explicitly establish the relation between the Landau-Ginsburg Hamiltonians and superconformal models. An important ingredient in the demonstration of this equivalence is the $N = 2$ non-renormalization theorem which is valid in the ϵ -expansion we have used. These results will be reported in a separate publication.

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Fig. 1 : The contribution to the β function

Fig. 2 : The contribution to the wave-function renormalization

Fig. 3 : The renormalization of the operator insertion ϕ^n