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BUBBLE FORMATION IN ANISOTROPIC COSMOLOGIES

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ABSTRACT

We discuss the decay of the false vacuum in homogeneous cosmologies, both isotropic and anisotropic. If the false vacuum is isotropic, it is expected that the decay rate is dominated by spherically symmetric bubbles. However, if the universe during its past evolution was trapped in an anisotropic false vacuum one expects this anisotropy to be present in the decay products (bubbles) of that vacuum. In this paper we show the existence of two new, anisotropic types of bubble solution of Kantowski-Sachs type for a general double well potential. The first represents the decay of four dimensional de Sitter space, and the second the decay of the product of the two-sphere and two dimensional de Sitter space. We also derive qualitative features of the classical space-times emerging after tunneling.

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1. INTRODUCTION

The problem of vacuum decay in the early universe is most easily and frequently studied by considering bubble formation. In this approach one discusses semi-classical vacuum decay, in which the universe tunnels out of the false vacuum of an asymmetric potential with (at least) two minima. One imagines the universe as being everywhere in the false vacuum phase. Immediately after tunneling, a bubble (loosely referred to as containing the true vacuum) exists in space-time and then proceeds to expand at a speed approaching the speed of light. In this way a substantial region of true vacuum phase is created in space-time.

This description of vacuum decay has been extensively studied in the context of phase transitions in the early universe. During its early evolution the universe underwent several phase transitions, such as the GUT transition, the quark-gluon phase transition, and perhaps the inflationary phase transition. Each transition is characterized by an order parameter, typically a scalar Higgs field or some bose condensate of fermions, whose expectation value changes during the phase transition. For simplicity of discussion, we will here assume that the order parameter is given by a singlet scalar field, ϕ . The effective potential energy density is a function of the expectation value of ϕ .

If the transition is first order, the universe may supercool below the critical temperature. In this circumstance, it is trapped in a metastable (false) minimum of the potential, prevented from evolving to the stable (true) minimum by a potential barrier separating the two minima. The transition occurs by the scalar field tunneling through the potential barrier from the false to the true vacuum (see Fig. 1). The tunneling rate is proportional to e^{-B} , where B is the action for a ‘bubble solution’ which describes the spontaneous decay of a region of space-time from false to true vacuum [1]. The bubble solution satisfies the semi-classical, Euclideanized equations of motion. Different bubble solutions correspond to different ‘tunneling modes.’

In curved space-time, two tunneling modes are well known: (a) The Coleman-DeLuccia (CD) mode [1] in which the bubble solution (as a function of radial distance from the bubble center) has a core of true phase ($\phi \approx \phi_-$) separated by a thin wall ($\phi \approx \phi_{TOP}$) from an exterior of metastable phase ($\phi \approx \phi_+$). After nucleation, the bubble wall grows radially outward, converting false to true vacuum. (b) The Hawking-Moss (HM) mode [2], which can be interpreted as the limiting case of ‘bubbles’ with a very thick wall — $\phi \approx \phi_{TOP}$ everywhere [3].

Previous treatments have been confined to the maximally symmetric $SO(4)$ invariant case: it has generally been assumed that the tunneling rate is dominated by ‘spherical’ bubbles. Indeed, this has been shown to be so in flat Minkowski space-time. When the effect of gravitation on the bubble is included, it is not clear that this result is generally true. In the case where the universe is trapped in a very symmetric false vacuum, such as de Sitter space, it seems reasonable that the bubble be spherical. But if the false vacuum has less symmetry it is not so clear. For example if gravity is strong such as in the early universe and the false vacuum is anisotropic, why should a bubble tunneling from this vacuum be spherically symmetric? Here we wish to concentrate on exactly this question, and attempt to extend the description of tunneling to a specific class of anisotropic models, described by the Kantowski-Sachs metrics. Although there are some similarities with the $SO(4)$ case there are also significant differences, especially in the classical space-time which emerges after tunneling.

The rest of this paper is organized as follows. In section 2 we outline the models under consideration and fix our notation. In section 3 we review the isotropic case and derive the classical future space-time by analytic continuation. Section 4 contains material specific to the anisotropic models we consider. We discuss the classical vacua and derive qualitative features of the space-times. In section 5 we consider an alternative approach to the classical evolution and derive results linking the geometry of the Euclidean solution relevant to the semi-classical tunneling to the geometry of the ‘bubble’ space-time. The consequences of this approach are in agreement with earlier sections. Section 6 contains a discussion. We have relegated to appendices our work on the numerical derivation of solutions and various technical proofs required in the main body of the paper.

2. EINSTEIN SCALAR FIELD THEORY

In this paper we shall consider a scalar field ϕ coupled minimally to gravity. We assume that ϕ is self-interacting in a double well potential of the shape like the one shown in Fig. 1. The action of this system is

$$\mathcal{S} = -\frac{M_P^2}{16\pi} \int_{\mathcal{M}} (\mathcal{R} - 2\Lambda) \sqrt{|g|} d^4x - \int_{\mathcal{M}} \mathcal{L} \sqrt{|g|} d^4x - \frac{M_P^2}{8\pi} \int_{\partial\mathcal{M}} K \sqrt{|h|} d^4x \quad (2.1)$$

Here Λ is the cosmological constant, K_{ij} is the second fundamental form of $\partial\mathcal{M}$ in $\mathcal{M}$, h_{ij} the metric induced on $\partial\mathcal{M}$, and $K = h^{ij} K_{ij}$. $\mathcal{L}$ is the Lagrangian density for the matter,

which in this paper simply is the scalar field, so

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \quad (2.2)$$

Here V is a scalar field potential such as the one shown in Fig. 1.

Varying the action (2.1) we find the following field equations

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} &= \frac{8\pi}{M_P^2}\mathcal{T}_{\mu\nu} \\ \frac{1}{\sqrt{|g|}}\partial_\mu g^{\mu\nu}\sqrt{|g|}\partial_\nu\phi &= V'(\phi) \end{aligned} \quad (2.3)$$

(here we have absorbed the cosmological constant Λ into the scalar potential). The energy-momentum tensor $\mathcal{T}_{\mu\nu}$ for the scalar field is,

$$\mathcal{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right) \quad (2.4)$$

We note that all the equations in this section hold for Euclidean as well as Lorentzian signature of the metric $g_{\mu\nu}$.

3. THE ISOTROPIC CASE

We shall first consider a situation with the scalar field trapped in a false vacuum which exhibits the maximal degree of symmetry, namely the symmetry of a de Sitter space-time. The decay through bubble nucleation of such a vacuum is expected to break at least some of the symmetry because of the bubble core consists of true phase, while at far distances from the center we have false phase, *i.e.* the bubble breaks translation symmetry. However there is no apriori reason why the bubble solution should break rotational invariance (isotropy). In fact, for the special case of vacuum decay in flat Minkowski space ($M_P = \infty$) it has already been demonstrated that the nucleation rate is dominated by rotationally invariant bubbles, or more precisely $SO(4)$ -symmetric bubbles [4]. Such a result does not exist in the case where gravity is present, but for the case of tunneling from de Sitter space considered in this section we shall make the assumption that the lowest action bubble is $SO(4)$ -symmetric. As argued by Coleman and DeLuccia [5,1], the bubble solution can now be found by varying

the *Euclidean* action and finding the ‘solution of least resistance’ interpolating between the false vacuum and the true vacuum. In the remainder of this section we shall use the $SO(4)$ -assumption to derive the equations for the bubble, and show the bubble can be understood as a physical solution in the space-like part of Minkowski space. This is remarkable since the bubble was found as a solution to the *Euclidean* equations of motion, and it is not obvious that it also satisfies the *Lorentzian* field equations.

The $SO(4)$ symmetry is implemented by the following ansatz for the line element,

$$ds^2 = d\xi^2 + \beta^2(\xi)d\Omega_3^2 \quad (3.1)$$

where $d\Omega_3$ is the line element on the three-sphere. This is consistent if the scalar field only depends on ξ , $\phi = \phi(\xi)$. Substituting this ansatz into (2.1) we derive the following (Euclidean) field equations,

$$\begin{aligned} \dot{\beta}^2 &= 1 + \frac{8\pi}{3M_P^2} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \beta^2 \\ \ddot{\phi} + 3\frac{\dot{\beta}}{\beta}\dot{\phi} &= V'(\phi) \end{aligned} \quad (3.2)$$

Here a dot indicates differentiation with respect to ξ . Note that this set of equations describes the motion of a Newtonian particle in a potential $-V(\phi)$ under the influence of a time-dependent friction force $3\dot{\beta}\dot{\phi}/\beta$. It can be shown that any solution β satisfying these equations has at least two zeroes, and therefore the solution has the topology of the four-sphere S^4 [6]. Now, let two consecutive zeroes of β occur at, say, $\xi = 0$ and $\xi = \xi_{max}$, then in order for the solution to be non-singular, the friction term demands that $\dot{\phi}(0) = 0$ and $\dot{\phi}(\xi_{max}) = 0$, i.e. the boundary conditions on (3.2) are,

$$\beta(0) = \dot{\phi}(0) = \beta(\xi_{max}) = \dot{\phi}(\xi_{max}) = 0 \quad (3.3)$$

The solutions one obtains from these boundary conditions are solutions in which the scalar field interpolates between the two vacua in the sense that $\phi(0) \approx \phi_-$ and $\phi(\xi_{max}) \approx \phi_+$ (region I in Fig. 2). So the solutions describe bubbles with a core of true phase and which at far distances from the center of the bubble approach the false phase.

We can get a picture of the growth of the bubble in physical space by continuing the solution to (3.2) to Lorentzian space – this continuation will carry the Euclidean manifold into a manifold with Lorentzian signature of the metric [1,6].

To do this it is convenient to write the solution in conformally flat coordinates; defining a ‘conformal Euclidean time’ η through $d\eta/\eta = d\xi/\beta$, and $f = \beta/\eta$, we rewrite (3.1),

$$\begin{aligned} ds^2 &= f^2(\eta)(d\eta^2 + \eta^2(d\chi^2 + \cos^2 \chi d\Omega_2^2)) \\ &= f^2(\sqrt{\tau^2 + r^2})(d\tau^2 + dx^2 + dy^2 + dz^2) \end{aligned} \quad (3.4)$$

where $(\tau, r) = \eta(\sin \chi, \cos \chi)$. In this form we can continue the metric simply by making the substitution $\tau = it$ (or equivalently $\chi = i\rho$). This carries the metric into the Lorentzian metric,

$$ds^2 = f^2(\sqrt{r^2 - t^2})(-dt^2 + dx^2 + dy^2 + dz^2) \quad (3.5)$$

Now, since $f(\sqrt{r^2 + \tau^2})$ and $\phi(\sqrt{r^2 + \tau^2})$ satisfy the Euclidean field equations (3.2), we conclude that $f(\sqrt{r^2 - t^2})$ and $\phi(\sqrt{r^2 - t^2})$ satisfy Einstein’s equations in the region where $\sqrt{r^2 - t^2}$ is *real*, i.e. in the space-like region of coordinates. In terms of our original spherical coordinates this means that $\beta(\xi)$ and $\phi(\xi)$ satisfying (3.2) also satisfy Einstein’s equations in the space-like region of the Lorentzian manifold, in fact (3.2) *are* Einstein’s equations for space-like separations. The line element (3.5) takes the following form,

$$ds^2 = d\xi^2 + \beta^2(\xi)(-d\rho^2 + \cosh^2 \rho d\Omega_2^2) \quad (3.6)$$

where the time coordinate ρ is defined above. It is clear from (3.6) that for any fixed time the bubble has the topology of the three-sphere and the bubble wall has the geometry of S^2 . We also note that all dependence on coordinates enters through $\sqrt{r^2 - t^2}$, so that the bubble accelerates rapidly to a speed approaching the speed of light.

It has been shown that in addition to space-like part of the bubble solution (region I) there are two ‘time-like’ parts of the full bubble solution (regions II and III, see Fig. 2) [6]. These correspond to the rolling of the field from $\phi = \phi(0)$ into the true vacuum at ϕ_- , and from $\phi = \phi(\xi_{max})$ into the false vacuum at ϕ_+ . Since region II and III have the same structure, we shall just consider region II here. From (3.4) we see that one can continue to

the time-like region by making the formal substitution $\tau = \tilde{t}$, $x = i\tilde{x}$, $y = i\tilde{y}$, $z = i\tilde{z}$, and $f = i\tilde{f}$. This gives the following line element in region II,

$$ds^2 = \tilde{f}^2(\sqrt{\tilde{t}^2 - \tilde{r}^2})(-d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2)$$

$\tilde{f}(\sqrt{\tilde{t}^2 - \tilde{r}^2})$ and $\phi(\sqrt{\tilde{t}^2 - \tilde{r}^2})$ now satisfy the Lorentzian field equations (2.3). Returning to the spherical coordinates we notice that the above substitution corresponds to the substitution $\xi = i\zeta$ and $\beta(\xi) = iR(\zeta)$. With this substitution we find the following line element in ‘spherical coordinates’,

$$ds^2 = -d\zeta^2 + R^2(\zeta)d\Omega_T^2 \quad (3.7)$$

where $d\Omega_T$ is the line element of an open Robertson-Walker cosmology, and R satisfies the open Robertson-Walker equations [6],

$$\begin{aligned} \frac{\dot{R}^2}{R^2} - \frac{1}{R^2} &= \frac{8\pi}{3M_P^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \\ \ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} &= -V'(\phi) \end{aligned} \quad (3.8)$$

where dot indicates differentiation with respect to ζ . Equation (3.8) describes the evolution of the universe for $\tilde{t}^2 - \tilde{r}^2 \geq 0$, as the scalar field ϕ rolls from $\phi = \phi(0)$ into the true vacuum at $\phi = \phi_-$. The boundary between region I and II is the surface with $\tilde{t}^2 - \tilde{r}^2 = r^2 - t^2 = 0$, *i.e.* the Minkowski light-cone. It is easy to check that across this boundary the scalar field as well as the metric are continuously differentiable and so may be joined. So if we fix the time of nucleation to be at time $t = 0$ then the full solution for positive times to Einstein’s equations is given by the solution to (3.2) in the space-like region of coordinates, and by (3.8) in the time-like region.

Finally, the bubble nucleation rate per unit volume per unit time is given by,

$$\Gamma \approx \exp(-B) \quad (3.9)$$

where B is the difference in Euclidean action between the bubble solution and the false vacuum solution, *i.e.* $B = S - S_+$. From (3.2), the false vacuum solution $\phi \equiv \phi_+$, corresponds to $\beta(\xi) = H_+^{-1} \sin H_+ \xi$, where $H_+^2 = 8\pi V(\phi_+)/3M_P^2$ is the square of the Hubble constant

of the false vacuum. From this we find $S_+ = -3M_P^4/8V(\phi_+)$, and the line element (3.1) becomes that of S^4 ,

$$ds^2 = H_+^{-2} d\Omega_4^2 \quad (3.10)$$

This is not surprising as the four-sphere is exactly the analytical continuation of the Lorentzian vacuum solution – the de Sitter space-time.

4. ANISOTROPIC TUNNELING

In the previous section we have seen how to calculate the tunneling rate and the evolution of the bubble in the case of a false vacuum which has the highest possible degree of symmetry – the de Sitter vacuum with 10 Killing vectors, 3 corresponding to spatial homogeneity (translations), 3 to spatial isotropy (rotations), 3 to invariance under boosts, and one due to invariance under time translations. In this section we shall relax the assumption that the false vacuum have maximal symmetry, and consider a scalar field trapped in a vacuum of less symmetry.

Our universe today is very isotropic, and on large scales also very homogeneous. However, the universe may well have undergone a period of inflation some time during its past evolution [7]. If this is the case then there is little reason why the universe should have been homogeneous or isotropic before inflation, since a large enough period of inflation would cause today’s observed universe to be both homogeneous and isotropic [8]. So before inflation the universe may well have been very inhomogeneous and very anisotropic. In this section we shall analyse a situation where the universe is trapped in a false vacuum such as the one in Fig. 1, in which we allow for some degree of anisotropy, but which still remains homogeneous. Since the false vacuum at ϕ_+ has higher energy than the true vacuum at ϕ_- , we expect it to decay through bubble nucleation, just as in the isotropic case of last section. However it is clear that the formalism used there is dependant on the large degree of symmetry of the pure de Sitter vacuum, and has to be modified for less symmetric cases. For example the assumption that the bubble have $SO(4)$ symmetry seems unnatural if the vacuum has a symmetry which does not contain $SO(4)$. In the case of tunneling from pure de Sitter space, it is reasonable that the bubble be $SO(4)$ -symmetric since the Euclidean version of de Sitter space is $SO(5)$ -symmetric and thus contains $SO(4)$. (In physical space this amounts to saying that de Sitter space has $SO(4,1)$ symmetry and thus can accommodate

an $SO(3,1)$ -symmetric bubble.) However, if the vacuum is, say, $SO(3) \times SO(3)$ -symmetric it would be very surprising if the bubble should have $SO(4)$ symmetry. Below we shall analyze the decay of a scalar field coupled minimally to an anisotropic metric of a particular form – the Kantowski-Sachs metric – in which the spatial hypersurfaces have the symmetry of $S^1 \times S^2$. We will find two possible vacua, one with $SO(5)$ symmetry – the de Sitter vacuum – and one with $SO(3) \times SO(3)$ symmetry, and while both of these vacua may decay through nucleation of anisotropic bubbles, only the $SO(5)$ symmetric one can decay through nucleation of isotropic $SO(4)$ symmetric bubbles.

To find a bubble solution and nucleation rate, we shall use the same strategy as in the previous section: search for Euclidean bounce solutions of finite action and then continue these to Lorentzian space, to give the physical bubble solution, and finally use the boundary conditions given by the bubble solution to evolve it forwards in time and into the true vacuum after the instant of tunneling.

We begin with the (Lorentzian) line element of the Kantowski-Sachs cosmology [9],

$$ds^2 = -dt^2 + X^2(t)d\chi^2 + Y^2(t)d\Omega^2 \quad (4.1)$$

where $d\chi$ is the line element on S^1 or the real line and $d\Omega$ is the line element on the 2-sphere or 2-hyperboloid. The Euclidean continuation of (4.1) is,

$$ds^2 = d\xi^2 + X^2(\xi)d\chi^2 + Y^2(\xi)(d\theta^2 + \sin^2\theta d\varphi^2) \quad (4.2)$$

where χ is the coordinate on S^1 , and θ and φ are those on S^2 , so that the surface of homogeneity in (4.2) is compact. It has the symmetry of $SO(2) \times SO(3)$. Substituting (4.2) into (2.3) gives us the following (Euclidean) field equations,

$$\begin{aligned} 2\frac{\dot{X}\dot{Y}}{XY} - \frac{1 - \dot{Y}^2}{Y^2} &= \frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \\ 2\frac{\ddot{Y}}{Y} - \frac{1 - \dot{Y}^2}{Y^2} &= -\frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \\ \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} &= -\frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \\ \ddot{\phi} + 3\frac{\dot{\beta}}{\beta}\dot{\phi} &= V'(\phi) \end{aligned} \quad (4.3)$$

where a dot indicates differentiation with respect to ξ and where β is the ‘scale factor’ defined through $\beta^3 = XY^2$. Just as in the previous section it follows that the scale factor β

has two zeroes (see Appendix A), so we derive boundary conditions analogous to (3.3),

$$\beta(0) = \dot{\phi}(0) = \beta(\xi_{max}) = \dot{\phi}(\xi_{max}) = 0 \quad (4.4)$$

In order to avoid conical singularities in the (4.2) we shall also require that whenever $X = 0$, the first derivative $\dot{X} = \pm 1$ at that point (for Y the similar condition is a consequence of the equations of motion). One can now apply a simple overshoot/undershoot argument to show that solutions do exist (see Appendix B). As in the previous section it follows that these solutions produce bubbles with a core of true phase, but which approaches the false phase at distances far from the center of the bubble. However, unlike in the previous section these bubbles are not $SO(4)$ symmetric; they retain only the symmetry of (4.1), *i.e.* an $SO(2) \times SO(3)$ symmetry.

The solutions to (4.3) can be divided into two classes according to the number of zeroes of X and Y . We have mentioned above that β has two zeroes; this means that either X or Y have two zeroes, or they have one each. In Appendix C we have shown that the case where Y has two zeroes (solutions with $S^1 \times S^3$ topology) cannot occur. The two remaining cases correspond to,

- (i) X and Y have *one* zero each: The manifold has S^4 -topology,
- (ii) X has *two* zeroes: The manifold has $S^2 \times S^2$ -topology.

In case (i) the underlying manifold is the same as in the previous section treating isotropic tunneling, however the solution (i) is anisotropic, not isotropic like the Coleman-DeLuccia solutions. In case (ii) the manifold has $S^2 \times S^2$ topology, and is therefore inherently anisotropic; this manifold cannot, therefore, describe an isotropic Coleman-DeLuccia bubble solution.

Before we proceed to the analytical continuation to Lorentzian space of the bubble solutions (i) and (ii) let us first consider the possible vacuum solutions to (4.3). The vacuum solutions corresponds to $\dot{\phi} \equiv 0$ and $V \equiv \Lambda/\sqrt{8\pi} = \text{constant}$. Substituting this into the (4.3) we can integrate the Y equation, and find (see Appendix D) that only two non-singular solutions exist, corresponding to the two possible topologies. The solution belonging to case

(i) has

$$\begin{aligned} X &= \sqrt{\frac{3}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3}} (\xi - \xi_0) \\ Y &= \sqrt{\frac{3}{\Lambda}} \cos \sqrt{\frac{\Lambda}{3}} (\xi - \xi_0) \end{aligned} \tag{4.5}$$

This is the $SO(5)$ symmetric solution: substituting (4.5) into (4.2) yields in the line element of S^4 . The solution belonging to case (ii) has

$$\begin{aligned} X &= \frac{1}{\sqrt{\Lambda}} \sin \sqrt{\Lambda} (\xi - \xi_0) \\ Y &= \frac{1}{\sqrt{\Lambda}} \end{aligned} \tag{4.6}$$

This solution turns the line element into that of $S^2 \times S^2$ – the $SO(3) \times SO(3)$ symmetric solution. So (4.2) allows for two vacuum solutions, S^4 and $S^2 \times S^2$. The first of these, the four-sphere is the analytical continuation of the de Sitter space-time. This suggests that solutions of class (i) should be understood as describing the decay of a pure de Sitter space via formation of anisotropic bubbles. Similarly, $S^2 \times S^2$ is the analytical continuation of the direct product of two dimensional de Sitter space and S^2 , so by analogy solutions of class (ii) should be understood as describing the decay of an $(SO(2,1)/SO(2)) \times S^2$ vacuum via formation of anisotropic bubbles.

Both of these cases are interesting since for case (i) we know already from the previous section, that a metastable de Sitter vacuum can decay via nucleation of isotropic, $SO(4)$ -symmetric bubbles, and here we have found a new decay mode through nucleation of anisotropic bubbles. Different decay modes, in general, have different decay rates, so for case (i) we are in a position to compare the decay rate of the $SO(4)$ bubble to the one of the anisotropic bubble, and check the conjecture that the $SO(4)$ bubble (when existent) dominates the decay rate. In case (ii) no $SO(4)$ -symmetric decay mode exists, thus the decay rate is dominated by the *anisotropic* $SO(2) \times SO(3)$ -symmetric bubbles found above. So case (ii) provides an example of a vacuum that cannot decay through nucleation of $SO(4)$ bubbles.

Now, let us proceed to show how the Euclidean bubble solutions really exist in physical space-time by showing how they can be continued to Lorentzian solutions of the Einstein equations. We shall begin with case (ii) since it most resembles the isotropic case considered

in an earlier section. To carry out the continuation we shall write the part of the metric to be continued in coordinates that look conformally flat:

$$\begin{aligned}
 ds^2 &= d\xi^2 + X^2(\xi) + Y^2(\xi)d\Omega_2^2 \\
 &= \frac{X^2}{\eta^2}(d\eta^2 + \eta^2 d\chi^2) + Y^2 d\Omega_2^2 \\
 &= f^2(\sqrt{x^2 + \tau^2})(d\tau^2 + dx^2) + Y^2 d\Omega_2^2
 \end{aligned} \tag{4.7}$$

where η is defined through $d\eta/\eta = d\xi/X$, $f = X/\eta$, and $(\tau, x) = \eta(\sin \chi, \cos \chi)$. This line element can be continued by making the substitution $\tau = it$ (or equivalently $\chi = i\rho$), we then find the Lorentzian line element,

$$ds^2 = f^2(\sqrt{x^2 - t^2})(-dt^2 + dx^2) + Y^2 d\Omega_2^2 \tag{4.8}$$

or in polar coordinates,

$$ds^2 = -X^2(\xi)d\rho^2 + d\xi^2 + Y^2(\xi)d\Omega_2^2$$

As in section 3, since $f(\sqrt{x^2 + \tau^2})$ and $\phi(\sqrt{x^2 + \tau^2})$ satisfy the Euclidean field equations (4.3), we conclude that $f(\sqrt{x^2 - t^2})$ and $\phi(\sqrt{x^2 - t^2})$ satisfy Einstein's equations in the region where $\sqrt{x^2 - t^2}$ is *real*, i.e. in the space-like region. Returning to the polar coordinates this means that $X(\xi)$, $Y(\xi)$ and $\phi(\xi)$ satisfying (4.3) also satisfy Einstein's equations, in fact (4.3) are Einstein's equations for a Kantowski-Sachs metric in the space-like region. Also in this case we notice that the bubble expands with a speed rapidly approaching the speed of light.

The future evolution of the bubble into the true and false vacuum after the time of nucleation can be found by carrying out the following continuation of (4.3), let $x = i\tilde{x}$, $X = i\tilde{X}$ and $f = i\tilde{f}$. In the polar coordinates this is the same as the substitution $\xi = i\zeta$, $\chi = i\rho$ and $X = i\tilde{X}$, then the line element takes the form,

$$ds^2 = -d\zeta^2 + \tilde{X}^2(\zeta)d\rho^2 + Y^2(\zeta)d\Omega_2^2 \tag{4.9}$$

and $\tilde{X}$, Y and ϕ satisfy the *Lorentzian* field equations (closed Kantowski-Sachs equations),

$$\begin{aligned}
2\frac{\dot{\tilde{X}}\dot{Y}}{\tilde{X}Y} + \frac{1 + \dot{Y}^2}{Y^2} &= \frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \\
2\frac{\ddot{Y}}{Y} + \frac{1 + \dot{Y}^2}{Y^2} &= -\frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \\
\frac{\ddot{\tilde{X}}}{\tilde{X}} + \frac{\ddot{Y}}{Y} + \frac{\dot{\tilde{X}}\dot{Y}}{\tilde{X}Y} &= -\frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \\
\ddot{\phi} + \left(\frac{\dot{\tilde{X}}}{\tilde{X}} + 2\frac{\dot{Y}}{Y} \right) \dot{\phi} &= -V'(\phi)
\end{aligned} \tag{4.10}$$

where a dot indicates differentiation with respect to ζ . This solution describes the evolution of the universe in the time-like region, *i.e.* for $t^2 - x^2 \geq 0$. It is easy to check that on the two dimensional light cone, $t^2 - x^2 = 0$ region I (the space-like region) and region II (the time-like region) joins smoothly: both the metric and the scalar field are continuously differentiable. The boundary between region I and III has the same structure as the one between region I and II, and it follows that the time-like piece of the solution corresponding to the scalar field rolling into the false vacuum joins up smoothly to the space-like part of the bubble solution.

We have just demonstrated the existence of a bubble configuration which describes the decay of a scalar field trapped in a false vacuum with the symmetry of two dimensional de Sitter space cross S^2 . After tunneling, the universe is anisotropic, described by the Kantowski-Sachs metric (4.10). The decay rate is finite, and can be calculated according to (3.9).

Next we shall show how to continue case (i) to Lorentzian space. This is the case corresponding to an Euclidean bubble with S^4 topology. Then X and Y each have one zero; the zero of Y always appears on the true vacuum side of the barrier, and the zero of X on the false side. In this case there are two non-equivalent analytical continuations of the bubble solution, one for which Y is continued to imaginary values, and one for which X is continued to imaginary values. We shall see that the first one can be interpreted as the decay of the *false* vacuum, while the latter corresponds to the ‘decay’ of the *true* vacuum. Let us here consider in detail the decay of false vacuum; the decay of the true vacuum can be treated in a way similar to this and to the above continuation of $S^2 \times S^2$. To carry out

the continuation across the boundary where Y vanishes, we recast the part of the metric to be continued in conformally flat coordinates, then (4.2) takes the form,

$$\begin{aligned} ds^2 &= d\xi^2 + X^2(\xi)d\chi^2 + Y^2(\xi)(d\theta^2 + \cos^2\theta d\varphi^2) \\ &= \frac{Y^2}{\eta^2}(d\eta^2 + \eta^2(d\theta^2 + \cos^2\theta d\varphi^2)) + X^2 d\chi^2 \\ &= h^2(\sqrt{x^2 + y^2 + \tau^2})(d\tau^2 + dx^2 + dy^2) + X^2 d\chi^2 \end{aligned} \quad (4.11)$$

where η is defined by $d\eta/\eta = d\xi/Y$, $h = Y/\eta$ and (τ, x, y) are cartesian coordinates. This can easily be continued to a Lorentzian line element by continuing τ to imaginary values $\tau = it$ (or equivalently $\theta = i\rho$),

$$ds^2 = h^2(\sqrt{x^2 + y^2 - t^2})(-dt^2 + dx^2 + dy^2) + X^2 d\chi^2$$

in the polar coordinates this corresponds to,

$$ds^2 = d\xi^2 + X^2(\xi)d\chi^2 + Y^2(\xi)(-d\rho^2 + \cosh^2\rho d\varphi^2) \quad (4.12)$$

Again, we can conclude that since X , Y and ϕ satisfy the Euclidean field equations, they also satisfy the Lorentzian field equations in the space-like region (region I).

The equations determining the future evolution of the field into the true vacuum may be found by making the following substitution in (4.11), $x = i\tilde{x}$, $y = i\tilde{y}$, and $h = i\tilde{h}$. In polar coordinates this corresponds to $\xi = i\zeta$, $Y = i\tilde{Y}$ and $\theta - \pi/2 = i\rho$. The line element becomes,

$$ds^2 = -d\zeta^2 + X^2(\zeta) + \tilde{Y}^2(\zeta)(d\rho^2 + \sinh^2\rho d\varphi^2)$$

and X , $\tilde{Y}$ and ϕ satisfy the Lorentzian (open Kantowski-Sachs equations),

$$\begin{aligned} 2\frac{\dot{X}\dot{\tilde{Y}}}{X\tilde{Y}} - \frac{1 - \dot{\tilde{Y}}^2}{\tilde{Y}^2} &= \frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \\ 2\frac{\ddot{\tilde{Y}}}{\tilde{Y}} - \frac{1 - \dot{\tilde{Y}}^2}{\tilde{Y}^2} &= -\frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \\ \frac{\ddot{X}}{X} + \frac{\ddot{\tilde{Y}}}{\tilde{Y}} + \frac{\dot{X}\dot{\tilde{Y}}}{X\tilde{Y}} &= -\frac{8\pi}{M_P^2} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) \\ \ddot{\phi} + \left(\frac{\dot{X}}{X} + 2\frac{\dot{\tilde{Y}}}{\tilde{Y}} \right) \dot{\phi} &= -V'(\phi) \end{aligned} \quad (4.13)$$

here a dot indicates differentiation with respect to ζ . This solution describes the region

of time-like separations (region II) where $t^2 - x^2 - y^2 \geq 0$. It is easy to check that on the two dimensional light cone, regions I and II join smoothly.

It is interesting to note that region III does not exist in this case – no analytical continuation is possible across the other boundary where $X = X(\xi_{max})$ vanishes. This is because this boundary is not a three surface, but only a two surface. This can be seen by substituting in (4.12) $\bar{\xi} = \xi_{max} - \xi$ and expanding $X = \bar{\xi} + O(\bar{\xi}^3)$, then (4.12) takes the form,

$$ds^2 = d\bar{\xi}^2 + \bar{\xi}^2 d\chi^2 + Y^2(-d\rho^2 + \cosh^2 \rho d\varphi^2)$$

For $\bar{\xi} = 0$ this becomes,

$$ds^2 = Y^2(\xi_{max})(-d\rho^2 + \cosh^2 \rho d\varphi^2)$$

clearly a two surface. In fact, the manifold consisting of region I and II is geodesically complete for $t > 0$, and cannot be extended to a region III (see the next section).

Instead of having continued the metric across the boundary at $\xi = 0$ where Y vanishes, we could also have continued across the boundary at ξ_{max} where X vanishes. Then we would have found a different bubble solution consisting only of regions I and III. This solution can be interpreted as a tunneling from the true vacuum into the false vacuum.

In this section, we have found two new bubble solutions describing the decay of a false vacuum (and one describing the decay of the true vacuum) through formation of *anisotropic* bubbles of $SO(2) \times SO(3)$ symmetry. The bubbles have the topology of S^4 in the case where the vacuum is de Sitter space, and $S^2 \times S^2$ for a vacuum with the symmetry of the product of two dimensional de Sitter space and S^2 . For the first case another tunneling solution is already known, namely the $SO(4)$ -symmetric solution of Coleman and DeLuccia. It is not immediately clear if that, or the anisotropic S^4 -bubble gives the dominant contribution to the tunneling rate Γ , however, this question has been studied numerically in a later section, and we have found that the $SO(4)$ symmetric solution always has the lower action and therefore gives the dominant contribution to Γ . For the latter case no $SO(4)$ -symmetric bubble solution can be found, thus the anisotropic solution found here gives the dominant contribution to the tunneling rate.

5. ANALYTIC CONTINUATION FROM INITIAL DATA SETS

In this section we wish to discuss an alternative approach to finding the classical space-times after tunneling has occurred. A suggestion often found in the literature [2,10], is that one should find a suitable three surface in the Riemannian manifold with vanishing second fundamental form and such that the normal derivative of the matter fields vanishes and use this as an initial data configuration. Now the normal-normal and normal-tangential components of the Euclidean field equations are

$$-{}^3R + \frac{1}{2}({}^3\nabla\phi)^2 + V(\phi) = K^{ab}K_{ab} - K^2 + \frac{1}{2}({}^4\nabla_n\phi)^2 \quad (5.1)$$

and

$$2K_a{}^b{}_{;b} - 2K_b{}^b{}_{;a} + \frac{1}{2}({}^4\nabla_n\phi)^3 \nabla_a\phi = 0. \quad (5.2)$$

In the case where the three surface is a ‘moment of imaginary time symmetry’, K_{ab} vanishes so that the surface is totally geodesic in the four manifold and we will require that the normal derivative ${}^4\nabla_n\phi$ vanishes too. Now the Lorentzian constraint equations are identical to (5.1) and (5.2) with the exception that the right hand side of (5.1) acquires a relative minus sign. Thus we see that we have obtained a data set for the Lorentzian initial value problem. This may then in principle be evolved forward in time to obtain the classical Lorentzian space-time. It is this approach that we wish to apply in this section to obtain results which agree with the previous sections. However in the highly symmetrical cases we are considering here we may further note the following point. Typically any three surface which is a totally geodesic submanifold of a four manifold has an induced metric possessing an isometry group of smaller dimension than that of the four manifold. Nonetheless we would expect to see this greater degree of symmetry reflected in the analytically continued space-time and we will here relate the isometry groups of the Euclidean and Lorentzian spaces.

We first prove a result justifying the imposition of the geometrical symmetries on the scalar field. Suppose for some solution of the Euclidean scalar field equations we have a continuous isometry generated by a Killing vector $K^a \frac{\partial}{\partial x^a}$. Then the Ricci tensor is invariant

under this action and it follows from the field equations that

$$\mathcal{L}_K T_{ab} = 0$$

which implies that

$$2\nabla_{(a}\mathcal{L}_K\phi\nabla_{b)}\phi = g_{ab}(V'(\phi)\mathcal{L}_K\phi + \nabla_c\mathcal{L}_K\phi\nabla^c\phi) \quad (5.3)$$

where $\mathcal{L}_K$ is the Lie derivative with respect to K . Thus both sides of (5.3) must vanish, which implies that either ϕ or $\mathcal{L}_K\phi$ are constant. This shows that $\mathcal{L}_K\phi$ must vanish and thus ϕ has the symmetry of the space-time. This situation does not hold for gauge fields where the notion of a symmetry is more subtle (see for example Ref. [11]). In the case of S^4 there are many gauge fields with vanishing stress momentum tensor arising from solutions of the self-duality equations. In general these solutions which solve Einstein-Yang-Mills theory with a cosmological term have no nice properties with respect to the isometry group $SO(5)$ of S^4 .

We are now in a position to relate isometries of the Euclidean space to that of the evolved Lorentzian space-time. We first take a Killing vector ξ in the Euclidean space and decompose it into parts normal and tangential to the relevant hypersurface Σ ; $\xi = \beta + \xi^0 n$ where n is the unit normal to Σ . We then project both Killing's equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \quad (5.4)$$

and its implied identity

$$\nabla_\alpha \nabla_\beta \xi_\gamma = -\mathcal{R}_{\beta\gamma\alpha\delta} \xi^\delta \quad (5.5)$$

to obtain

$$D_a \beta_b + D_b \beta_a = 2\xi^0 K_{ab} \quad (5.6)$$

and

$$D_a D_b \xi^0 = \xi^0 ({}^3R_{ab} - h_a^c h_b^d \mathcal{R}_{cd} - K K_{ab} + 2K_{ac} K_b^c) + \beta^c D_c K_{ab} + K_{ac} D_b \beta^c + K_{bc} D_a \beta^c \quad (5.7)$$

where D is the covariant derivative on Σ (see Ref. [12] for related equations in the Lorentzian case). In the case where K_{ab} vanishes we see that we can decompose the Killing vectors

into those tangential and those normal to Σ for these projections separately solve (5.6) and (5.7). This is intuitively clear since $K_{ab} = 0$ and $\nabla_n \phi = 0$ imply a time reversal symmetry of the space and hence a grading of the isometry group. It is crucial to note that despite appearances (5.7) depends only on the initial data (*i.e.* g_{ab} , K_{ab} , ϕ , $\nabla_n \phi$) assuming that the field equations hold, since we may replace the term in $\mathcal{R}_{cd}$ with the stress momentum tensor of the scalar field which is re-expressible in terms of the initial data. Thus we may express the Lie algebra of the isometry group $g = t \oplus n$ as a direct sum of the orthogonal and tangential Killing vectors. Now these components may be interpreted as the restriction of a Killing vector field in space to a hypersurface or as a parametrization of a surface deformation moving the hypersurface in the four dimensional space. In this second interpretation the work of Teitelboim tells us how to find the effects of performing successive such deformation [13]. The vector and scalar on the surface Σ generate an infinitesimal deformation in the coordinate of the surface $y^A(x^r)$ by

$$y^A \rightarrow y^A + \delta \xi^0 n^A + \delta \beta^r y_{,r}^A.$$

Then the commutator of two such deformations $(\delta \xi^0, \delta \beta^r)$ $(\delta \eta^0, \delta \gamma^r)$ is given by:

$$\delta \zeta^0 = -(\delta \gamma^r \delta \xi_{,r}^0 - \delta \beta^r \delta \eta_{,r}^0) \quad (5.8)$$

$$\delta \alpha^r = \epsilon g^{rs} (\delta \eta^0 \delta \xi_{,s}^0 - \delta \xi^0 \delta \eta_{,s}^0) + (\delta \beta^s \delta \gamma_{,s}^r - \delta \gamma^s \delta \beta_{,s}^r) \quad (5.9)$$

where $\epsilon = g^{AB} n_A n_B$ and n^A is the unit normal to the hypersurface. Thus the isometry algebra has the symmetric space form, $\dim g = N$, $i = 1 \dots r = \dim t$, $\alpha = 1 \dots N - r = \dim n$,

$$\begin{aligned} [t^i, t^j] &= C^{ij}_k t^k \\ [t^i, n^\alpha] &= D^{i\alpha}_\beta n^\beta \\ [n^\alpha, n^\beta] &= \epsilon E^{\alpha\beta}_i t^i \end{aligned} \quad (5.10)$$

Our scheme for finding the Lorentzian space-time is to take the data given by the time-symmetric slice in the Euclidean space and evolve it forward. How do we find the Killing vectors of the evolved space-time? It turns out that these may be obtained by examining the data set. A property that a potential Killing vector (β, ξ^0) must possess is that the

initial data (both coordinates and conjugate momenta) must be preserved under the translation generated by $\beta^\alpha H_\alpha + \xi^0 H$ where H_α, H are the supermomenta and super Hamiltonian respectively. Provided that the equations of motion hold and that the initial data value of the scalar field is invariant under $\beta^\alpha \frac{\partial}{\partial x^\alpha}$ (which we will assume) then the data is infinitesimally invariant precisely when (5.4) and (5.5) hold (in the case of vanishing K_{ab}). One may then ask if this infinitesimal condition suffices to yield a Killing vector for at least some region of the evolved Lorentzian space-time. A result of Moncrief's generalized to include space-times with matter shows that one does obtain such a Killing vector [14]. Essentially in the case of spatial Killing vector inspection of Hamilton's equations for the theory shows that the Killing vector exists in the space-time, whereas for the case of normal pointing vector, choosing the function ξ^0 as the lapse function produces the required Killing vector. Thus we may summarize our results as follows; provided that the initial data hypersurface has $\pi = K_{ab} = 0$ where π is the momentum conjugate to ϕ and our data obeys

$$\mathcal{L}_\beta \phi = \mathcal{L}_\beta g = D_a D_b \xi^0 - ({}^3R_{ab} - h_a^c h_b^d \mathcal{R}_{cd}) \xi^0 = 0 \quad (5.11)$$

then $(\beta^a + \xi^0 n^a)$ may be integrated to give a local Killing vector field. Moreover the algebra so obtained is related to (5.10) merely by the change of sign of ϵ . It is easy to check that such a change of sign is consistent and that the Jacobi identity for the new structure constants still holds. The relation to more conventional analytic continuations is perhaps made clearer by showing that setting $n^A = i\tilde{n}^A$ leads to the new algebra.

We now show how this type of continuation reproduces the results of the previous sections. We first discuss the solution of Coleman and DeLuccia which was also studied by Guth and Weinberg [1,6]. The Euclidean solution has the form of equation (3.1) with $\phi = \phi(\xi)$. We now seek a three submanifold which is totally geodesic and has $\nabla_n \phi$ vanishing. The first point to note is that it is extremely unlikely that such a submanifold would be a constant ξ surface for then the derivatives of β and ϕ would have to vanish at the same ξ value. Instead we take an equatorial two sphere in the three sphere to obtain as data:

$$\begin{aligned} d\tilde{s}^2 &= d\xi^2 + \beta^2(\xi) d\Omega_2^2 \\ \phi &= \phi(\xi). \end{aligned} \quad (5.12)$$

It is clear that this will evolve into the space-time:

$$\begin{aligned} ds^2 &= d\xi^2 + \beta^2(\xi)(-d\rho^2 + \cosh^2 \rho d\Omega_2^2) \\ \phi &= \phi(\xi) \end{aligned} \quad (5.13)$$

which is identical to (3.6), and solves the Lorentzian equations (3.2). Since $-d\rho^2 + \cosh^2 \rho d\Omega_2^2$ is the metric induced on the hyperboloid $x^2 + y^2 + z^2 - t^2 = 1$ in Minkowski space-time, we see that the isometry group of the Lorentzian space is $SO(3,1)$ (unless β has a very special form, in which case this is only a subgroup). The Killing vectors of the Euclidean space were the generators of $SO(4)$ of which an $SO(3)$ subgroup is the isometry group of the initial data set, generated by J^i say. The generators of $SO(4)$ orthogonal to the equatorial two sphere are L^i and obey

$$\begin{aligned} [J^i, J^j] &= \epsilon^{ijk} J^k \\ [J^i, L^j] &= \epsilon^{ijk} L^k \\ [L^i, L^j] &= \epsilon \epsilon^{ijk} J^k \end{aligned} \quad (5.14)$$

On taking $\epsilon = -1$ in the final relation we indeed find the Lie algebra of $SO(3,1)$.

As we are evolving forward in time we obtain the part of the space-time (5.13) with $\rho > 0$. The spatial boundary is the moment just after tunneling. The Penrose diagram is given in Fig. 3. Region I corresponds to the space-time found above, but we must extend across the boundaries as a geodesic may reach them in finite affine parameter time. The extension is identical to that in the previous section. Region I is given by:

$$ds^2 = \Omega^2(\xi)(d\bar{\xi}^2 + \bar{\xi}^2(-dt^2 + \cosh^2 t d\Omega_2^2)) \quad (5.15)$$

with $0 \leq \bar{\xi} < \infty$ and where $\xi(\bar{\xi} = \infty) = \xi_{max}$. This region is conformal to the whole of the space-like future region of the origin in Minkowski space with nowhere vanishing conformal factor.

As in Ref. [15] we bring the boundary to finite coordinate values by setting $\tan p = \bar{\xi} e^t$ and $\tan q = -\bar{\xi} e^{-t}$ with the angular variables having the ranges $\frac{1}{2}\pi > p > 0 > q > -\frac{1}{2}\pi$. Now we let $u = p + q$ and $r = p - q$ and obtain

$$ds^2 = \Omega^2(\xi) \frac{1}{4} \sec^2 p \sec^2 q (-du^2 + dr^2 + \sin^2 r d\Omega_2^2) \quad (5.16)$$

where our region satisfies $r > u$ and $\pi > r + u > 0$. Then we continue across the boundary

$\bar{\xi} = 0$ to obtain:

$$\begin{aligned} ds^2 &= -d\zeta^2 + R^2(\zeta)(d\rho^2 + \sinh^2 \rho d\Omega_2^2) \\ &= \tilde{\Omega}^2(\zeta)(-d\bar{u}^2 + \bar{u}^2(d\rho^2 + \sinh^2 \rho d\Omega_2^2)) \end{aligned} \quad (5.17)$$

where $iR(\zeta) = \beta(i\zeta)$. This gives region II with a space-like infinity and region III similarly as the geometry of region I is conformal to that obtained by reflection in ξ_{max} . We stress here that these continuations are necessary to obtain a manifold which is geodesically complete in the future of the initial moment.

We now turn to discuss the space-time obtained from the Kantowski-Sachs Euclidean solutions and first consider those solutions possessing S^4 topology. This is rather different from the $SO(4)$ invariant case of Guth and Weinberg.

First recall that the metric on S^4 in Kantowski-Sachs form is given by equation (4.2). We have in principle two 3-submanifolds obeying the $K_{ab} = \nabla_n \phi = 0$ condition. The first is given by $\theta = \frac{1}{2}\pi$ and has the metric

$$ds^2 = d\xi^2 + X^2(\xi)d\chi^2 + Y^2(\xi)d\varphi^2 \quad (5.18)$$

whilst the second given by $\chi = 0$ is

$$ds^2 = d\xi^2 + Y^2(\xi)d\Omega_2^2 \quad (5.19)$$

where $0 < \xi < \xi_{max}$. We take the scalar field to be $\phi = \phi(\xi)$. The metric given by equation (5.18) evolves into

$$d\xi^2 + X^2(\xi)d\chi^2 + Y^2(\xi)(-d\rho^2 + \cosh^2 \rho d\varphi^2) \quad (5.20)$$

whilst that given by equation (5.19) evolves into

$$-X^2(\xi)d\rho^2 + d\xi^2 + Y^2(\xi)d\Omega_2^2 \quad (5.21)$$

for the region $\rho > 0$. These two regions may be analytically continued across $\xi = \xi_{max}$ in the first case and $\xi = 0$ in the second, but not across the other region where this boundary is in both cases merely a coordinate singularity and hence no extension is required as we

now show. Consider first the space-time (5.20) and (after first transforming $\xi \rightarrow \xi_{max} - \xi$) we obtain

$$ds^2 = \Omega^2(d\bar{\xi}^2 + \bar{\xi}^2(-d\rho^2 + \cosh^2 \rho d\varphi^2) + \left(\frac{X}{\Omega}\right)^2 d\chi^2) \quad (5.22)$$

where $d\bar{\xi} = \Omega^{-1}d\xi$ and $\bar{\xi} = \Omega^{-1}Y$. But now in addition to having ρ positive we also have $0 < \bar{\xi} < \bar{\xi}_{max}$. We first transform this region to the product of a circle and the region between the light cone and a time-like hyperboloid in Minkowski space by setting:

$$\hat{x} = \bar{\xi} \cosh \rho \sin \varphi \quad \hat{y} = \bar{\xi} \cosh \rho \cos \varphi \quad \hat{t} = \bar{\xi} \sinh \rho \quad (5.23)$$

with $0 < \hat{x}^2 + \hat{y}^2 - \hat{t}^2 < \bar{\xi}_{max}^2$ and $\hat{t} > 0$.

We further change coordinates:

$$\hat{x} = r \cos \theta \quad \hat{y} = r \sin \theta \quad \tan p = \hat{t} + r \quad \tan q = \hat{t} - r \quad (5.24)$$

with $\frac{1}{2}\pi > p > q > -\frac{1}{2}\pi$ and $0 > \tan p \tan q > -\bar{\xi}_{max}^2$ to obtain:

$$ds^2 = \Omega^2 \sec^2 p \sec^2 q \left(-dpdq + \frac{1}{4} \sin^2(p - q) d\theta^2 + \left(\frac{X(\xi)}{\Omega(\xi) \sec p \sec q} \right)^2 d\chi^2 \right) \quad (5.25)$$

We show this in the (α, β) plane in Fig. 4 where $\alpha = p + q$ and $\beta = p - q$. The line $\alpha = \beta$ corresponds to $Y = 0$ and it is easy to show that this is reached by geodesics in a finite affine parameter time and so we must continue as in section 4 to obtain

$$ds^2 = -d\zeta^2 + \tilde{Y}^2(\zeta)(d\rho^2 + \sinh^2 \rho d\varphi^2) + \tilde{X}^2(s)d\chi^2 \quad (5.26)$$

where

$$\begin{aligned} \tilde{X}(\zeta) &= X(i\zeta) \\ i\tilde{Y}(\zeta) &= Y(i\zeta) \end{aligned} \quad (5.27)$$

and $i\zeta = (\xi_{max} - \xi)$ Now near $\bar{\xi} = 0$ we obtain a metric which is the product of the circle and the region on the space-like side of the light-cone in 3 dimensional Minkowski space whereas the region near $\zeta = 0$ is the space-like side of the light-cone which can thus be joined on smoothly as above. In this region ϕ takes values on the true vacuum side of the

bounce solution. Thus the region II in Fig. 4 is obtained. Because the other ‘boundary’ of region I is merely a coordinate artifact, no region III is obtained. Thus we are nowhere in the false vacuum in the future space-time. The space-time evolving from (5.19) is obtained similarly and we find the Penrose diagram as in Fig. 5. Here there is no true vacuum and we obtain only regions I and III as again the other ‘boundary’ is merely a coordinate singularity. The isometry group of the original Euclidean configuration is clearly $SO(3) \times SO(2)$. In the Lorentzian spaces we obtain (in region I) the groups $SO(2,1) \times SO(2)$ and $R \times SO(3)$. These have algebras of the anticipated forms. In the case of the Euclidean manifold having topology $S^2 \times S^2$ it is easy to see that the global structure of the evolved space-time is similar to the $SO(4)$ Guth and Weinberg case, but applied to a two dimensional metric, the product factor of S^2 playing essentially no role in the structure.

6. NUMERICAL ANALYSIS OF ANISOTROPIC TUNNELING

In the previous section we have demonstrated the existence of anisotropic tunneling configurations, and we have shown explicitly how to continue the Euclidean bubble solution to Lorentzian space-time. One cannot, however, find analytical expressions for the solution of (4.3) for a general scalar potential. Here we shall present the results we have obtained numerically for a model with a quartic potential of the form,

$$V(\phi) = V_0 - \frac{1}{2}m^2\phi - \epsilon\frac{m}{3}\phi^3 + \frac{\lambda}{4}\phi^4 \quad (6.1)$$

A potential of this type has a double-well shape of Fig. 1, and the energy difference between the two vacua is determined by the asymmetry parameter ϵ . We have used (6.1) to search for solutions to (4.3) for X , Y and ϕ , and found two different classes, (i) and (ii), as we expected from the results of section 4. Case (i) is characterized by X and Y having one zero each – this is the solution with S^4 topology. Case (ii) is characterized by X having two zeroes – the solution with $S^2 \times S^2$ topology.

For case (i) we have found numerically, that non-trivial (anisotropic) solutions exist only when $m^2 > 10H_0^2$, where $H_0^2 = 8\pi V(0)/3M_P^2$ is the Hubble constant corresponding to the top of the potential barrier. This is in agreement with the result of Appendix B. (It is interesting to note that this limit is the same as the limit found for the first ‘excited bubble state’ for CD bubbles [3].) By varying m we have also found that as m^2 approaches $10H_0^2$ from above,

the bubble solution converges onto the Hawking-Moss solution. This constraint on m is stronger than the constraint for $\text{SO}(4)$ -symmetric CD bubbles to exist, $m^2 > 4H_0^2$ [2,3]. The action of the anisotropic bubble has been found always to be smaller than the action of the HM bubble, but larger than the CD bubble. This means that for $m^2 > 10H_0^2$ the decay of the de Sitter vacuum is dominated by the CD bubbles, but rare anisotropic bubbles will also be present, as well as very rare HM bubbles. For $4H_0^2 < m^2 \leq 10H_0^2$ no anisotropic bubbles can exist, and the tunneling rate is still dominated by the CD bubbles with rare HM bubbles. For $m^2 \leq 4H_0^2$ no non-trivial bubble solutions exist, and the tunneling rate is dominated by the HM bubbles.

In case (ii), which describes the decay of two dimensional de Sitter space cross S^2 , only anisotropic bubbles exist. We have confirmed numerically that non-trivial solutions exist only when $m^2 > 4H_0^2$. For $m^2 \leq 4H_0^2$ only the trivial (anisotropic) solution corresponding to $\phi = 0$ exist – this is the equivalent of the HM solution in case (i), so we shall use that name here also. In this case, as m^2 tends to $4H_0^2$ from above, the non-trivial solution converges onto the HM solution. For $m^2 > 4H_0^2$ the action of the non-trivial solution has numerically been found always to be smaller than the HM action, and thus dominates the tunneling rate, while for $m^2 \leq 4H_0^2$ the HM bubble dominates.

7. DISCUSSION

We have considered the problem of bubble formation and subsequent evolution in homogeneous cosmologies both isotropic and anisotropic. We find that vacuum decay by anisotropic bubbles is permitted and computed decay rates. For all the cases we have considered the decay to a de Sitter space occurs at a slower rate when mediated by anisotropic bubbles than that given by the isotropic Coleman-DeLuccia bubble. This much was anticipated although no analytical proof exists as yet. We also found that decay is possible of the direct product of the two-sphere and a two dimensional de Sitter space. This latter result is essentially new and we believe that we have found the dominant decay mechanism. However even in the comparatively well understood case of tunneling of four dimensional de Sitter space the global structure of the evolution of the bubble after formation is rather different and we find two possible final states. There is no contradiction here, it is just a question of different transitions being mediated by the same Euclidean configuration in the semi-classical approximation. In each of these two possible four dimensional de Sitter like

states one finds that in the future the scalar field is either nowhere in the true vacuum or nowhere in the false vacuum. This is significantly different from the future for the tunneling of the scalar field in flat space, but perhaps differs less from the isotropic gravitational case, where the bubble at the time of nucleation nowhere contains true vacuum (*i.e.* the scalar field does not take the value ϕ_+). However it does lead to a different global structure of the space-time.

One feature of our bubble solutions that is somewhat intriguing is that being at constant ξ values they have the topology (in Euclidean space) of the product of the two sphere with the circle. However there exists a proof that in a certain formulation of the thin wall approximation the only bubbles which separate two constant curvature spaces and further obey certain topological constraints must have spherical topology. Clearly our case does not satisfy the version of the thin wall approximation used, which essentially models the bubble and scalar field by a distributional source for the stress momentum tensor with spaces of constant curvature on either side. However we believe that departures from the conditions of the result outlined above are ‘small’. It is clear that this notion needs to be sharpened in order to resolve these difficulties and this may have some bearing on the validity of previous approximations of bubble dynamics. We hope to study some of these questions in the near future.

APPENDIX A

In this appendix we shall show that β has at least two zeroes. We shall show this in synchronous gauge for a general inhomogeneous cosmology. In synchronous gauge the metric takes the form $g_{00} = 1$, $g_{0i} = 0$, where latin letters run from 1 to 3. We define the three-metric h_{ij} by $h_{ij} = g_{ij}$. In this gauge the *Euclidean* Raychaudhuri equation is,

$$\frac{\ddot{\beta}}{\beta} = -\frac{8\pi}{3M_P^2}(\dot{\phi}^2 + V(\phi)) - \frac{2}{3}\sigma^2 \quad (\text{A1})$$

where the scale factor β is defined through $\beta = (\det h_{ij})^{1/6}$ and the shear $\sigma^2 \equiv \sigma_{ij}\sigma^{ij}/2$ is defined in terms of the trace free part of $\dot{h}_{ij}$,

$$2\sigma_{ij} = \dot{h}_{ij} - 2\frac{\dot{\beta}}{\beta}h_{ij}$$

Since β positive implies $\ddot{\beta}$ negative, it is easy to show that β has at least one zero, say

$\beta(0) = 0$. To show that β has yet another zero, let ξ_0 be chosen so $\beta(\xi_0) > 0$ – without loss of generality we may take $\xi_0 > 0$. First assume that another point $\xi_1 > \xi_0$ exists with $0 < \beta(\xi_1) < \beta(\xi_0)$. Then the line through $(\xi_0, \beta(\xi_0))$ and $(\xi_1, \beta(\xi_1))$ has negative slope, $-\alpha$, and by the mean value theorem there exists another point ξ_2 between ξ_0 and ξ_1 , so $\dot{\beta}(\xi_2) = -\alpha$. Now, let l be the tangent to β through this point, then because the slope of l is negative, l has a zero at some $\xi_3 > \xi_2$. Since $\ddot{\beta} < 0$ it follows that $\beta(\xi) < l(\xi)$ for $\xi > \xi_2$, so β has another zero between ξ_2 and ξ_3 . Next assume that no such point ξ_1 exists, then $\beta(\xi) \geq \beta(\xi_0)$ for $\xi > \xi_0$. Now consider the following identity for β ,

$$\begin{aligned}\beta(\xi) &= \beta(\xi_0) + \int_{\xi_0}^{\xi} d\bar{\xi} \left(\dot{\beta}(\xi_0) + \int_{\xi_0}^{\bar{\xi}} d\bar{\xi} \ddot{\beta}(\bar{\xi}) \right) \\ &= \beta(\xi_0) + \dot{\beta}(\xi_0)(\xi - \xi_0) + \int_{\xi_0}^{\xi} d\bar{\xi} \int_{\xi_0}^{\bar{\xi}} d\bar{\xi} \ddot{\beta}(\bar{\xi})\end{aligned}$$

At this point we may use (A1) to estimate the right hand side,

$$\begin{aligned}\beta(\xi) &\leq \beta(\xi_0) + \dot{\beta}(\xi_0)(\xi - \xi_0) - \beta(\xi_0)H_-^2 \int_{\xi_0}^{\xi} d\bar{\xi}(\bar{\xi} - \xi_0) \\ &\leq \beta(\xi_0) + \dot{\beta}(\xi_0)(\xi - \xi_0) - \frac{1}{2}\beta(\xi_0)H_-^2(\xi - \xi_0)^2\end{aligned}$$

where $H_-^2 = 8\pi V(\phi_-)/3M_P^2$. For large enough ξ the right hand side becomes smaller than $\beta(\xi_0)$. This shows that a point ξ_1 defined above always exists, and from the first part it then follows that β always has a second zero.

APPENDIX B

We now derive the constraints on $V(\phi)$ in order for non-trivial solutions to (4.3) to exist. The argument we give applies for solutions with S^4 topology. A similar argument can be applied for solutions with $S^2 \times S^2$ topology. The potentials we consider are those of the double-well type shown in Fig. 1. The only features of V we use here is that $V(\phi_+)$ is positive and that it is quadratic around the top of the barrier, *i.e.* $V(\phi) \approx m^2(\phi - \phi_{TOP})^2/2$ for ϕ close to ϕ_{TOP} . (We assume here, that $\phi_{TOP} = 0$.) An *overshoot* solution is a solution to (4.3) with the boundary condition that $\dot{\phi}(0) = 0$ and $\dot{\phi}(\xi) \geq 0$ for all $\xi > 0$. An *undershoot* solution is a solution also with $\dot{\phi}(0) = 0$, but which has $\dot{\phi}(\xi_0) \leq 0$ for some $\xi_0 > 0$. In both cases we require the solution to be non-trivial, *i.e.* we exclude the solution $\phi = \text{constant}$. A *bubble* solution is *both* an undershoot solution and an overshoot solution.

Let us first demonstrate that overshoot solutions always exist. Now, choose a release point, $\phi(0)$ very close to the false vacuum at ϕ_+ . V is roughly constant here, so we may use the vacuum solution (4.5) for X and Y . With $\Phi = \phi - \phi_+$ we find,

$$\ddot{\Phi} + H_+(\cot H_+\xi - 2 \tan H_+\xi)\dot{\Phi} = V''(\phi_+)\Phi \quad (\text{B1})$$

Notice that the friction term turns into a push as ξ approaches $\pi/2$. Close to this point the friction force becomes negative, *i.e.* turns into a push, and diverges like

$$-\left(\frac{\pi}{2H_+} - \xi\right)^{-1} \dot{\Phi} \quad (\text{B2})$$

Due to this large push, it is possible to obtain that $\dot{\Phi}$ be arbitrarily large and still be within the approximation of (B1) – this can be done by choosing $\Phi(0)$ close enough to zero. Now, if $\dot{\Phi}$ is large enough, we can ensure that Φ reaches the true vacuum, and $\dot{\Phi}$ remain positive – this is an overshoot solution.

To analyze undershoot solutions, let us release ϕ at a value $\phi(0) < 0$ close to the top of the barrier, then (4.3) becomes

$$\ddot{\phi} + H_0(\cot H_0\xi - 2 \tan H_0\xi)\dot{\phi} = -m^2\phi \quad (\text{B3})$$

where we have defined $m^2 \equiv -V''(0)$. For $m \gg H_0$ the friction term can be approximated in the region where $\xi \ll \pi/2H_0$ with $\dot{\phi}/\xi$. The solution to (B3) is then well approximated by

the Bessel function $J_0(m\xi)$, so $\dot{\phi}$ changes sign for $m\xi \approx 3.8317$. So for large m a release point close to the top of the barrier results in an undershoot solution. For $m \ll H_0$, however, the potential force on ϕ is very small so as ξ approaches $\pi/2H_0$ the dominant force on ϕ comes from the friction term, which diverges in a way similar to (B2) and gives an overshoot solution. From these two observations we can conclude that solutions to (B3) with the boundary condition $\dot{\phi}(0) = 0$ are characterized by a number n , of oscillation periods (*i.e.* the number of zeroes of $\dot{\phi}$). For $m \ll H_0$, $n = 1$ corresponding to the initial zero of $\dot{\phi}$ at $\xi = 0$. As m increases one finds solutions with higher values of n . The limiting values $m \equiv m_n$ occurs as ‘eigenvalues’ in (B3) exactly when the additional boundary condition $\dot{\phi}(\pi/2H_0) = 0$ is satisfied. To find the lowest eigenvalue m_1 , above which only undershoot solutions to (4.3) exist, we notice that solutions to (B3) which satisfy the boundary conditions $\dot{\phi}(0) = \dot{\phi}(\pi/2H_0)$ are periodic and can be represented as

$$\phi = \sum_{N=0}^{\infty} a_N \cos 2NH_0\xi$$

It is easy to see that the lowest eigenvalue is $m_1^2 = 10H_0^2$ corresponding to the ‘ground state’

$$\phi = \phi_0(1 - 5 \cos 2H_0\xi)$$

For $m^2 < 10H_0^2$ any release point close to the top of the barrier, $\phi(0)$ results in an overshoot, while for $m^2 > 10H_0^2$ any release point gives an undershoot solution. This allows us to conclude, that if $m^2 \geq 10H_0^2$ bubble solutions to (4.3) can be found.

Similar arguments for solutions with $S^2 \times S^2$ topology would have given a value for m_1^2 of $4H_0^2$. This is the same limit one finds for existence of non-trivial (SO(4)-symmetric) solutions to (3.2).

APPENDIX C

In this section we show that the Euclidean Kantowski-Sachs equations (4.3) do not admit solutions of $S^1 \times S^3$ topology that is solutions such that X does not vanish and that Y has two zeroes. For suppose that Y has a zero at $\xi = 0$ then it follows without loss of generality that $\dot{Y} = 1$ there. We may thus directly integrate (4.3) to obtain

$$\dot{Y} = \frac{X}{X(0)} \exp \left(-\frac{4\pi}{M_P^2} \int_0^\xi d\xi \dot{\phi}^2 \frac{Y}{\dot{Y}} \right) \quad (C1)$$

We now derive a contradiction by showing that X must vanish at the top of the bounce where $\dot{Y}$ vanishes. Expanding about this point we obtain :

$$\ddot{Y}_{TOP}(\xi_{TOP} - \xi) = \frac{X_{TOP}}{X(0)} \exp(-A)(\xi_{TOP} - \xi)^\nu + O((\xi_{TOP} - \xi)^{\nu+1}) \quad (C2)$$

where A is a constant and $\nu = 4\pi/M_P^2 (\dot{\phi}_{TOP})^2 Y_{TOP}/(-\ddot{Y}_{TOP})$. Clearly X is only non-zero if ν is unity, but equations (4.3) then show that $\nu = 1 - (\ddot{Y}_{TOP} Y_{TOP})^{-1}$. Thus X must vanish which is a contradiction. Thus only those topologies for which solutions have been found numerically are possible.

APPENDIX D

Here we shall show that the non-singular vacuum solutions to (4.3) are given by (4.5) and (4.6). For a constant scalar field we can integrate the Y -equation in (4.3). We find (for Y not a constant),

$$(\dot{Y}^2 - 1)Y = -\frac{\Lambda}{3}Y^3 + \frac{\Lambda}{3}a \quad (D1)$$

where a is the constant of integration. Thus if Y vanishes, a must be zero. In this case integration of (D1) gives,

$$Y = \sqrt{\frac{3}{\Lambda}} \cos \sqrt{\frac{\Lambda}{3}}(\xi - \xi_0)$$

The equation for X can now be solved,

$$X = C \sin \sqrt{\frac{\Lambda}{3}}(\xi - \xi_0)$$

where C is a constant which can be fixed by requiring (4.2) be non-singular: close to the

point where X vanishes, (4.2) takes the form,

$$ds^2 = d\xi^2 + C^2 \frac{\Lambda}{3} \xi^2 d\chi^2 + Y^2(0)(d\theta^2 + \sin^2 \theta d\varphi^2)$$

But this expression has a conical singularity, unless the first two terms add up to the usual line element in two dimensions, *i.e.* unless $C^2 = 3/\Lambda$. In which case, the solution becomes the S^4 solution as listed in (4.5).

For a non zero (D1) corresponds to the motion of a Newtonian particle Y , with energy $1/2$ in a potential

$$U(Y) = \frac{\Lambda}{6} \left(Y^2 - \frac{a}{Y} \right) \quad (\text{D2})$$

If a is positive $\dot{Y}$ diverges, and the solution is singular, so a must be negative. Now (4.3) imply that X is proportional to $\dot{Y}$. In order to remove possible conical singularities at the two zeroes of X which correspond to the limits of oscillation requires that the values of $\dot{X}$ be equal and opposite at the two zeroes. It is easy to show that this can only occur if Y is constant. In this case the Y equation of (4.3) shows that $Y = 1/\sqrt{\Lambda}$ and $X = D \sin \sqrt{\Lambda}(\xi - \xi_0)$ follows easily and the value of D is fixed by enforcing non-singularity such that $D^2 = \Lambda^{-1}$ and we obtain the $S^2 \times S^2$ solution given in (4.6). It is clear that this result must have been known to workers on gravitational instantons.

The Lorentzian vacuum solutions with a cosmological term are also easily found. We first discuss the case where the surfaces of homogeneity have the geometry of $S^1 \times S^2$. We obtain the first integral:

$$(\dot{Y}^2 + 1)Y = \frac{\Lambda}{3}Y^3 - b \quad (\text{D3})$$

where b is a constant of integration. For b positive we have expanding solutions with both X and Y increasing from $\xi = 0$, starting with vanishing X and non vanishing Y . For b negative we have two classes of behavior according to whether $-b\sqrt{\Lambda}$ is less than or greater than $\frac{2}{3}$. The former case corresponds to expansion from zero X and non-zero Y . The latter case to the asymptotic behavior with $Y \propto t^{2/3}$ and $X \propto t^{-1/3}$ at small t and both X and Y proportional to $\exp \sqrt{\Lambda/3}t$ at late times. The case $b = 0$ has the solutions of four dimensional Lorentzian de Sitter space or where Y is a constant of the direct product of two dimensional de Sitter space and the two sphere. For the case of non-compact surfaces of homogeneity, we have expanding solutions.

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FIGURE CAPTIONS

1. $V(\phi)$ shown versus ϕ . The false vacuum at ϕ_+ is separated from the true vacuum at ϕ_- by a potential barrier at ϕ_{TOP} .
2. $V(\phi)$ shown versus ϕ . For a typical bubble solution the scalar field interpolates between $\phi(0)$ and $\phi(\xi_{max})$. This corresponds to region I. Region II corresponds to the future evolution of the field into the true vacuum, and region III to the future evolution into the false vacuum.
3. Penrose diagram for SO(4) invariant bubble. The wavy line denotes the moment of bubble formation, solid lines the trajectories of comoving points (*i.e.* in I points on the bubble) and dashed lines denote coordinate singularities).
4. Penrose diagram for the S^4 Kantowski-Sachs universe with Y continuation.
5. Penrose diagram for the S^4 Kantowski-Sachs universe with X continuation.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5