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Perturbative Aspects of the Chern-Simons Field Theory

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Abstract

The quantization of the non-Abelian Chern-Simons theory in three dimensions is performed in the framework of the BRS formalism. General covariance is preserved on the physical subspace. The perturbative analysis at two loops confirms that the model is finite. To this order, the radiative corrections to the two- and three-point proper vertices vanish.

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There have recently been important developments in the understanding of conformal field theories [1]. It has been shown that the state space of a three-dimensional non-Abelian Chern-Simons field theory describes the conformal blocks of an underlying two-dimensional conformal field theory. At the same time, the expectation values of the Wilson line operators, which are the basic observables of the Chern-Simons theory, are strictly related to the link invariants of knot theory. So a surprising connection exists between two-dimensional conformal field theories and knot theory in three dimensions! The main ideas concerning this remarkable relationship have been exposed in [1,2], but the details have not been fully established. In particular, many field theory aspects of the pure non-Abelian Chern-Simons theory have not been explored. One fundamental issue for example is the calculability of the vacuum expectation values of the Wilson line operators associated with knots (and more generally, with links) in three dimensions. As is well known, this problem is twofold. First, one has to construct well-defined correlation functions of the gauge potential; in particular, one should study the possible coupling constant and wave function renormalizations. Second, one should give a definite meaning to the Wilson line operators as composite operators preserving, as much as possible, their geometrical meaning.

In the present paper we address the first aspect of the problem and set up the basic formalism to be used in the explicit computation of the vacuum expectation values of the Wilson line operators [3]. We quantize the Chern-Simons model in the framework of the BRS formalism and show that the general covariance is maintained in the physical space. We show that the theory is finite at two loops (at least) and, more important, that the self-energy and three-point proper vertex corrections vanish identically at this order. Finally, we discuss the contribution of the ghosts and gauge fixing fields in the variation of the quantum action with respect to the gauge potential. Inserted in the correlation functions, this variation gives rise to certain identities which play a crucial role in analysing [4] the quantum holonomy. In this particular context, we show that the ghost and gauge-fixing field contributions cancel, which rigorously proves the correctness of our derivation of the skein relation given in [4].

We start by recalling the expression of the classical action of the Chern-Simons (C-S) model [5,6]

$$S_{CS} = \frac{k}{4\pi} \int_{M^3} d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho \right) , \quad (1)$$

where $A_\mu = A_\mu^a T^a$, T^a being the generators of a compact simple Lie group G . In what follows, we take $G = SU(N)$ with

$$\text{Tr } T^a T^b = \frac{1}{2} \delta^{ab} . \quad (2)$$

The general case can be treated with obvious modifications. Under a gauge transformation $\Omega(x) \in SU(N)$,

$$A_\mu(x) \mapsto A_\mu^\Omega(x) \equiv \Omega^{-1}(x) A_\mu(x) \Omega(x) - i \Omega^{-1}(x) \partial_\mu \Omega(x) , \quad (3)$$

the action (1) transforms as

$$S_{CS}[A^\Omega] = S_{CS}[A] + 2\pi k S_{WZ}[\Omega] , \quad (4)$$

where

$$S_{WZ}[\Omega] = \frac{1}{24\pi^2} \int_{M^3} d^3x \epsilon^{\mu\nu\rho} \text{Tr} [\Omega^{-1}(x) \partial_\mu \Omega(x) \Omega^{-1}(x) \partial_\nu \Omega(x) \Omega^{-1}(x) \partial_\rho \Omega(x)] . \quad (5)$$

With the normalization (2), the Wess-Zumino functional S_{WZ} takes integer values [7]. Therefore for integer k , the C-S model is invariant also under large gauge transformations.

Another important property of the action (1) is its general covariance; being the integral of a three-form on a three-dimensional manifold, S_{CS} is independent of the metric that one can introduce on M^3 . This feature is crucial in relating the C-S model with two-dimensional conformal field theory and knot theory [1,4,8,9,10].

The classical action (1) is degenerate and in order to perform the quantization we adopt the standard Faddeev-Popov procedure. Namely, we consider the total action

$$S = S_{CS} + S_{GF} + S_{FP} , \quad (6)$$

where

$$S_{GF} = \frac{k}{4\pi} \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} A_\mu^a \partial_\nu B^a , \quad (7)$$

$$S_{FP} = - \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} \partial_\mu \bar{c}^a (D_\nu c)^a , \quad (8)$$

and

$$(D_\mu c)^a = \partial_\mu c^a - f^{abc} A_\mu^b c^c , \quad (9)$$

f^{abc} being the structure constants of $SU(N)$. In eqs.(7,8) $g_{\mu\nu}$ is some metric on M^3 , B^a are Lagrange multipliers enforcing the gauge constraint $\partial_\mu(\sqrt{g}g^{\mu\nu}A_\nu^a) = 0$ and c and $\bar{c}$ are the usual Faddeev-Popov ghosts. It is easily seen that the action S is invariant under the BRS transformations [11]

$$\delta A_\mu^a = (D_\mu c)^a \quad , \quad \delta B^a = 0 \quad , \quad (10)$$

$$\delta c^a = \frac{1}{2}f^{abc}c^b c^c \quad , \quad \delta \bar{c}^a = \frac{k}{4\pi}B^a \quad . \quad (11)$$

The BRS charge operator Q , associated with (10,11), is off-shell nilpotent. This property can be easily maintained at any order of the loop expansion; it is a general fact that there are no gauge anomalies in three dimensions.

Now we have a non-degenerate action (6) but the term $S_{GF} + S_{FP}$ unavoidably spoils the general covariance. Of course, there are no reasons to worry about this; the metric dependence enters only in the gauge-fixing procedure and does not affect the physical space. Indeed, the symmetric energy-momentum tensor $\Theta_{\mu\nu}$ associated with (6) is

$$\Theta_{\mu\nu} = \frac{k}{4\pi}(A_{\{\mu}^a \partial_{\nu\}} B^a - g_{\mu\nu} A_\rho^a \partial^\rho B^a) - \partial_{\{\mu} \bar{c}^a (D_{\nu\}} c)^a + g_{\mu\nu} \partial_\rho \bar{c}^a (D^\rho c)^a \quad (12)$$

and can be rewritten in the form

$$\Theta_{\mu\nu} = [Q, \partial_{\{\mu} \bar{c}^a A_{\nu\}}^a - g_{\mu\nu} \partial^\rho \bar{c}^a A_\rho^a] \quad . \quad (13)$$

Since Q annihilates the physical states, the mean value of $\Theta_{\mu\nu}$ between physical states vanishes. This implies general covariance on the physical space, as there should be.

At this stage one can perform the quantization. We shall do this in the framework of standard perturbation theory. From now on we consider $M^3 = R^3$ (or S^3) with the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. We start by deriving the propagators following from the action (6). The quadratic part of S reads

$$S_{quadr} = \int d^3x \left(\frac{k}{8\pi} \epsilon^{\mu\nu\rho} A_\mu^a \partial_\nu A_\rho^a + \frac{k}{4\pi} A_\mu^a \partial^\mu B^a + \bar{c}^a \partial^\mu \partial_\mu c^a \right) \quad . \quad (14)$$

The ghost propagator is immediately obtained

$$\langle c^a(x) \bar{c}^b(y) \rangle = -\frac{i}{4\pi} \delta^{ab} \frac{1}{|x-y|} \equiv i\delta^{ab} \Delta(x-y) \quad , \quad (15)$$

$$\Delta(x) = -\frac{1}{4\pi|x|} = \int \frac{d^3p}{(2\pi)^3} e^{ipx} \frac{1}{p^2} , \quad (16)$$

whereas the inversion of the differential operator in the bosonic part of S_{quad} needs some care because it is not diagonal in the $\{A_\mu^a, B^a\}$ space. One finds

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{i}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} = i \frac{4\pi}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \partial_x^\rho \Delta(x-y) , \quad (17)$$

$$\langle A_\mu^a(x) B^b(y) \rangle = -\frac{i}{k} \delta^{ab} \frac{(x-y)_\mu}{|x-y|^3} = -i \frac{4\pi}{k} \delta^{ab} \partial_\mu^x \Delta(x-y) , \quad (18)$$

$$\langle B^a(x) B^b(y) \rangle = 0 . \quad (19)$$

The action (6) is renormalizable by power counting. The relevant proper vertices which are potentially divergent are: the vector self-energy $\Pi_{\mu\nu}^{ab}$, the ghost self-energy Π^{ab} , the three-vector vertex $\Gamma_{\mu\nu\rho}^{abc}$ and the ghost-vector vertex Γ_μ^{abc} . As mentioned before, on general grounds the gauge invariance implies that the coupling constant k must be an integer. For this reason one expects vanishing of the β -function. In order to verify this statement, an explicit computation is necessary. The analysis reported below shows that the β -function vanishes at two loops at least; even more, the one- and two-loop corrections to $\Pi_{\mu\nu}^{ab}$ and $\Gamma_{\mu\nu\rho}^{abc}$ are not only finite, but vanish identically!

The relevant one-loop graphs are shown in Fig.1 and Fig.2. The dashed and the continuous lines represent the gauge and ghost fields respectively. Here and in what follows it is understood that the ghost loop includes both the possible orientations. The individual diagrams in Fig.1 and Fig.2 are divergent and need to be regularized. One possible regularization is the following. Since all the propagators depend on $\Delta(x)$ and there are at most linear (actually logarithmic) divergences, one may simply regularize by substituting

$$\frac{1}{p^2} \mapsto \frac{1}{p^2} - \frac{1}{p^2 + M^2} \quad (20)$$

in eq.(16). Here M is a sort of Pauli-Villars mass. As we shall see in a moment, the theory is at least two-loop finite and then the $M \rightarrow \infty$ limit can be trivially performed (there is no need of counterterms). The values of the graphs displayed in Fig.1 are:

$$(1.a) = c_v \delta^{ab} \partial_\mu^x \Delta(x-y) \partial_\nu^x \Delta(x-y) , \quad (21)$$

$$(1.b) = -c_v \delta^{ab} \partial_\mu^x \Delta(x-y) \partial_\nu^x \Delta(x-y) , \quad (22)$$

$$(1.c) = \frac{4\pi}{k} c_v \delta^{ab} \epsilon_{\mu\nu\rho} \partial_\mu^x \partial_\nu^x \Delta(x-y) \partial_\rho^x \Delta(x-y) = 0 . \quad (23)$$

Here c_v is the Casimir operator in the adjoint representation ($c_v = N$ for $SU(N)$). Equations (21-23) obviously imply that at one loop

$$\Pi_{\mu\nu}^{(1)ab}(x-y) = 0 \quad , \quad \Pi^{(1)ab}(x-y) = 0 \quad . \quad (24)$$

Concerning the graphs contributing to $\Gamma_{\mu\nu\rho}^{abc}$ (Fig.2a,b) one has

$$(2.a) = f^{lam} f^{mbn} f^{ncl} [\delta^{\mu\alpha} \delta^{\rho\beta} \delta^{\nu\gamma} + \delta^{\nu\alpha} \delta^{\mu\beta} \delta^{\rho\gamma} + \delta^{\rho\alpha} \delta^{\nu\beta} \delta^{\mu\gamma} - \delta^{\rho\alpha} \delta^{\mu\beta} \delta^{\nu\gamma}] \partial_\alpha^x \Delta(x-y) \partial_\beta^y \Delta(y-z) \partial_\gamma^z \Delta(z-x) \quad , \quad (25)$$

$$(2.b) = -f^{lam} f^{mbn} f^{ncl} [\delta^{\mu\alpha} \delta^{\nu\beta} \delta^{\rho\gamma} + \delta^{\nu\alpha} \delta^{\rho\beta} \delta^{\mu\gamma}] \partial_\alpha^x \Delta(x-y) \partial_\beta^y \Delta(y-z) \partial_\gamma^z \Delta(z-x) \quad . \quad (26)$$

The sum of (25) and (26) is proportional to the completely antisymmetrized product

$$\partial_{[\alpha}^x \Delta(x-y) \partial_\beta^y \Delta(y-z) \partial_{\gamma]}^z \Delta(z-x) \quad , \quad (27)$$

which can be easily shown to vanish. Therefore,

$$\Gamma_{\mu\nu\rho}^{(1)abc}(x-y, y-z, z-x) = 0 \quad . \quad (28)$$

Finally, for the ghost vertex Γ_μ^{abc} one finds (see Fig.2c,d)

$$(2.c) = \frac{4\pi}{k} f^{lam} f^{mcn} f^{nbl} \epsilon_{\nu\rho\sigma} \partial_\mu^x \partial_\nu^x \Delta(x-y) \partial_\rho^y \Delta(y-z) \partial_\sigma^z \Delta(z-x) \quad , \quad (29)$$

$$(2.d) = -\frac{4\pi}{k} f^{lam} f^{mcn} f^{nbl} \epsilon_{\nu\rho\sigma} [\partial_\mu^x \partial_\nu^x \Delta(x-y) \partial_\rho^y \Delta(y-z) \partial_\sigma^z \Delta(z-x) + \partial_\mu^y (\partial_\nu^x \Delta(x-y) \partial_\rho^y \Delta(y-z) \partial_\sigma^z \Delta(z-x))] \quad . \quad (30)$$

The sum of (29) and (30) is also proportional to (27) and therefore

$$\Gamma_\mu^{(1)abc}(x-y, y-z, z-x) = 0 \quad . \quad (31)$$

The above results imply that the theory is one-loop finite. In addition, the two- and three-point functions remain the *free* ones. This remarkable property reflects the geometrical character of the theory. For instance, the free two-point function (17) is related to the Gauss (linking) number [12]. The result (24) simply means that there are no "radiative corrections" to the linking number, as it should be.

We turn now to the full expression of the one-loop effective action $\Gamma[A]$. The determinant resulting from the integration of the ghost variables is $\text{Det}(\partial^\mu D_\mu)$, whereas the determinant coming from the bosonic variables is

$$\text{Det}^{-\frac{1}{2}}(\mathcal{D}(A)) \quad , \quad (32)$$

where

$$\mathcal{D}(A) = \begin{pmatrix} \epsilon^{\mu\nu\rho} D_\nu^{ab} & \delta^{ab} \partial^\mu \\ -\delta^{ab} \partial^\rho & 0 \end{pmatrix} \quad . \quad (33)$$

Therefore the one-loop effective action is given by

$$\exp(i\Gamma[A]) = \text{Det}(\partial^\mu D_\mu) \text{Det}^{-\frac{1}{2}}(\mathcal{D}) \quad . \quad (34)$$

In the Abelian case, the expression (34) is related to the Ray-Singer torsion of the three-manifold [5]. Let us now elaborate on the A_μ -dependence of $\Gamma[A]$. Introducing for convenience the matrix $\hat{A}_\mu$ defined by

$$\hat{A}_\mu^{ab} \equiv f^{acb} A_\mu^c \quad , \quad (35)$$

one finds for the ghost contribution

$$-i \text{Tr} \ln \left(1 - \partial^{-2} \partial^\rho \hat{A}_\rho \right) \quad . \quad (36)$$

Analogously, the contribution coming from $\mathcal{D}$ is

$$\begin{aligned} & \frac{i}{2} \text{Tr} \ln \left[1 + \begin{pmatrix} \partial^{-2} \partial_\nu \hat{A}_\mu - \delta_{\mu\nu} \partial^{-2} \partial_\rho \hat{A}^\rho & 0 \\ -\epsilon_{\nu\rho\sigma} \partial^{-2} \partial^\rho \hat{A}^\sigma & 0 \end{pmatrix} \right] = \\ & = \frac{i}{2} \text{Tr} \ln \left(\delta_{\mu\nu} + \partial^{-2} \partial_\nu \hat{A}_\mu - \delta_{\mu\nu} \partial^{-2} \partial_\rho \hat{A}^\rho \right) \quad . \end{aligned} \quad (37)$$

Finally, in compact form

$$\Gamma^{(1)}[A] = \frac{i}{2} \text{Tr} \ln \left(\delta_{\mu\nu} + \partial^{-2} \partial_\nu \hat{A}_\mu - \delta_{\mu\nu} \partial^{-2} \partial_\rho \hat{A}^\rho \right) - i \text{Tr} \ln \left(1 - \partial^{-2} \partial^\rho \hat{A}_\rho \right) \quad . \quad (38)$$

In spite of the relative simplicity of expression (38), we have not found any direct argument (apart from the explicit computation) for the vanishing of $\Pi_{\mu\nu}^{(1)ab}$ and $\Gamma_{\mu\nu\rho}^{(1)abc}$. One may ask whether a complete cancellation between ghosts and gauge contributions takes place also for the n -point A_μ -proper vertices with $n \geq 4$. This would be very

amusing, but unfortunately it does not happen. For instance, the four-point function is of the form

$$\Gamma_{\mu\nu\rho\sigma}^{(1)abcd} = T^{abcd} F_{\mu\nu\rho\sigma}(x, y, z, w) \quad , \quad (39)$$

where

$$T^{abcd} = f^{lam} f^{mbn} f^{ncp} f^{pdl} \quad (40)$$

and $F_{\mu\nu\rho\sigma}$ is non-vanishing. The symmetry properties of T^{abcd} are useful at two-loops.

The diagrams contributing to the two-loop self-energy $\Pi_{\mu\nu}^{(2)ab}$ are shown in Fig.3. The graph (3.a) vanishes because of eq.(24). It is clear that the graphs (3.b) and (3.c) cancel. In fact, the sum of the subdiagrams inside the boxes gives precisely $\Pi_{\mu\nu}^{(1)ab}$. Analogously, diagrams (3.d) and (3.e) cancel because of the vanishing of $\Gamma_{\mu\nu\rho}^{(1)abc}$. The same cancellation mechanism works also for the diagrams (3.f) and (3.g) because of equation (31). Therefore

$$\Pi_{\mu\nu}^{(2)ab}(x - y) = 0 \quad . \quad (41)$$

In the same way one can show that

$$\Pi^{(2)ab}(x - y) = 0 \quad . \quad (42)$$

Eqs.(41,42) simply follow from the Schwinger-Dyson equations and the one-loop vanishing.

Let us consider now the two-loop vertex $\Gamma_{\mu\nu\rho}^{(2)abc}$. The cancellation mechanism explained above immediately applies to the planar graphs contributing to $\Gamma_{\mu\nu\rho}^{(2)abc}$. The remaining graphs are shown in Fig.4. Each of them vanishes because of the peculiar structure in the internal indices. Indeed one can always select a three-vector vertex (as shown in Fig.4) contracted with a diagram contributing to $\Gamma_{\mu\nu\rho\sigma}^{(1)abcd}$. Any such diagram is proportional to the tensor T^{abcd} defined in eq.(40) and the contraction is of the form $T^{abcd} f^{ace}$. This tensor structure vanishes because $T^{abcd} = T^{cbad}$. Therefore

$$\Gamma_{\mu\nu\rho}^{(2)abc}(x - y, y - z, z - x) = 0 \quad . \quad (43)$$

This concludes the proof of the two-loop finiteness of the C-S model.

Summarizing, we have shown that it is possible to quantize the C-S model in such a way that the symmetries of the classical action (1) are realized on the physical subspace defined by the BRS cohomology. Moreover, the quantum C-S theory has a meaningful perturbative expansion. Not only the theory is two-loop finite but, at this order, there

are no corrections to the two- and three-point functions at all. We expect that this nice feature persists at higher orders of the loop expansion.

As in any gauge theory, the natural question which arises at this stage is: what are the physical states? In order to answer this question, consider the equations of motion

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c = 0 \quad , \quad (44)$$

following from the classical action (1). Eq.(44) shows one of the most striking features of the theory: roughly speaking, one has a gauge invariant theory in which even the transverse parts of A_μ^a are eliminated because of the equations of motion. So, what are the physical degrees of freedom of this theory? In fact, there are no physical degrees of freedom in the usual sense of field theory. For instance, in the canonical quantization on $S^2 \otimes R$ there is only one physical state - the vacuum, so, the C-S model does not have a standard particle content. Yet, the physical Hilbert space is not necessarily trivial. Consider for example the equations of motion (44) on the three-manifold $\Sigma \otimes R$ where Σ is a Riemann surface of non-trivial genus. The set $\{A_\mu^a\}$ of the solutions of eq.(44) modulo gauge transformations determines a non-trivial physical phase space [1,13].

Another interesting situation occurs in the presence of Wilson line operators $W(L)$. In this case, Σ may have punctures arising as intersection points of the link L with Σ . Consequently, the physical space is non-trivial even if $\Sigma = S^2$ [1,13]. In general, the expectation values $\langle W(L) \rangle$ of the Wilson line operators are related to link invariants [1,4,9] and one of the main problems is their explicit computation. A detailed discussion of this point will be reported in [3]. Here, we conclude with a few comments on this problem. We give also a rigorous proof of the main formula which has been used in [4] to show that $\langle W(L) \rangle$ defines a regular isotopy link invariant and satisfies (in the fundamental representation) the skein relation of a S_L polynomial.

At first sight, the problem of computing $\langle W(L) \rangle$ seems to have a trivial solution. A naive use of eq.(44) would suggest that "on-shell" there is no curvature and therefore $\langle W(L) \rangle$ is trivial. The failure of this argument can be seen in different ways. In the canonical approach, the R.H.S. of eq.(44) gets modified by pointlike source terms. This is precisely the origin of the punctures with a given representation of the group assigned to each of them [1]. In the functional integral approach, eq.(44) is not identically satisfied when $F_{\mu\nu}^a$ is inserted into the correlation functions. This fact plays an important role in showing how $\langle W(L) \rangle$ behaves under deformations of the contour on which the path ordering is defined [4]. Let us elaborate on this point in some detail.

As is well known, an infinitesimal deformation of the path associated with a Wilson line operator leads to an $F_{\mu\nu}$ -insertion. More precisely, one has

$$\frac{\delta U(1,2)}{\delta \sigma^{\mu\nu}(x)} = iU(1,x)R^a F_{\mu\nu}^a(x)U(x,2) \quad , \quad (45)$$

where

$$U(1,2) = P \exp i \int_1^2 dx^\mu R^a A_\mu^a(x) \quad (46)$$

is the holonomy operator between the points 1 and 2 in the R representation of the group $SU(N)$. Differently from the standard Yang-Mills theories, the $F_{\mu\nu}$ -insertions are easily treated in the C-S model. In fact, since

$$F_{\mu\nu}^a(x) = \frac{4\pi}{k} \epsilon_{\mu\nu\rho} \frac{\delta S_{CS}}{\delta A_\rho^a(x)} \quad , \quad (47)$$

a simple integration by parts in the Feynman path integral gives

$$\begin{aligned} & \int DA \exp(iS_{CS}) \text{Tr}[U(1,x)R^a F_{\mu\nu}^a(x)U(x,1)]O_1 \cdots O_n = \\ & = i \frac{4\pi}{k} \epsilon_{\mu\nu\rho} \int DA \exp(iS_{CS}) \frac{\delta}{\delta A_\rho^a(x)} (\text{Tr}[U(1,x)R^a U(x,1)]O_1 \cdots O_n) \quad , \end{aligned} \quad (48)$$

$O_1, \dots, O_n$ being gauge invariant observables. Therefore, in the C-S model the $F_{\mu\nu}$ -insertion (45) can be equivalently replaced by a functional derivative with respect to A_μ . The result (48) is correct, but needs a more rigorous argument. The point is that, because of gauge invariance, one has to take into account gauge fixing and ghost contributions. So, instead of equation (47) one has actually

$$F_{\mu\nu}^a(x) = \frac{4\pi}{k} \epsilon_{\mu\nu\rho} \frac{\delta S}{\delta A_\rho^a(x)} + f_{\mu\nu}^a(x) \quad , \quad (49)$$

where

$$f_{\mu\nu}^a(x) = \frac{4\pi}{k} \epsilon_{\mu\nu\rho} f^{abc} \partial^\rho \bar{c}^b(x) c^c(x) - \epsilon_{\mu\nu\rho} \partial^\rho B^a(x) \quad . \quad (50)$$

At this stage, in order to prove eq.(48), i.e.

$$\begin{aligned} & < \text{Tr}[U(1,x)R^a F_{\mu\nu}^a(x)U(x,1)] O_1 \cdots O_n > = \\ & = i \frac{4\pi}{k} \epsilon_{\mu\nu\rho} \frac{\delta}{\delta A_\rho^a(x)} < \text{Tr}[U(1,x)R^a U(x,1)] O_1 \cdots O_n > \quad , \end{aligned} \quad (51)$$

one must show that

$$< \text{Tr}[U(1, x) R^a f_{\mu\nu}^a(x) U(x, 1)] O_1 \cdots O_n > = 0 \quad . \quad (52)$$

The L.H.S. of (52) can be rewritten in the form

$$\epsilon_{\mu\nu\rho} < \text{Tr}[U(1, x) R^a (f^{abc} \partial^\rho \bar{c}^b(x) c^c(x) - [Q, \partial^\rho \bar{c}^a(x)]) U(x, 1)] O_1 \cdots O_n > \quad . \quad (53)$$

Because of the BRS invariance of the vacuum and of $O_1, \dots, O_n$, the action of the BRS charge Q in (53) can be transferred on $U(1, x)$ and $U(x, 1)$. Using the covariant properties of U under BRS transformations, one finds that eq.(52) is indeed satisfied, which proves the validity of eq.(51).

In conclusion, one of the open problems quoted in [1], namely whether the non-Abelian C-S action defines a sensible quantum theory, has an affirmative answer. In the BRS approach, general covariance is preserved on the physical space and the model has a meaningful perturbative expansion. We have explicitly checked the finiteness at two loops. The more delicate question is how to deal with the Wilson line operators. Essentially the problem is how to consistently extend the framing procedure beyond the first order in $\frac{1}{k}$. Our preliminary computations suggest that this is possible. We shall report on that in [3].

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FIGURE CAPTIONS

Fig.1 One-loop self-energy graphs.

Fig.2 One-loop vertex graphs.

Fig.3 Two-loop self-energy graphs.

Fig.4 Two-loop non-planar vertex graphs.

Fig. 1

Fig. 2

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Fig. 3

(a)

(b)

(c)

(d)

Fig. 4