CERN-TH.5330/89

**Quantum Yang-Mills Theory on Riemann Surfaces
and Conformal Field Theory**

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Abstract

It is shown that Yang-Mills theory on a smooth surface, when suitably quantized, is a topological quantum field theory. This topological gauge theory is intimately related to two-dimensional conformal field theory. It is conjectured that all conformal field theories may be obtained from Yang-Mills theory on smooth surfaces

CERN-TH.5330/89

March 1989

Two-dimensional conformally invariant quantum field theory [1] has assumed an important position in contemporary theoretical physics. Conformal field theory has significant connections with string theory, statistical mechanics and certain areas of pure mathematics. These connections are mostly well known. In the context of string theory, conformal field theories are classical solutions of the string equations of motion. Many statistical mechanical systems correspond, at their critical points, to conformal field theories and the conformal invariance can provide important information about their critical behaviour (see [2], for example). There are several areas of mathematics which also appear to have fundamental connections with conformal field theory. The representation theory of loop groups and of the group of diffeomorphisms of the circle are closely related to conformal field theory [3]. The Fischer-Griess monster group is the group of automorphisms of a conformal field theory. The elliptic cohomology theory of Landweber and Stong [4] and the index theorem for the Dirac operator on loop space [5] appear also to be related to conformal field theory [6]. There appear to be many mysterious connections, involving quantum groups and the Yang-Baxter equation, between integrable systems and conformal field theory. The deep connections between conformal field theory and invariants of knots and links (see [7]) have recently been elucidated in the work of Witten [8]. It is clear then that conformal field theories are interesting objects and that they are worthy of study. If, however, one's primary goal is to understand the fundamental structure of string theory then it is less clear what significance should be attached to the study of conformal field theories. Understanding the structure of string theory will crucially depend upon understanding the correct degrees of freedom and the correct symmetries of the theory. A more geometrical understanding of conformal field theory may lead to a deeper understanding of these issues. At the moment, however, the various definitions of conformal field theory which have been proposed [1] [9] [10] are all essentially algebraic, or algebro-geometric, in nature. Although this algebraic approach has led to important progress towards the classification of rational conformal field theories [11] [12] [13] it seems likely that an alternative, more geometrical viewpoint, will be important in understanding the deeper structure of string theory. The recent very interesting work of Witten has shown that generally covariant quantum field theories in 2+1 dimensions are closely related to conformal field theories in 1+1 dimensions [8]. The connection between topological quantum field theory and conformal field theory may be of great importance in understanding the geometrical structure of string theory. In [8] it was shown that the topological quantum field theory in 2+1 dimensions defined by the Chern-Simons action for a gauge theory is directly related to current algebra in 1+1 dimensions. In fact it may be shown that all known rational conformal field theories in 1+1 dimensions may be obtained from the Chern-Simons topological gauge theory in 2+1 dimensions by an appropriate choice of gauge group [14]. In this paper we show that two-dimensional Yang-Mills theory, when suitably quantized, is a topological quantum field theory which is intimately related to two-dimensional conformal field theory. This connection may be of importance in attempting to understand how the circle of ideas relating topological quantum field theory and conformal field theory elucidates the structure and symmetries of string theory.

We consider a gauge theory defined on a compact smooth surface Σ . Let G be a compact simple Lie group. A gauge theory on Σ , with gauge group G , is described geometrically by a principal G -bundle $P \rightarrow \Sigma$. Associated to P by the adjoint action of G on itself is the bundle of groups

$$\text{Ad}(P) = P \times_G G$$

and the bundle associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$

$$\text{ad}(P) = P \times_G \mathfrak{g}.$$

The space of $\text{ad}(P)$ -valued p -forms on Σ is

$$\Omega^p(\Sigma; \text{ad}(P)) = \Gamma(\Lambda^p T^* \Sigma \otimes \text{ad}(P)).$$

A gauge potential is a connection 1-form on P . The space $\mathcal{A}$ of all connections on $P \rightarrow \Sigma$ is an affine space with associated vector space $\Omega^1(\Sigma; \text{ad}(P))$. Let $\mathcal{G}$ denote the group of all gauge transformations of $P \rightarrow \Sigma$, i.e., the group of all automorphisms of P which induce the identity diffeomorphism on Σ . The Yang-Mills action on Σ is the natural L^2 -norm on $\Omega^2(\Sigma; \text{ad}(P))$ defined by

$$S[A] = \int_{\Sigma} \text{Tr}(F_A \wedge *F_A), \quad (1)$$

where $F_A \in \Omega^2(\Sigma; \text{ad}(P))$ is the curvature of the connection A , $*$ is the Hodge duality operator induced by a fixed Riemannian structure on Σ and Tr represents an invariant bilinear form on $\mathfrak{g}$. The Yang-Mills action $S[A]$ is invariant under the natural action of $\mathcal{G}$ on $\mathcal{A}$ and hence gives a well-defined function on the orbit space $\mathcal{A}/\mathcal{G}$.

To exhibit two-dimensional Yang-Mills theory as a topological quantum field theory (in spite of its apparent dependence on the Riemannian structure on Σ) we will give a covariant Hilbert space quantization of the theory. To understand the Hilbert space quantization of the theory defined by (1) it is first necessary to understand the classical phase space of Yang-Mills theory. Abstractly a classical phase space (see [15], for example) is a *Poisson manifold*, i.e., a manifold X together with an antisymmetric map $\{, \} : C^\infty(X) \times C^\infty(X) \rightarrow C^\infty(X)$ on the ring of smooth functions on X which satisfies the Jacobi identity. The map $\{, \}$ is called the *Poisson bracket* on X . The most natural geometrical way to obtain a Poisson bracket on a manifold X is that it be induced by a symplectic structure on X . If X has a symplectic structure then there exists a closed, non-degenerate 2-form ω_X on X , which gives a natural isomorphism $\# : T^*X \rightarrow TX$ between 1-forms and vector fields on X . The Poisson bracket of two functions $f, g \in C^\infty(X)$ is given, in terms of this isomorphism, by

$$\{f, g\} = (df)^\#(g). \quad (2)$$

The Poisson bracket $\{, \}$ induces a Lie algebra structure $\mathcal{F}(X)$ on the space $C^\infty(X)$ of smooth functions on X .

In this paper we wish to give a Hilbert space quantization of Yang-Mills theory on a two-dimensional surface Σ of arbitrary topology. However conventional canonical quantization requires that Σ should have the topology $S^1 \times \mathbb{R}$. To quantize Yang-Mills theory on arbitrary surfaces we shall employ a covariant Hilbert space quantization scheme.

Consider a local field theory, defined on a manifold Z , where the fields are denoted by φ . Assume that the dynamical behaviour of the theory is determined by a Lagrangian density $L = L(\varphi, \nabla\varphi, q)$, for $q \in Z$. In the path integral approach to quantum field theory the starting point is the action

$$I = \int_Z L \, dv \quad (3)$$

and the quantization procedure is manifestly covariant. Alternatively, one can violate covariance by the explicit choice of a time coordinate t and introduce a constant time hypersurface $S \subset Z$ (defined at $t = 0$, say). One then considers the time-dependent Lagrangian

$$\int_S L d\sigma, \quad (4)$$

where $d\sigma$ is the volume element on S . This is the standard starting point for canonical quantization. In this approach the phase space is simply the space of positions and momenta at time zero. It may seem from this that the notion of phase space is inherently non-covariant. This, however, is not the case. It is possible to give an equivalent definition of phase space which is manifestly covariant (this is well-known in the context of geometric quantization (see [16]) and has been more recently discussed in [17]). The phase space may be defined as the space Z of solutions of the classical equations of motion. This definition is clearly equivalent to the conventional definition, as specifying the positions and momenta at time zero precisely determines a classical solution. It is then possible to define a symplectic structure on Z by integrating a conserved symplectic current over a spacelike hypersurface $S \subset Z$ (as described in [16], [17]).

In the case of two-dimensional Yang-mills theory the equations of motion which follow from the action (1) are the Yang-Mills equations

$$D_A * F_A = 0, \quad (5)$$

where $D_A : \Omega^p(\Sigma; \text{ad}(P)) \rightarrow \Omega^{p+1}(\Sigma; \text{ad}(P))$ is the covariant derivative associated to the gauge connection A . Thus the covariant phase space of Yang-Mills theory on a smooth surface Σ can be defined as

$$\mathfrak{R} = \{A \in \mathcal{A} : D_A * F_A = 0\} / \mathcal{G}. \quad (6)$$

The structure of the phase space $\mathfrak{R}$ will be discussed below in some detail and, in particular, it will be shown that $\mathfrak{R}$ has a mathematically natural symplectic structure.

Given a symplectic manifold X , which is the phase space of some dynamical system, geometric quantization (see [16], for example) attempts to identify the quantum Hilbert space of the theory with the space of sections of a complex line bundle $\mathcal{L} \rightarrow X$, with a Hermitian connection, which are covariantly constant on the leaves of a Lagrangian foliation of X . To be more precise, consider a *quantizable* symplectic manifold X . A symplectic manifold X with symplectic form ω_X is quantizable if the cohomology class $[\omega_X] \in H^2(X; \mathbf{R})$ of ω_X is integral, i.e., if $[\omega_X] \in \text{im } i_*$, where $i_* : H^2(X; \mathbf{Z}) \rightarrow H^2(X; \mathbf{R})$. If X is quantizable then there exists a complex line bundle $\mathcal{L} \rightarrow X$ such that the first Chern class $c_1(\mathcal{L}) = [\omega_X]$, and there exists a Hermitian connection ∇ on $\mathcal{L}$ with curvature 2-form $R^\nabla = \omega_X$. We now wish to interpret the quantum Hilbert space as the space of sections, of some suitable type, of $\mathcal{L}$. As the wave function in quantum theory does not depend upon both the position *and* momentum variables it is necessary to separate the phase space into different degrees of freedom representing “positions” and “momenta”. Such a separation is provided by giving a *polarization* of X , i.e., a smooth map P which assigns to each point $x \in X$ a Lagrangian subspace $P_x \subset T_x^{\mathbf{C}} X$ of the complexified tangent space at X . Thus P is a Lagrangian

foliation of X . Relative to a fixed polarization P of X , a local section $s : U \rightarrow \mathcal{L}$ is *polarized* if $\nabla_\xi s = 0$ for all complex vector fields ξ on U for which $\overline{\xi(x)} \in P_x$, for all $x \in X$. For a given polarization P of X one attempts to define the quantum Hilbert space $\mathcal{H}$ to be the space of all P -polarized sections of $\mathcal{L} \rightarrow X$. The fundamental problem which confronts geometric quantization is that a general (quantizable) symplectic manifold X has no natural polarization. If we simply pick an arbitrary polarization P on X then it is not clear that the quantization is independent of P . Thus, there is no known way to quantize an arbitrary symplectic manifold. Certain symplectic manifolds, however, have additional geometric structure which defines a canonical polarization and these symplectic manifolds may be quantized. The class of symplectic manifolds which have a natural polarization that will be of interest in this paper are Kähler manifolds. If X is a Kähler manifold with complex structure J then a canonical polarization is defined by

$$P_x = \{\xi_x \in T_x^C X : J_x \xi_x = i \xi_x\}, \quad (7)$$

for $x \in X$. This polarization corresponds to separating the phase space X into holomorphic and anti-holomorphic degrees of freedom. For a Kähler manifold with the complex polarization (7) the Hermitian line bundle $\mathcal{L} \rightarrow X$ has a holomorphic structure and the quantum Hilbert space $\mathcal{H} = \Gamma(\mathcal{L})$, the space of holomorphic sections of $\mathcal{L}$.

We will now apply these general ideas to the quantization of Yang-Mills theory on a smooth surface Σ . We will see that the resulting theory is a topological quantum field theory. This may seem, at first sight, paradoxical; the Yang-Mills action (1) makes explicit reference to a Riemannian structure on Σ (through the Hodge $*$), but a topological quantum field theory is, by definition, independent of any metric on Σ . The crucial observation, which resolves this paradox, is that in two dimensions the solutions of the Yang-Mills equations do not depend upon the metric, but only upon the global topology of Σ . In fact the solutions are essentially determined by representations of $\pi_1(\Sigma)$. Thus the phase space $\mathfrak{R}$, defined as the space of solutions of the Yang-Mills equations, is independent of any choice of metric on Σ and depends only on the topology of Σ . Consequently, when $\mathfrak{R}$ is quantized we obtain a topological quantum field theory.

The phase space $\mathfrak{R}$ is defined in (6) to be the space of equivalence classes of solutions of the Yang-Mills equations (5). Thus we wish to study those connections A for which $D_A * F_A = 0$. To describe such connections we will introduce, as in [18], a suitable central extension of the fundamental group $\pi_1(\Sigma)$ of Σ (a more detailed discussion of these ideas will be given in [19]). Recall that the fundamental group $\pi_1(\Sigma)$ of a surface Σ of genus g is a free group on $2g$ generators $a_1, b_1, \dots, a_g, b_g$ subject to the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

For $g \geq 1$, the group $\pi_1(\Sigma)$ has a universal central extension π :

$$1 \rightarrow \mathbf{Z} \rightarrow \pi \rightarrow \pi_1(\Sigma) \rightarrow 1, \quad (8)$$

where π is generated by $a_1, b_1, \dots, a_g, b_g$ and a central element j , subject to the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = j. \quad (9)$$

The central extension Π of $\pi_1(\Sigma)$ of interest here is obtained from π by extending the centre to $\mathbf{R}$. Thus, we have a central extension

$$1 \rightarrow \mathbf{R} \rightarrow \Pi \rightarrow \pi_1(\Sigma) \rightarrow 1. \quad (10)$$

It is not difficult to show (see [18]) that given any homomorphism $\rho : \Pi \rightarrow G$ one obtains a connection A_ρ which satisfies the Yang-Mills equations (5). Conversely, as is proved in [18], all connections satisfying the Yang-Mills equations may be obtained in this way. Hence, there is a one-to-one correspondence between equivalence classes of connections which are solutions of the Yang-Mills equations on Σ and conjugacy classes of homomorphisms $\rho : \Pi \rightarrow G$. The phase space of Yang-Mills theory may, therefore, be identified with the space $\text{Hom}(\Pi, G)/G$ of homomorphisms from Π to G modulo conjugation, i.e.,

$$\mathfrak{R} = \text{Hom}(\Pi, G)/G. \quad (11)$$

It should be noted that the space $\text{Hom}(\Pi, G)/G$ describes solutions of the Yang-Mills equations on bundles over Σ of all topological type. Thus, the phase space $\mathfrak{R}$ defined in (11) should be regarded as a universal phase space. In the remainder of this paper we shall use $\mathfrak{R}$ to denote both the universal phase space $\text{Hom}(\Pi, G)/G$ and a generic component of this space.

We have seen that the phase space $\mathfrak{R}$ depends only on the topology of Σ ; thus, when $\mathfrak{R}$ is quantized the resulting theory will be a topological quantum field theory. As has been noted above, in order to quantize a symplectic manifold it is necessary for the manifold to have a Kähler structure. The phase space $\mathfrak{R}$ of Yang-Mills theory does not have a canonical Kähler structure; however $\mathfrak{R}$ does have a Kähler structure once a complex structure is chosen on Σ . Once a complex structure is chosen on Σ the space $\mathfrak{R}$ may be identified with the moduli space of holomorphic vector bundles over Σ , and this space has a natural Kähler structure.

Consider a complex vector bundle $E \rightarrow \Sigma$ of rank n and degree r ($\text{degree}(E) = c_1(E)$, the Chern class of E). Let A be a unitary connection on E and let

$$D_A : \Omega^p(\Sigma; E) \rightarrow \Omega^{p+1}(\Sigma; E) \quad (12)$$

be the covariant derivative associated to A , where $\Omega^p(\Sigma; E)$ denotes the space of p -forms on Σ with values in E . The Hodge $*$ operator associated to a Riemannian structure on Σ gives a decomposition

$$\Omega_{\mathbb{C}}^1(\Sigma; E) = \Omega^{1,0}(\Sigma; E) \oplus \Omega^{0,1}(\Sigma; E) \quad (13)$$

relative to which the covariant derivative operator D_A splits into

$$\mathcal{D}_A : \Omega^0(\Sigma; E) \rightarrow \Omega^{1,0}(\Sigma; E) \quad (14)$$

and

$$\overline{\mathcal{D}}_A : \Omega^0(\Sigma; E) \rightarrow \Omega^{0,1}(\Sigma; E). \quad (15)$$

This decomposition defines a holomorphic structure on E , such that the local frames s satisfy $\overline{\mathcal{D}}_A s = 0$. Let $\mathcal{C}$ denote the space of all holomorphic structures on E . There is a map $\mathcal{A} \rightarrow \mathcal{C}$ given by taking the $\overline{\mathcal{D}}_A$ component of the covariant derivative D_A and this map is an isomorphism (see [18]).

The group of automorphisms $\text{Aut}(E)$ of E acts on $\mathcal{C}$ and the orbit space $\mathcal{C}/\text{Aut}(E)$ is, by definition, the moduli space of holomorphic bundles on Σ with rank n and degree r . Denote this moduli space by $M_{n,r}$, i.e.,

$$M_{n,r} = \mathcal{C}/\text{Aut}(E). \quad (16)$$

Unfortunately, the moduli space $M_{n,r}$ is a non-Hausdorff space. To obtain a “good” moduli space it is necessary to consider only the stable bundles over Σ . Let $\mu(E) = c_1(E)/\text{rank}(E)$. A holomorphic bundle E is *stable* if for every proper holomorphic sub-bundle $F \subset E$, $\mu(F) < \mu(E)$. If $\bar{\mathcal{C}}$ denotes the space of all stable holomorphic bundles over Σ then the moduli space

$$\bar{M}_{n,r} = \bar{\mathcal{C}}/\text{Aut}(E) \quad (17)$$

of stable bundles of rank n and degree r is a Hausdorff space. The most natural way to exhibit a Kähler structure on $\bar{M}_{n,r}$ is by using the moment map (see [18]).

We recall the general definition of the moment map in symplectic geometry (see [20], for instance). Let X be a symplectic manifold with symplectic form ω_X . A diffeomorphism $f : X \rightarrow X$ is *symplectic* if $f^*\omega_X = \omega_X$. An *infinitesimal symplectic transformation* is given by a vector field ξ on X , for which

$$L_\xi \omega_X = 0, \quad (18)$$

where $L_\xi = d i_\xi + i_\xi d$ (with d the exterior derivative and i_ξ interior product with ξ) is the Lie derivative. Let K be a Lie group which acts on X by symplectic transformations. In addition, let $\mathfrak{k}$ denote the Lie algebra of K and let $\mathfrak{k}^*$ denote the dual of the Lie algebra. A vector field on a symplectic manifold corresponding under the isomorphism $\# : T^*X \rightarrow TX$ to the differential of a function is called a Hamiltonian vector field. If we denote the Lie algebra of Hamiltonian vector fields on X by $\text{Ham}(X)$ then the map $f \rightarrow (df)^\#$ defines a canonical extension of Lie algebras

$$0 \rightarrow \mathbf{R} \rightarrow \mathcal{F}(X) \xrightarrow{h} \text{Ham}(X) \rightarrow 0, \quad (19)$$

where $\mathcal{F}(X)$ is the Lie algebra of smooth functions on X and the map $\mathbf{R} \rightarrow \mathcal{F}(X)$ is the inclusion of $\mathbf{R}$ as the constant functions on X .

The action of the Lie algebra $\mathfrak{k}$ on X by symplectic transformations defines a Lie algebra homomorphism

$$P : \mathfrak{k} \rightarrow \text{Ham}(X). \quad (20)$$

A lifting $\hat{P}$ of P is a Lie algebra homomorphism

$$\hat{P} : \mathfrak{k} \rightarrow \mathcal{F}(X), \quad (21)$$

such that $P = h \circ \hat{P}$. Such a lifting (if it exists) is called a *co-moment map* on X . It is well known (see [20]) that there exists a unique lifting $\hat{P}$ of P if $H^1(\mathfrak{k}) = H^2(\mathfrak{k}) = 0$. A co-moment map $\hat{P}$ defines, dually, a *moment map*

$$\mu : X \rightarrow \mathfrak{k}^*. \quad (22)$$

The moment map μ is K -equivariant, i.e.,

$$\mu(k(x)) = (\text{ad } k)^*(\mu(x)), \quad (23)$$

where $k \in K$ and $x \in X$. Thus, for every orbit $\theta \subset \mathfrak{k}^*$ the inverse image $\mu^{-1}(\theta) \subset X$ is K -invariant. The quotient space

$$W = \mu^{-1}(\theta)/K \quad (24)$$

is the *symplectic quotient* of X by K . Under suitable non-degeneracy conditions, W inherits a natural symplectic structure from X .

Returning to the moduli space $\overline{M}_{n,r}$ of stable bundles on Σ , we can now see that $\overline{M}_{n,r}$ inherits a natural Kähler structure (see [18]). The space of connections $\mathcal{A}$ has a natural symplectic structure defined by

$$\omega(\alpha, \beta) = \int_{\Sigma} \text{Tr}(\alpha \wedge \beta), \quad (25)$$

where $\alpha, \beta \in T\mathcal{A} = \Omega^1(\Sigma; \text{ad}(P))$. With respect to this symplectic structure $\mathcal{G}$ acts on $\mathcal{A}$ by symplectic transformations. The associated moment map

$$\mu : \mathcal{A} \rightarrow \text{Lie}(\mathcal{G})^* \quad (26)$$

is given by

$$\mu(A) = F_A, \quad (27)$$

where $A \in \mathcal{A}$. Taking $\theta \in \text{Lie}(\mathcal{G})^*$ to be the orbit defined by the Yang-Mills minimum gives

$$\overline{M}_{n,r} = \mu^{-1}(\theta)/\mathcal{G}. \quad (28)$$

Thus $\overline{M}_{n,r}$ inherits a natural symplectic structure; furthermore, $\overline{M}_{n,r} = \overline{\mathcal{C}}/\text{Aut}(E)$ has a natural complex structure induced from $\overline{\mathcal{C}}$. Thus $\overline{M}_{n,r}$ has a natural Kähler structure. In fact, when r is prime to n the moduli space $\overline{M}_{n,r}$ is compact and, in this case, $\overline{M}_{n,r}$ is a projective algebraic variety.

The essential connection between the phase space of Yang-Mills theory, defined in (11) to be $\mathfrak{R} = \text{Hom}(\Pi; G)/G$ (where Π is the central extension of $\pi_1(\Sigma)$ given in (10)), and the moduli space of holomorphic vector bundles over Σ is given by the theorem of Narasimhan and Seshadri [21], as reformulated by Atiyah and Bott [18]. This theorem states that a holomorphic $G_{\mathbb{C}}$ -bundle over a compact Riemann surface Σ is stable if and only if it arises from an irreducible representation $\rho : \Pi \rightarrow G$, and furthermore, isomorphic bundles correspond to equivalent representations. Thus we can identify the phase space $\mathfrak{R} = \text{Hom}(\Pi, G)/G$ with the moduli space $\overline{M}_{n,r}$ of stable bundles over Σ . It follows from this identification that the phase space $\mathfrak{R}$ has a natural Kähler structure. It should be noted that the space $\overline{M}_{n,0}$ occurred in [8] as the non-covariant phase space of Chern-Simons theory in $2+1$ dimensions.

To quantize the phase space $\mathfrak{R}$, with the Kähler polarization defined by (7), we require a holomorphic line bundle $\mathcal{L} \rightarrow \mathfrak{R}$ with $c_1(\mathcal{L}) = \omega_{\mathfrak{R}}$, where $\omega_{\mathfrak{R}}$ is the Kähler form on $\mathfrak{R}$. The Hilbert space $\mathcal{H}$ obtained by covariantly quantizing Yang-Mills theory will then be given by the space $\Gamma(\mathcal{L})$ of holomorphic sections of $\mathcal{L}$. Given a vector bundle $E \rightarrow \Sigma$ there is a $\bar{\partial}$ operator $\bar{\partial} : \Omega^{0,0}(\Sigma; E) \rightarrow \Omega^{0,1}(\Sigma; E)$, associated to each holomorphic structure on E . Over a point in $\mathcal{C}$ corresponding to the operator $\bar{\partial}$ we may define the determinant line

$$\bigwedge^{\max}(\ker \bar{\partial})^* \otimes \bigwedge^{\max}(\ker \bar{\partial}^*). \quad (29)$$

This family of determinant lines defines a holomorphic line bundle $\text{Det } \bar{\partial}$ over $\mathcal{C}$ [22]. The line bundle $\text{Det } \bar{\partial}$ has a Hermitian metric and a compatible connection, the curvature of which is given by the 2-form on $\mathcal{C}$ (suggested by the index theorem)

$$\Omega = \left\{ \int_{\Sigma} \text{ch}(E) \hat{A}(\Sigma) \right\}_{(2)}. \quad (30)$$

It is proved in [22] that the curvature Ω is equal to the Kähler form on $\mathcal{C}$. The determinant line bundle $\text{Det } \bar{\partial}$ descends to $\mathfrak{R} = \mathcal{C}/\text{Aut}(E)$ and the curvature form $\Omega_{\mathfrak{R}}$ of $\text{Det } \bar{\partial} \rightarrow \mathfrak{R}$ satisfies $\Omega_{\mathfrak{R}} = \omega_{\mathfrak{R}}$. More generally, we may consider the Kähler form $\hat{\omega}_{\mathfrak{R}} = k\omega_{\mathfrak{R}}$ (for k an integer) and work with the line bundle $\hat{\mathcal{L}} = (\text{Det } \bar{\partial})^{\otimes k}$, whose curvature $\hat{\Omega}_{\mathfrak{R}}$ satisfies $\hat{\Omega}_{\mathfrak{R}} = \hat{\omega}_{\mathfrak{R}}$. If σ is a complex structure on Σ then we have seen that σ induces a Kähler structure on $\mathfrak{R}$. Let $\mathcal{L}_{\sigma} = \text{Det } \bar{\partial}$ denote the determinant line bundle on $\mathfrak{R}$ for this choice of σ . The quantum Hilbert space of Yang-Mills theory is given by $\mathcal{H}_{\Sigma, \sigma} = \Gamma(\mathcal{L}_{\sigma})$. In the coprime case $(n, r) = 1$, $\mathfrak{R}$ is a compact manifold and it follows from the Riemann-Roch theorem that $\mathcal{H}_{\Sigma, \sigma} = \Gamma(\mathcal{L}_{\sigma})$ is finite dimensional.

The quantization of Yang-Mills theory on a smooth surface Σ , with complex structure σ , associates to Σ and σ a Hilbert space $\mathcal{H}_{\Sigma, \sigma}$, which in the coprime case is finite dimensional. Let $\mathcal{M}_g$ denote the moduli space of complex structures on the surface Σ of genus g . As σ varies over $\mathcal{M}_g$ the family of Hilbert spaces $\{\mathcal{H}_{\Sigma, \sigma}\}_{\sigma \in \mathcal{M}_g}$ forms a holomorphic vector bundle $\mathcal{V} \rightarrow \mathcal{M}_g$. The vector bundle $\mathcal{V}$ is of finite rank in the coprime case $(n, r) = 1$. For each choice of complex structure $\sigma \in \mathcal{M}_g$ the phase space $\mathfrak{R}$ of Yang-Mills theory is quantized with its canonical polarization; it follows, therefore, that the quantum theory should not depend upon σ . Thus, for any two distinct complex structures $\sigma_1, \sigma_2 \in \mathcal{M}_g$, the projective Hilbert spaces $\mathbf{P}(\mathcal{H}_{\Sigma, \sigma_1})$ and $\mathbf{P}(\mathcal{H}_{\Sigma, \sigma_2})$ should be canonically isomorphic. Recall that if V is a vector bundle over a manifold Y and if γ is a curve in Y with end points $y_1, y_2 \in Y$, then parallel transport along γ will yield a canonical isomorphism between the fibres $V|_{y_1}$ and $V|_{y_2}$, which is independent of the curve γ up to homotopy, if and only if V has a canonical flat connection. Therefore if we denote the projectivization of the vector bundle $\mathcal{V} \rightarrow \mathcal{M}_g$ by $\mathbf{P}(\mathcal{V}) \rightarrow \mathcal{M}_g$ (i.e., the fibre $\mathbf{P}(\mathcal{V})|_{\sigma}$ is $\mathbf{P}(\mathcal{V}|_{\sigma}) = \mathbf{P}(\mathcal{H}_{\Sigma, \sigma})$, for $\sigma \in \mathcal{M}_g$) the isomorphism $\mathbf{P}(\mathcal{H}_{\Sigma, \sigma_1}) \cong \mathbf{P}(\mathcal{H}_{\Sigma, \sigma_2})$ is equivalent to the assertion that the bundle $\mathbf{P}(\mathcal{V}) \rightarrow \mathcal{M}_g$ is *flat*. The projective bundle $\mathbf{P}(\mathcal{V}) \rightarrow \mathcal{M}_g$ is flat if and only if the bundle $\mathcal{V} \rightarrow \mathcal{M}_g$ is *projectively flat*. Projectively flat bundles over $\mathcal{M}_g$ are the key ingredient of the modular geometry approach to conformal field theory [9]. Thus we have obtained the basic geometrical structure of conformal field theory from the quantization of Yang-Mills theory on Σ . The quantization of Chern-Simons theory in $2 + 1$ dimensions also results in projectively flat bundles over $\mathcal{M}_g$; for a discussion of this closely related topic see [8] and also [23].

The quantization of Yang-Mills theory on a Riemann surface Σ (i.e., a smooth surface Σ together with a fixed complex structure σ on Σ) yields a Hilbert space $\mathcal{H}_{\Sigma} = \Gamma(\text{Det } \bar{\partial})$. The association $\Sigma \rightarrow \mathcal{H}_{\Sigma}$ is a modular functor in the sense of Segal [10]. It is proved in [10] that any modular functor $\Phi : \Sigma \rightarrow \mathcal{H}_{\Sigma}$ results in a projectively flat vector bundle over the moduli space $\mathcal{M}_g$ of complex structures on the surface of genus g .

A smooth projectively flat Hermitian vector bundle of rank N over a manifold Y is classified by a representation $\phi : \pi_1(Y) \rightarrow PU(N)$ of $\pi_1(Y)$ on the projective unitary group $PU(N)$. The moduli space $\mathcal{M}_g$ is simply connected for all $g \geq 1$ [24] and hence there are no non-trivial representations of $\pi_1(\mathcal{M}_g)$. However, this does not imply that there are no

non-trivial flat bundles over $\mathcal{M}_g$. The moduli space $\mathcal{M}_g$ is an orbifold, with singularities corresponding to those Riemann surfaces which admit a non-trivial group of automorphisms, and thus the bundles naturally defined over $\mathcal{M}_g$ are V -bundles. If $\mathcal{M}_g^s$ denotes the smooth complement of the set of orbifold points in $\mathcal{M}_g$ then it may be shown that $\pi_1(\mathcal{M}_g^s) = \Gamma_g$, where Γ_g is the mapping class group of the surface of genus g . Thus a projectively flat V -bundle of rank N over $\mathcal{M}_g$ is classified by a representation $\phi : \Gamma_g \rightarrow PU(N)$. Hence, as we have seen above from a different viewpoint, a projectively flat bundle over $\mathcal{M}_g$ depends only on the topology of the surface Σ .

It is possible to generalize the situation discussed above by considering Yang-Mills theory on a surface Σ with static non-Abelian charges at some marked points $P_i \in \Sigma$. In this case the natural modification of the geometrical structure is that the phase space $\mathfrak{R}$ becomes the moduli space of *parabolically stable* $G_{\mathbb{C}}$ -bundles over Σ . This moduli space is a Kähler manifold and the theory can be quantized geometrically.

The quantization of Yang-Mills theory on a smooth surface Σ results in a projectively flat vector bundle $\mathcal{V}$ over the moduli space $\mathcal{M}_g$ of complex structures on Σ and this is the essential geometrical structure of conformal field theory. In fact quantum Yang-Mills theory in two dimensions is intimately related to two-dimensional current algebra. (For a closely related discussion in the context of Chern-Simons theory in $2 + 1$ dimensions see [23]).

The phase space $\mathfrak{R}$ of Yang-Mills theory (with gauge group G) on a smooth surface Σ may be identified, once a complex structure σ is chosen on Σ , with the moduli space of holomorphic $G_{\mathbb{C}}$ -bundles over the Riemann surface defined by Σ together with the complex structure σ . Denote the Riemann surface (Σ, σ) simply by Σ , and let $x \in \Sigma$ be a distinguished point. Let $D \subset \Sigma$ be the unit disc around x and let $\Sigma_0 = \Sigma - D$ denote the complement of D in Σ . A holomorphic $G_{\mathbb{C}}$ -bundle V over Σ is clearly trivial (both topologically and holomorphically) over D . Furthermore, V is also trivial over Σ_0 . This is because Σ_0 has no 2-cell and thus any bundle is topologically trivial over Σ_0 ; in addition, Σ_0 is a Stein manifold and therefore if a bundle is topologically trivial over Σ_0 it is also holomorphically trivial. Thus V is determined by the clutching data γ on $S^1 = D \cap \Sigma_0$; or, equivalently, V is determined by an element $\gamma \in LG_{\mathbb{C}}$ of the loop group of $G_{\mathbb{C}}$. If $L_D G_{\mathbb{C}}$ and $L_{\Sigma_0} G_{\mathbb{C}}$ denote the groups of loops in $LG_{\mathbb{C}}$ which extend to holomorphic maps $D \rightarrow G_{\mathbb{C}}$ and $\Sigma_0 \rightarrow G_{\mathbb{C}}$, respectively, then the moduli space of holomorphic $G_{\mathbb{C}}$ bundles on Σ is given by the double coset space [3]

$$\mathfrak{R} = L_{\Sigma_0} G_{\mathbb{C}} \backslash LG_{\mathbb{C}} / L_D G_{\mathbb{C}}. \quad (31)$$

The loop group $LG_{\mathbb{C}}$ has a universal central extension $\widehat{LG}_{\mathbb{C}}$ by the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers [3]

$$\mathbb{C}^\times \rightarrow \widehat{LG}_{\mathbb{C}} \rightarrow LG_{\mathbb{C}}. \quad (32)$$

Taking the double quotient space

$$\mathcal{P} = L_{\Sigma_0} G_{\mathbb{C}} \backslash \widehat{LG}_{\mathbb{C}} / L_D G_{\mathbb{C}} \quad (33)$$

defines a principal $\mathbb{C}^\times$ -bundle $\mathcal{P} \rightarrow \mathfrak{R}$. Associated to $\mathcal{P}$ by the action of $\mathbb{C}^\times$ on $\mathbb{C}$ is the holomorphic line bundle

$$\mathcal{L} = \mathcal{P} \times_{\mathbb{C}^\times} \mathbb{C}. \quad (34)$$

The holomorphic line bundle $\mathcal{L} \rightarrow \mathfrak{R}$ so defined may be identified with the determinant line bundle $\text{Det } \bar{\partial} \rightarrow \mathfrak{R}$. The quantum Hilbert space of Yang-Mills theory on Σ is $\mathcal{H}_\Sigma = \Gamma(\text{Det } \bar{\partial}) \cong \Gamma(\mathcal{L})$. However, it follows from the representation theory of loop groups [3] that the space $\Gamma(\mathcal{L})$ of holomorphic sections of $\mathcal{L}$ may be identified with the subspace of the fundamental representation of $\widehat{LG}_\mathbb{C}$ which is invariant under $L_{\Sigma_0} G_\mathbb{C}$. By considering $\mathcal{L} = (\text{Det } \bar{\partial})^{\otimes k}$ one obtains the irreducible representation of $\widehat{LG}_\mathbb{C}$ at level k . Thus two-dimensional current algebra is associated to quantum Yang-Mills theory on Σ .

We have seen that the quantization of Yang-Mills theory on a smooth surface Σ results in the general geometrical structure of conformal field theory and, in particular, is intimately related to current algebra. In the coprime case $(n, r) = 1$, the quantum Hilbert space of Yang-Mills theory on Σ is finite dimensional and we obtain a connection with rational conformal field theory. At the modular functor level the connection between Yang-Mills theory on Σ and rational conformal field theory is essentially identical to the connection between Chern-Simons theory in $2+1$ dimensions and rational conformal field theory. It has been shown [14] that all known rational conformal field theories can be obtained from Chern-Simons theory in $2+1$ dimensions. It should, therefore, follow that all known rational conformal field theories can be obtained from Yang-Mills theory on Σ . It is natural to conjecture (as has been conjectured in the context of Chern-Simons theory in [14]) that all rational conformal field theories may be obtained from Yang-Mills theory on Σ . In fact it is possible to consider a stronger conjecture. In the non-coprime case $(n, r) \neq 1$, the quantum Hilbert space of Yang-Mills theory on Σ will be infinite dimensional and we obtain a non-rational modular geometry. Thus quantum Yang-Mills theory on Σ naturally includes non-rational conformal field theory. Hence, one might conjecture that all conformal field theories (both rational and non-rational) may be obtained from Yang-Mills theory on a smooth surface Σ .

Acknowledgements

I would like to thank G. B. Segal for discussions.

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