**INTEGRABILITY OF THE SELF-DUAL MEMBRANES
IN 4+1 DIMENSIONS AND THE TODA LATTICE**

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ABSTRACT

We introduce the notion of self-duality for membranes in 4+1 dimensions and we discuss their relation with $SU(\infty)$ self dual Y-M fields. We find that the self-dual spherical membranes is an integrable system isomorphic to the infinite Toda lattice and we determine the corresponding linear system. Simple solutions are found in the spherically symmetric case.

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In a recent paper [1] it was pointed out that in the large N limit, SU(N) Yang-Mills (Y-M) theories have, at the classical level, a simple geometrical structure with the SU(N) matrix potentials $A_\mu(X)$ replaced by c-number functions of two additional coordinates θ, ϕ of an internal sphere S^2 at every space time point, while the SU(N) symmetry is replaced by the infinite dimensional algebra of area preserving diffeomorphisms of the sphere S^2 called SDiff(S^2). More explicitly the SU(N) fields:

$$A_\mu(x) = A_\mu^\alpha(x) t^\alpha, \quad t^\alpha \in SU(N), \quad \alpha = 1, 2, \dots, N^2-1; \quad \mu = 0, 1, 2, 3 \quad (1)$$

become c-number functions

$$A_\mu(x, \theta, \phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} A_\mu^{\ell m}(x) Y_{\ell m}(\theta, \phi) \quad (2)$$

where $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics on S^2 . The local gauge transformations

$$\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega] \quad ; \quad \omega = \omega^\alpha t^\alpha \quad (3a)$$

and

$$\delta F_{\mu\nu} = [F_{\mu\nu}, \omega] \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (3b)$$

are replaced by

$$\delta A_\mu(x, \theta, \phi) = \partial_\mu \omega(x, \theta, \phi) + \{A_\mu, \omega\} \quad (4a)$$

$$\delta F_{\mu\nu}(x, \theta, \phi) = \{F_{\mu\nu}, \omega\} \quad (4b)$$

where

$$F_{\mu\nu}(x, \theta, \phi) = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\} \quad (5)$$

and the Poisson bracket on S^2 is defined:

$$\{f, g\} = \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \cos \theta} - \frac{\partial g}{\partial \phi} \frac{\partial f}{\partial \cos \theta} \quad (6)$$

So the commutators are replaced by Poisson brackets according to

$$\lim_{N \rightarrow \infty} N [A_\mu, A_\nu] = \{A_\mu, A_\nu\} \quad (7)$$

The Y-M action in the large N limit becomes

$$S_\infty = \frac{1}{16\pi g^2} \int_{S^2} d\Omega \int d^4x F_{\mu\nu}(x, \theta, \phi) F^{\mu\nu}(x, \theta, \phi) \quad (8)$$

where

$$g = \lim_{N \rightarrow \infty} g_N / N \quad (9)$$

This large N limit of SU(N) Y-M theories was found by making use of the relation between a particular $N \rightarrow \infty$ and $\text{SDiff}(S^2)$, proven explicitly in [2].

Although the relation, if any, of this particular limit of SU(N) and the planar Y-M theories remains to be found, we think it is instructive to study solutions of the classical equations of motion (given by the action (8)),

$$\partial_\mu F^{\mu\nu} = \{F^{\mu\nu}, A_\mu\} \quad (10)$$

relevant to the topology of the vacuum structure of this theory. So in this work we analyse in some detail the simplest case, i.e. gauge potentials independent of space, which satisfy self-duality conditions. If we work in the axial gauge

$$A_0 = 0 \quad (11)$$

then the self-duality equations are:

$$E_i = \pm B_i \quad (12)$$

where

$$E_i = \frac{d}{dt} A_i, \quad i=1,2,3 \quad (13a)$$

$$B_i = \frac{1}{2} \epsilon_{ijk} \{A_j, A_k\} \quad (13b)$$

are the electric and the magnetic $\text{SU}(\infty)$ colour fields. Here ϵ_{ijk} is the antisymmetric tensor in 3 dimensions. In the gauge (11), the equations of motion for the gauge fields A_i are (summation over j)

$$\ddot{A}_i = \{A_j, \{A_j, A_i\}\} \quad (14)$$

and the Gauss law becomes (summation over j)

$$\{\dot{A}_j, A_j\} = 0 \quad (15)$$

We observe that the above system of space homogeneous potentials is exactly the same as the spherical membrane in 4+1 dimensions, after gauge fixing and elimination of the constraints solving for $X_- = X_0 - X_4$. Here $X_i = A_i$, $i=1,2,3$. The Gauss law (15) turns out to be the compatibility condition of the reparametrization constraints, which express the reparametrization invariance in the internal coordinates of the sphere [2] (see also [3]). The electric and the magnetic $\text{SU}(\infty)$ homogeneous colour fields (13a-b) have very simple geometrical meaning in the membrane "language". They are the velocity and the normal vector of every point of the membrane. The self-duality equation (12) simply states that the velocity of each point of the membrane is equal or opposite to the normal vector to the membrane. Thus the self-dual membranes either explode or implode. The self-dual membranes (membranes which satisfy (12)), satisfy the Euclidean, (t - it), equations of motion (14) and the Gauss law (15), due to Jacobi identities. Since the Gauss law (15) is satisfied we can construct the other coordinates X_4 and X_0 of the membrane [13].

The self-dual membranes were first introduced in [4] but in 2+1 dimensions (see also [5,6]).

where the equations of motion become trivial and the membrane theory is a topological three dimensional field theory [4,5,7]. In this work we study the self-dual membranes in 4+1 dimensions, which in the light-cone gauge are equivalent to 3+1 dimensional self -dual Y-M in the large N limit for homogeneous (space independent) potentials. This exercise has a direct connection with the study of the vacuum structure of pure SU(N) Y-M theory, inside a periodic box, with $N \rightarrow \infty$ [8,9,10].

After these preliminaries we turn to the study of the self-dual system (the anti self-dual system can be treated in a similar way).

$$\dot{A}_i = \frac{1}{2} \epsilon_{ijk} \{A_j, A_k\} \quad i, j, k = 1, 2, 3. \quad (16)$$

The coordinates (Y-M potentials) of the self-dual membrane, A_i , $i=1,2,3$, are functions of the time and of the internal coordinates of the sphere (θ, ϕ) . Such a system (but with Poisson brackets replaced by commutators of matrices $A_1(t), A_2(t), A_3(t)$ of some Lie algebra) has appeared in the study of monopoles [11,12] as well in the connection of monopoles with the Toda molecule[13], and the Toda lattice [14]. Since our case is structurally the same, the above system of equations (16) is an integrable one. To show that it is indeed the case, below we provide the linear system for which equations (16) are the compatibility equations. First we rewrite eqs. (16) in the form:

$$\dot{A}_+ = i \{A_3, A_+\} \quad (17a)$$

$$\dot{A}_- = i \{A_3, A_-\} \quad (17b)$$

$$\dot{A}_3 = \frac{i}{2} \{A_+, A_-\} \quad (17c)$$

where

$$A_{\pm} = A_1 \pm i A_2 \quad (18)$$

Notice that eqs. (17a-c) have various constants of motion:

$$\int d\Omega A_{\pm}^n(t, \theta, \phi) = \text{constant in time} \quad (19a)$$

$$\int d\Omega A_3 A_{\pm} = \text{constant in time} \quad (19b)$$

in particular, "the centre of mass" of a self-dual membrane does not move:

$$\int d\Omega A_i(t, \theta, \phi) = \text{constant} , \quad i = 1, 2, 3 \quad (19c)$$

The linear system corresponding to (17a-c) is the following ($\lambda \neq 0$ is an arbitrary complex parameter):

$$\dot{\psi} = i \{ A_3 + \lambda A_- , \psi \} \quad (20a)$$

$$\dot{\psi} = i \left\{ \frac{1}{\lambda} A_+ - A_3 , \psi \right\} \quad (20b)$$

These eqs. can be written as

$$\dot{\psi} = \mathcal{L}_{A_3 + \lambda A_-} \psi \quad (21a)$$

$$\dot{\psi} = \mathcal{L}_{\frac{1}{\lambda} A_+ - A_3} \psi \quad (21b)$$

where the differential operators $\mathcal{L}_f$ are defined as:

$$\mathcal{L}_f \equiv i \left(\frac{\partial f}{\partial q} \frac{\partial}{\partial \cos \theta} - \frac{\partial f}{\partial \cos \theta} \frac{\partial}{\partial q} \right) \quad (22)$$

The compatibility condition of (21a-b) is

$$[\partial_t - \mathcal{L}_{A_3 + \lambda A_-} , \partial_t - \mathcal{L}_{\frac{1}{\lambda} A_+ - A_3}] = 0 \quad (23)$$

from which, comparing the two sides for the coefficients of the powers $1/\lambda, \lambda^0, \lambda^1$ of the spectral parameter λ , we find (17a-c). For the case of self-duality eqs. for $SU(N)$ Y-M the linear system was found in [15-16]. From the linear system (21a-b) using the inverse scattering method, one could in principle construct all solutions of the self-duality equations (17a-c) [15-16].

We find though more explicit and transparent the method of reference [13], which at the same time makes explicit use of the symmetry structure of the self-duality equations. In reference [13] spherically symmetric monopoles for the $SU(N)$ are related in the particular case of minimal embedding of $SU(2)$ in $SU(N)$ to the Toda molecule [14,17]. In our system, eqs. (16), there is an $O(3)$ global symmetry which is the rotation group of the embedding space of the membrane

$$\delta_{\vec{\omega}} A_i \Rightarrow (\vec{\omega} \cdot \vec{T})'_i A_i \quad (24)$$

where T_1, T_2, T_3 are generators of $O(3)$, and ω , are the parameters of the infinitesimal rotation, as well as reparametrizations of the membrane, with infinitesimal transformations,

$$\delta A_i = \{ h, A_i \} \equiv \mathcal{L}_h A_i \quad i = 1, 2, 3 \quad (25)$$

where h is an arbitrary function on the sphere. There is a subgroup of rigid rotations for the internal coordinates of the sphere $SU(2) \subset S\text{Diff}(S^2)$.

$$L_j = i \left[\frac{\partial e_j}{\partial \varphi} \frac{\partial}{\partial \cos \theta} - \frac{\partial e_j}{\partial \cos \theta} \frac{\partial}{\partial \varphi} \right], \quad j=1,2,3 \quad (26)$$

$$e_1 = \sin \theta \cos \varphi, \quad e_2 = \cos \theta \cos \varphi, \quad e_3 = \cos \theta \quad (27)$$

We call total angular momentum $\vec{J}$ of the membrane the sum

$$\vec{J} = \vec{T} + \vec{L} \quad (28)$$

The conditions of the spherical symmetry imply that the potentials $A_{\pm}, A_3$, transform as follows:

$$[L_3, A_{\pm}] = L_3 A_{\pm} = i \{e_3, A_{\pm}\} = \pm A_{\pm} \quad (29)$$

$$[L_3, A_3] = i \{e_3, A_3\} \quad (30)$$

This means that the potentials $A_{\pm}, A_3$, for the "spherically symmetric" membranes should be expressed as:

$$A_{\pm} = \sum_{\ell=1}^{\infty} a_{\ell}^{\pm}(t) Y_{\ell, \pm 1}(\vartheta, \varphi) \quad (31a)$$

$$A_3 = \sum_{\ell=1}^{\infty} a_{\ell}^3(t) Y_{\ell, 0}(\vartheta, \varphi) \quad (31b)$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics on S^2 . The spherical harmonics $Y_{l, \pm 1}$ correspond to the simple roots for the minimal $SU(2)$ embedding basis of $SU(N)$ and $Y_{l, 0}$ to the Cartan subalgebra. Using eqs. (17a-c) it is shown in ref. [13], in the case of $SU(N)$, that the corresponding a_l , satisfy some equations which can be reduced to the Toda system.

Here we shall exhibit how this works in the simple case where $a_l^{\pm} = a_l^3 = 0, l=2,3,\dots$. Introducing the functions $e_{\pm}, e_3$, proportional to $Y_{1, \pm 1}, Y_{1, 0}$:

$$\begin{aligned} e_{\pm} &= e_1 \pm i e_2 = e^{\pm i \varphi} \sin \theta \\ e_3 &= \cos \theta \end{aligned} \quad (32)$$

Equations (17a-c) imply the eqs. ($a_1^{\pm} = Q_{\pm}, a_1^3 = Q_3$)

$$\dot{Q}_{\pm} = Q_{\pm} Q_3 \quad (33a)$$

$$\dot{Q}_3 = Q_+ Q_- \quad (33b)$$

First we find that, due to an additional global U(1) symmetry of eqs. (17a-c),

$$A_{\pm} \rightarrow e^{\pm i\alpha} A_{\pm} \quad (34a)$$

$$A_3 \rightarrow A_3 \quad (34b)$$

the constant phase relation

$$\frac{d}{dt} \log \frac{Q_+}{Q_-} = 0 \quad (35)$$

implies that we may choose Q_+ , Q_- ,

$$Q_+ = Q_-^* = \frac{1}{\sqrt{2}} (1+i) R \quad (36)$$

and the quantity

$$\rho = \log Q_+ Q_- = 2 \log R \quad (37)$$

satisfies the Toda equation for SU(2),

$$\ddot{\rho} = \frac{d}{dt} \frac{d}{dt} \log Q_+ Q_- = \frac{d}{dt} 2Q_3 = 2e^{\rho} \quad (38)$$

This equation is integrated easily

$$\frac{1}{2} \dot{\rho}^2 = 2e^{\rho} + c \quad (39)$$

We find the three cases

$$R = \{c_0 / \sinh c_0(t-t_0)\} \quad , \quad Q_3 = c_0 \coth c_0(t-t_0) \quad ; \quad c = c_0^2 \quad (40a)$$

$$R = \{c_0 / \sin c_0(t-t_0)\} \quad , \quad Q_3 = c_0 \cot c_0(t-t_0) \quad ; \quad c = -c_0^2 \quad (40b)$$

$$R = \{R_0 / (1-R_0 t)\} \quad , \quad Q_3 = R \quad ; \quad c = 0 \quad (40c)$$

The first two cases represent axisymmetric ellipsoids in the three-dimensional space, collapsing to a string configuration and bouncing back to infinity. The third case represents a contracting sphere.

The analysis of more general solutions corresponding to the infinite Toda lattice, $a_l^{\pm}, a_l^3 \neq 0$, $l=1,2,\dots$ requires careful analysis of the root system of $SU(\infty)$ or $S\text{Diff}(S^2)$ algebra [18] but in the basis of spherical harmonics.

In conclusion, we have found a connection of the spherical self-dual membrane in 4+1 dimensions with the Toda infinite lattice system and we exhibited some simple solutions. We believe that this connection may provide us with some further interesting questions relating 3-dimensional manifolds with known integrable systems and extended conformal invariant field theories, in two dimensions [19].

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