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TOPOLOGICAL YANG-MILLS THEORY IS ABELIAN CHIRAL ANOMALY IN SUPERSPACE

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Abstract

Ghost moments of the BRST operator in the four dimensional topological Yang-Mills theory determine Gauss law generator of the Parisi-Sourlas extended Yang-Mills theory, with the superspace curvature two-form reproducing BRST transformations of the component fields. The superspace generalization of the abelian chiral anomaly contains Donaldson's invariants when expanded in the supercoordinates, and its cohomological nontriviality follows from standard arguments: The chiral anomaly is the superspace exterior differential of the Parisi-Sourlas generalized Chern-Simons three-form which fails to remain invariant under superspace gauge transformations.

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Recently it has been proposed [1,2] that the topology of low dimensional manifolds could be related to the properties of a topologically extended quantum Yang-Mills theory. In particular, Donaldson's invariants [3] for four manifolds can be extracted [2] from a relativistic Lagrangian field theory version of the approach in [1], where Floer groups of three manifolds are identified with the ground states of a second quantized hamiltonian.

Here we shall derive the topological quantities in [2] leading to Donaldson's invariants, directly from the abelian chiral anomaly in the Parisi-Sourlas extended Yang-Mills theory [4,5]. In analogy with the ordinary abelian chiral anomaly, we find that the superspace chiral anomaly is the exterior differential of the pertinent superspace nonabelian Chern-Simons three-form. Furthermore, since this three-form fails to be invariant under superspace gauge transformations, we conclude that the quantities presented in [2] are indeed cohomologically nontrivial.

The original discussion in [1,2] starts from the nilpotent BRST operator

$$\Omega = \psi_i^a \frac{\delta}{\delta A_i^a} \quad (1)$$

Here ψ_i^a is an anticommuting three vector and A_i^a is the three dimensional Yang-Mills field. Following [1,2] we are interested in the action of (1) on $\mathcal{A}/\mathcal{G}$, the space of three dimensional gauge fields modulo three dimensional gauge transformations. Locally, the cohomology of (1) then appears trivial - it selects A_i^a independent *i.e.* constant functionals - but since $\mathcal{A}/\mathcal{G}$ is noncontractible local considerations are insufficient to ensure nontriviality of the cohomology. Indeed, as shown in [1] the ground states of a hamiltonian based on (1) are related to Floer cohomology of three manifolds.

Following [6] we develop a canonical phase space version of [1,2], appropriate for reducible systems. For this we first reinterpret (1) as a phase space BRST operator, by realizing the functional derivative as the Poisson bracket action of the nonabelian electric field,

$$\{E_i^a(x), A_j^b(y)\} = -\delta_{ij} \delta^{ab} \delta(x-y) \quad (2)$$

Notice that Darboux theorem [7] justifies the apparently noncovariant form of the bracket: Equation (2) determines a symplectic structure on the phase space, and Darboux theorem guarantees the existence of canonical variables in a *neighborhood* on this phase space, such that Poisson brackets reduce to the simple form (2)¹. Indeed, the classical BRST formalism can be presented in a manifestly coordinate independent manner using symplectic

¹We (off)diagonalize the symplectic two-form, not the metric tensor. Also, in the following index contractions are understood with respect to the symplectic structure.

geometry on a graded phase space [8], and the manifest spacetime metric independence of our final results justifies *a posteriori* our brackets.

Similarly, we view ψ_i^a as a graded canonical coordinate and introduce its conjugate variable $\mathcal{X}_i^a$ with

$$\{\psi_i^a(x), \mathcal{X}_j^b(y)\} = -\delta_{ij}\delta^{ab}\delta(x-y) \quad (3)$$

where we again rely on Darboux theorem. Defining the ghost numbers of the doublet $\psi_i, \mathcal{X}_i$ to be 1 *resp.* -1 the ghost number of the BRST operator

$$\Omega = \psi^a \cdot E^a \quad (4)$$

is one. We are interested in the Poisson bracket action of Ω on gauge invariant functionals, *i.e.* functionals with vanishing Poisson brackets with the Gauss law generator

$$G^a = D^{ab} \cdot E^b \quad ; \quad \{G^a(x), G^b(y)\} = f^{abc}G^c(x)\delta(x-y) \quad (5)$$

On gauge invariant functionals the components of E_i^a satisfy $D \cdot E \sim 0$, hence on this subspace these components fail to be linearly independent. As a consequence on $\mathcal{A}/\mathcal{G}$ the BRST operator (4) should be improved to account for this reducibility:

Following [6] we introduce a pair of bosonic ghosts φ^a, p^a with ghost numbers 2 *resp.* -2, and Poisson bracket

$$\{p^a(x), \varphi^b(y)\} = -\delta^{ab}\delta(x-y) \quad (6)$$

and consider the following improvement of the BRST operator (4),

$$\Omega \rightarrow \Omega = \psi^a \cdot E^a + \varphi^a D^{ab} \cdot \mathcal{X}^b \quad (7)$$

Its action on the canonical coordinates is isomorphic to the BRST transformations in [2,9-11],

$$\delta_\Omega A = \{A, \Omega\} = \psi \quad \& \quad \delta_\Omega E = \{E, \Omega\} = [\varphi, \mathcal{X}] \quad (8.a)$$

$$\delta_\Omega \psi = \{\psi, \Omega\} = D\varphi \quad \& \quad \delta_\Omega \mathcal{X} = \{\mathcal{X}, \Omega\} = -E \quad (8.b)$$

$$\delta_\Omega \varphi = \{\varphi, \Omega\} = 0 \quad (8.c)$$

provided we identify E with the self-dual field b that appears in [9-11]. Furthermore,

$$\{\Omega, \Omega\} = -2\varphi^a(D_i^{ab}E_i^b + f^{abc}\psi_i^b\mathcal{X}_i^c) = -2\varphi^a(G^a + T^a) \quad (9)$$

where

$$T^a = f^{abc}\psi_i^b\mathcal{X}_i^c \quad \& \quad \{T^a(x), T^b(y)\} = f^{abc}T^c(x)\delta(x-y) \quad (10)$$

defines a representation of the gauge algebra (5). Hence Ω is nilpotent on the gauge invariant surface $G^a + T^a \sim 0$, while for the fields in (8) the operator (9) defines a gauge transformation with parameter φ

$$\delta_\Omega^2 A = \{A, \frac{1}{2}\{\Omega, \Omega\}\} = D\varphi \quad \& \quad \delta_\Omega^2 E = \{E, \frac{1}{2}\{\Omega, \Omega\}\} = \{E, \varphi\} \quad (11.a)$$

$$\delta_\Omega^2 \psi = \{\psi, \frac{1}{2}\{\Omega, \Omega\}\} = \{\psi, \varphi\} \quad \& \quad \delta_\Omega^2 \mathcal{X} = \{\mathcal{X}, \frac{1}{2}\{\Omega, \Omega\}\} = \{\mathcal{X}, \varphi\} \quad (11.b)$$

For the field p^a , not present in [2,9-11], the action is

$$\delta_\Omega^2 p = \{p, \frac{1}{2}\{\Omega, \Omega\}\} = D \cdot E + [\psi, \mathcal{X}] \equiv G + T \quad (11.c)$$

i.e. vanishes on the gauge invariant surface $G^a + T^a \sim 0$.

The relations (9) and (11) suggest that in order to construct an identically nilpotent improvement of the BRST operator (7) we should account for all representations of the gauge algebra (5). In addition to (5) and (10) we should then also include

$$Q^a = f^{abc} \varphi^b p^c ; \quad \{Q^a(x), Q^b(y)\} = f^{abc} Q^c(x) \delta(x - y) \quad (12)$$

Indeed, we find that the following improved BRST operator is identically nilpotent

$$\Omega \rightarrow \Omega = \psi^a \cdot E^a - \varphi^a \mathcal{P}^a + \varphi^a D^{ab} \cdot \mathcal{X}^b + \eta^a (G^a + T^a + Q^a) + \frac{1}{2} f^{abc} \eta^a \eta^b \mathcal{P}^c \quad (13)$$

Here

$$\{\eta^a(x), \mathcal{P}^b(y)\} = -\delta^{ab} \delta(x - y) \quad (14)$$

are the standard anticommuting ghosts for gauge symmetry.

The action of the BRST operator (13) on the canonical coordinates is isomorphic to the s -symmetry introduced in [12],

$$\delta_\Omega A = \{A, \Omega\} = \psi - D\eta \quad \& \quad \delta_\Omega E = \{E, \Omega\} = [\varphi, \mathcal{X}] + [E, \eta] \quad (15.a)$$

$$\delta_\Omega \psi = \{\psi, \Omega\} = D\varphi - [\eta, \psi] \quad \& \quad \delta_\Omega \mathcal{X} = \{\mathcal{X}, \Omega\} = -E - [\eta, \mathcal{X}] \quad (15.b)$$

$$\delta_\Omega \varphi = \{\varphi, \Omega\} = [\eta, \varphi] \quad \& \quad \delta_\Omega \eta = \{\eta, \Omega\} = \varphi - \frac{1}{2} [\eta, \eta] \quad (15.c)$$

provided we identify the self-dual field b in [12] with E .

From [6] we learn that the phase space considered thus far does not ensure manifest covariance. For this, it is necessary [6] to extend the phase space by the following canonical

coordinates: commuting Lagrange multipliers λ^a, π^a with ghost numbers 0 *resp.* 0 for Gauss law; commuting Lagrange multipliers λ_i^a, π_i^a with ghost numbers 0 *resp.* 0 for $E_i \sim 0$; anticommuting ghosts $\bar{\eta}^a, \bar{\mathcal{P}}^a$ with ghost numbers 1 *resp.* -1 for gauge fixing Gauss law; anticommuting ghosts $\bar{\psi}_i^a, \bar{\mathcal{X}}_i^a$ with ghost numbers 1 *resp.* -1 for gauge fixing $E_i \sim 0$; anticommuting Lagrange multipliers l^a, ρ^a with ghost numbers 1 *resp.* -1 for the ghost constraint [6]; commuting ghosts c^a, b^a with ghost numbers 2 *resp.* -2 for gauge fixing the ghost constraint; and finally commuting "extra ghosts" $\bar{\lambda}^a, \bar{\pi}^a$ with ghost numbers 0 *resp.* 0 and anticommuting "extra ghosts" $\bar{c}^a, \bar{b}^a$ with ghost numbers 1 *resp.* -1. Our final covariant phase space then has 24 canonical coordinates, with nontrivial Poisson brackets

$$\begin{aligned} \{\pi^a(x), \lambda^b(y)\} &= \{\bar{\eta}^a(x), \bar{\mathcal{P}}^b(y)\} = \{l^a(x), \rho^b(y)\} = \{c^a(x), b^b(y)\} = \\ &= \{\bar{\lambda}^a(x), \bar{\pi}^b(y)\} = \{\bar{c}^a(x), \bar{b}^b(y)\} = -\delta^{ab}\delta(x-y) \end{aligned} \quad (16.a)$$

$$\{\pi_i^a(x), \lambda_j^b(y)\} = \{\bar{\psi}_i^a(x), \bar{\mathcal{X}}_j^b(y)\} = -\delta^{ab}\delta_{ij}\delta(x-y) \quad (16.b)$$

for these additional variables. From [6] we find that the nilpotent BRST operator in this phase space is

$$\Omega = \psi^a \cdot E^a - \varphi^a \mathcal{P}^a + \varphi^a D^{ab} \cdot \mathcal{X}^b + \eta^a \{G^a + T^a + Q^a\} + \frac{1}{2} f^{abc} \eta^a \eta^b \mathcal{P}^c + \pi^a \bar{\eta}^a + \pi^a \cdot \bar{\psi}^a + \rho^a c^a + \bar{\pi}^a \bar{c}^a \quad (17)$$

and in order to compare with other formulations, we shall now consider certain canonical transformations of (17):

In [1,2] the following canonical conjugation is introduced,

$$\Omega \rightarrow e^{-\Phi} \Omega e^{\Phi} = \Omega + \{\Omega, \Phi\} + \frac{1}{2} \{\{\Omega, \Phi\}, \Phi\} + \dots \quad (18)$$

with $\Phi = t \cdot \omega_3(A)$, where t is a parameter and $\omega_3(A)$ the Chern-Simons three-form. For our present purpose we prefer not to introduce this conjugation at this stage: Its only effect is to shift $E \rightarrow E - tB$, with B the nonabelian magnetic field. When necessary, we can relate our results to those in [1,2] by this conjugation.²

Next, we consider the conjugation (18) with

$$\Phi = f^{abc} \eta^a \bar{\mathcal{P}}^b \lambda^c \quad (19)$$

²However, it is important to realize that $\omega_3(A)$ is not globally defined, *i.e.* it is not gauge invariant. As a consequence this conjugation does introduce nontrivial structure.

For (17) this yields

$$\begin{aligned} \Omega \rightarrow \Omega = & \psi^a \cdot E^a - \varphi^a \mathcal{P}^a + \varphi^a D^{ab} \cdot \mathcal{X}^b + \eta^a \{G^a + T^a + Q^a + U^a + V^a\} + \\ & + \frac{1}{2} f^{abc} \eta^a \eta^b \mathcal{P}^c + f^{abc} \bar{\mathcal{P}}^a \varphi^b \lambda^c + \pi^a \bar{\eta}^a + \pi^a \cdot \bar{\psi}^a + \rho^a c^a + \bar{\pi}^a \bar{c}^a \end{aligned} \quad (20)$$

where we have introduced the following realizations of the gauge algebra (5),

$$U^a = f^{abc} \bar{\mathcal{P}}^b \bar{\eta}^c \quad \& \quad V^a = f^{abc} \lambda^b \pi^c \quad (21)$$

In addition of (15) we then have the nontrivial relations

$$\delta_\Omega \lambda = \{\lambda, \Omega\} = -\bar{\eta} - [\lambda, \eta] \quad \& \quad \delta_\Omega \bar{\eta} = \{\bar{\eta}, \Omega\} = -[\eta, \bar{\eta}] - [\varphi, \lambda] \quad (22)$$

If we identify $\bar{\eta}$ and λ with the variables $\bar{\phi}$ and β in [12], our transformation (15), (22) becomes isomorphic to that presented there. However, for the present purposes we find it more illuminating to consider the simpler version (17) of the BRST operator. When necessary, our results can be extended to (20) by this conjugation.

Finally, we consider³

$$\Phi = \eta^a D^{ab} \cdot \mathcal{X}^b - \frac{1}{2} f^{abc} \eta^a \eta^b p^c \quad (23)$$

We find that the pertinent conjugation (18) relates the BRST operator (13) to an abelian BRST operator of the form discussed in [8,13]

$$\Omega = e^{-\Phi} \bar{\Omega} e^{\Phi} ; \quad \bar{\Omega} = \psi_i^a E_i^a - \varphi^a \mathcal{P}^a \quad (24)$$

We refer to [8,13] for further discussion on such abelian representations of the BRST operator and their relations to the corresponding nonabelian representations by conjugations of the form (18). Here we record that on the canonical coordinates the conjugation (23), (24) yields

$$\psi \rightarrow e^{-\Phi} \psi e^{\Phi} = \psi - D\eta \quad (25.a)$$

$$E \rightarrow e^{-\Phi} E e^{\Phi} = E + [\eta, \mathcal{X}] \quad (25.b)$$

$$\varphi \rightarrow e^{-\Phi} \varphi e^{\Phi} = \varphi - \frac{1}{2} [\eta, \eta] \quad (25.c)$$

³Notice that the functional Φ depends on the connection A_i . Hence this transformation can only be defined locally on the phase space.

On the abelian coordinates the action of the BRST operator $\tilde{\Omega}$ is then isomorphic to the corresponding transformations in [12],

$$\delta_{\tilde{\Omega}} A = \psi \quad \& \quad \delta_{\tilde{\Omega}} \psi = 0 \quad (26.a)$$

$$\delta_{\tilde{\Omega}} \mathcal{X} = -E \quad \& \quad \delta_{\tilde{\Omega}} E = 0 \quad (26.b)$$

$$\delta_{\tilde{\Omega}} \eta = \varphi \quad \& \quad \delta_{\tilde{\Omega}} \varphi = 0 \quad (26.c)$$

implying triviality of the *local* cohomology associated with (13).

We now make the following observation: If we introduce an exterior differential operator

$$\hat{d} = d + \delta_{\tilde{\Omega}} \quad (27)$$

with $\delta_{\tilde{\Omega}}$ defined in (13), we identify in (15) the following graded Yang-Mills structure

$$\mathcal{F} = F + \psi + \varphi = \hat{d}(A + \eta) + \frac{1}{2}[A + \eta, A + \eta] \quad \& \quad \hat{d}\mathcal{F} + [A + \eta, \mathcal{F}] = 0 \quad (28)$$

suggesting an underlying graded gauge symmetry in (13), (15). Indeed, we shall now proceed to demonstrate that in a Parisi-Sourlas superspace [4,5] obtained by extending the coordinates x^i into supercoordinates $y^\alpha = (x^i, \theta, \bar{\theta})$ with θ and $\bar{\theta}$ anticommuting $\theta^2 = \bar{\theta}^2 = \theta\bar{\theta} + \bar{\theta}\theta = 0$, the symmetry structure captured by the BRST operator (13) - and its improvement in (17) - determines the supergauge structure of the Parisi-Sourlas extended Yang-Mills theory [5]. For this we shall first establish that the full 24-plet of phase space variables introduced in (16) can be combined into canonical Yang-Mills variables $\mathcal{A}_\alpha(y)$, $\mathcal{E}_\alpha(y)$ with the nontrivial superspace Poisson bracket

$$\{\mathcal{E}_\alpha^a(y_1), \mathcal{A}_\beta^b(y_2)\} = -g_{\alpha\beta}\delta^{ab}\delta(y_1 - y_2) \quad (29)$$

Here $\delta(y_1 - y_2)$ is the δ -function in the Parisi-Sourlas superspace [5]. Also, Darboux theorem ensures that in a phase space neighborhood we can select $g_{ij} = \delta_{ij}$ and $g_{\theta\theta} = -g_{\theta\bar{\theta}} = 1$. The superspace Gauss law generator $\mathcal{G}^a$ then satisfies the superspace version of the gauge algebra (5),

$$\mathcal{G}^a = g^{\alpha\beta}\mathcal{D}_\alpha^{ab}\mathcal{E}_\beta^b ; \quad \{\mathcal{G}^a(y_1), \mathcal{G}^b(y_2)\} = f^{abc}\mathcal{G}^c(y_1)\delta(y_1 - y_2) \quad (30)$$

defining the gauge structure on the Parisi-Sourlas superspace. Here $\mathcal{D}_\alpha^{ab}$ is the superspace generalization of the gauge covariant derivative [5].

Expanding the superfields $\mathcal{A}_\alpha$ and $\mathcal{E}_\alpha$ in the coordinates θ and $\bar{\theta}$,

$$\mathcal{A}_\alpha(y) = \Lambda_\alpha(x) + \Upsilon_\alpha(x)\theta + \bar{\theta}\bar{\Upsilon}_\alpha(x) + \Gamma_\alpha\bar{\theta}\theta \quad (31.a)$$

$$\mathcal{E}_\alpha(y) = \Pi_\alpha(x) + \Sigma_\alpha(x)\theta + \bar{\theta}\bar{\Sigma}_\alpha(x) + \Delta_\alpha(x)\bar{\theta}\theta \quad (31.b)$$

we conclude from (29) that (Λ, Δ) , $(\Upsilon, \bar{\Sigma})$, $(\bar{\Upsilon}, \Sigma)$ and (Γ, Π) form canonically conjugated pairs. The phase space of the component fields then consists of 24 canonically conjugated coordinates, which *exactly* equals the number of canonically conjugated coordinates obtained earlier in our construction of the covariant phase space in the formalism of [6]. Indeed, if we define the ghost numbers of θ and $\bar{\theta}$ to be -1 *resp.* 1, we conclude that the following identifications are consistent with (29), and the grading and ghost number assignments:

$$\mathcal{A}_i^a(y) = A_i^a(x) + \theta\psi_i^a(x) + \bar{\theta}\bar{\chi}_i^a(x) + \theta\bar{\theta}\lambda_i^a(x) \quad (32.a)$$

$$\mathcal{A}_\theta^a(y) = \eta^a(x) - \varphi^a(x)\theta - \lambda^a(x)\bar{\theta} + \theta\bar{\theta}\bar{\eta}^a(x) \quad (32.b)$$

$$\mathcal{A}_{\bar{\theta}}^a(y) = \bar{b}^a(x) + \bar{\pi}^a(x)\theta + \bar{\theta}b^a(x) + \bar{\theta}\theta\rho^a(x) \quad (32.c)$$

$$\mathcal{E}_i^a(y) = \pi_i^a(x) + \theta\bar{\psi}_i^a(x) + \chi_i^a(x)\bar{\theta} + \theta\bar{\theta}E_i^a(x) \quad (32.d)$$

$$\mathcal{E}_\theta^a(y) = l^a(x) - c^a(x)\theta + \bar{\theta}\bar{\lambda}^a(x) + \bar{\theta}\theta\bar{c}^a(x) \quad (32.e)$$

$$\mathcal{E}_{\bar{\theta}}^a(y) = \bar{p}^a(x) + \pi^a(x)\theta - \bar{\theta}p^a(x) + \theta\bar{\theta}\mathcal{P}^a(x) \quad (32.f)$$

We remark that this identification is unique up to linear canonical transformations.

Notice that the multiplet corresponding to (13) emerges from the following, self-consistent truncation of (32),

$$\mathcal{A}_i^a(y) = A_i^a(x) + \theta\psi_i^a(x) \quad \& \quad \mathcal{A}_\theta^a(y) = \eta^a(x) - \varphi^a(x)\theta \quad (33.a)$$

$$\mathcal{E}_i^a(y) = \chi_i^a(x)\bar{\theta} + \theta\bar{\theta}E_i^a(x) \quad \& \quad \mathcal{E}_{\bar{\theta}}^a(y) = -\bar{\theta}p^a(x) + \theta\bar{\theta}\mathcal{P}^a(x) \quad (33.b)$$

If we substitute this into the superspace Gauss law operator (30) and expand $\mathcal{G}^a$ in θ and $\bar{\theta}$ we find that the components of $\mathcal{G}^a$ exactly reproduce the ghost moments of the BRST operator in (13),

$$\mathcal{G}^a(y) = -\{\mathcal{E}_{\bar{\theta}}^a, \Omega\} \quad (34)$$

with the nilpotency of Ω ensuring that $\mathcal{G}^a(y)$ is BRST invariant. From this we conclude that *the BRST invariance of the topological Yang-Mills theory determines superspace gauge invariance.*

We remark that the ghost moments in (34),

$$\mathcal{L}^a = \{p^a, \Omega\} \quad \& \quad T^a = \{\mathcal{P}^a, \Omega\} \quad (35)$$

determine the following extension of the gauge algebra (5)

$$\{T^a(x), T^b(y)\} = f^{abc} T^c(x) \delta(x - y) \quad (36.a)$$

$$\{T^a(x), \mathcal{L}^b(y)\} = f^{abc} \mathcal{L}^c(x) \delta(x - y) \quad \& \quad \{\mathcal{L}^a(x), \mathcal{L}^b(y)\} = 0 \quad (36.b)$$

The relation (34) suggests that the BRST operator (13) should also admit a simple superfield representation. Indeed, with (33) we find that (13) is given by

$$\Omega = \Omega_T + \Omega_G = \int d\bar{\theta} d\theta \left\{ g^{\alpha\beta} \partial_\theta \mathcal{A}_\alpha^a \mathcal{E}_\beta^a + \mathcal{A}_\theta^a \mathcal{D}_i^{ab} \mathcal{E}_i^b + \frac{1}{2} f^{abc} \mathcal{A}_\theta^a \mathcal{A}_\theta^b \mathcal{E}_\theta^c \right\} \quad (37)$$

$$\{\Omega, \Omega\} = \{\Omega_T, \Omega_T\} = \{\Omega_G, \Omega_G\} = 0 \quad (38)$$

Here the first term (Ω_T) generates phase space translations in the θ direction for the component fields, while the last two terms (Ω_G) have the functional form of a superspace BRST operator for (the spatial part of) superspace Gauss law, with $\mathcal{A}_\theta^a$ and $\mathcal{E}_\theta^a$ identified as the ghosts in the superspace.

We find that also for the *full* multiplet (32), (37) determines an identically nilpotent BRST operator in the superspace. This BRST operator reproduces the extended phase space BRST operator (17) modulo canonical transformations (18), where the generating functionals Φ have the functional form (19), (23) in the additional abelian variables in (17). Thus we identify (37) as the correct BRST operator for topological Yang-Mills theory in the extended phase space (32), and substituting in (34) we find

$$\mathcal{G}^a \sim \mathcal{D}_i^{ab} \mathcal{E}_i^b + g^{\theta\bar{\theta}} \mathcal{D}_\theta^{ab} \mathcal{E}_\theta^b = -\{\mathcal{E}_\theta^a, \Omega\} \quad (39)$$

This determines a selfconsistent truncation of the full Gauss law generator, corresponding to the superfields (32) in the absence of the "superspace antighosts" $\mathcal{A}_{\bar{\theta}}$ and $\mathcal{E}_{\bar{\theta}}$ ⁴.

The preceding results imply that the symmetry structure captured by the full BRST operator (37) of topological Yang-Mills theory determines supergauge invariance in the

⁴Since the BRST operator has ghost number one, it breaks ghost number symmetry. The superspace antighost fields $\mathcal{A}_{\bar{\theta}}$ and $\mathcal{E}_{\bar{\theta}}$, and the generator $\mathcal{D}_\theta^{ab} \mathcal{E}_\theta^b$ in $\mathcal{G}^a$ would appear if we would consider the superspace antiBRST operator.

Parisi-Sourlas superspace. We shall now proceed to demonstrate, that *Donaldson's invariants considered in [2] can be directly extracted from the component field expansion of the abelian chiral anomaly in the Parisi-Sourlas superspace.*

For this we first need to construct the Yang-Mills curvature tensor in the superspace: From [5] the Yang-Mills connection one-form is

$$\mathcal{A}(y) = dy^\alpha \mathcal{A}_\alpha(y) \tau^a \quad (40)$$

with τ^a generators of the underlying Lie algebra, *e.g.* $SU(N)$. The wedge product in this superspace is defined by

$$dy^\alpha \wedge dy^\beta = \frac{1}{2}(dy^\alpha \otimes dy^\beta - (-)^{\alpha\beta} dy^\beta \otimes dy^\alpha) \quad (41)$$

so that

$$dy^\alpha \wedge dy^\beta = -(-)^{\alpha\beta} dy^\beta \wedge dy^\alpha \quad (42)$$

and introducing superspace exterior derivative by $d = dy^\alpha \partial_\alpha$ the Yang-Mills curvature two-form is defined by [5]

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \quad (43)$$

If we substitute in (43) the truncated fields (33) we get

$$\begin{aligned} \mathcal{F} = & dx^i \wedge dx^j F_{ij} + \theta dx^i \wedge dx^j D_i \psi_j + d\theta \wedge dx^i (\psi_i - D_i \eta) \\ & + d\theta \wedge dx^i (D_i \varphi + [\psi_i, \eta]) \theta + d\theta \wedge d\theta (\varphi - \frac{1}{2}[\eta, \eta]) - d\theta \wedge d\theta [\varphi, \eta] \theta \end{aligned} \quad (44)$$

Comparing with (25), (26) we then observe that if we identify the canonical coordinates in (44) with the abelian coordinates in (25), the θ -dependent terms in (44) are exactly the BRST variations of the corresponding θ -independent terms in (44). Consequently we can identify $\delta_{\tilde{\eta}} \sim \partial_\theta$ *i.e.* the abelian BRST operator generates supertranslations in the Parisi-Sourlas superspace. Obviously, this could also be concluded by comparing (26) with the expansion (33) in the abelian variables.

The representation of the curvature tensor in the original variables of (13) follows when we implement the canonical conjugation (23), (24) in (44). This yields

$$\begin{aligned} \mathcal{F} \rightarrow e^{\tilde{\Phi}} \mathcal{F} e^{-\tilde{\Phi}} = & dx^i \wedge dx^j F_{ij} + dx^i \wedge dx^j (D_i \psi_j - D_i D_j \eta) \theta + d\theta \wedge dx^i \psi_i \\ & + d\theta \wedge dx^i (D_i \varphi + [\psi_i, \eta]) \theta + d\theta \wedge \theta \varphi - d\theta \wedge d\theta [\varphi, \eta] \theta \end{aligned}$$

$$\sim F + \psi + \varphi + \theta \delta_\Omega(F + \psi + \varphi) \quad (45)$$

Here the θ -independent terms exactly reproduce the structure in (28). Notice also how the curvature two-form neatly summarizes the effects of the BRST transformation (15). Furthermore, the identification $\delta_\Omega \sim \partial_\theta$ persists at this level.

We now consider the abelian chiral anomaly, *i.e.* the second Chern character in the superspace:

$$ch_2[\mathcal{A}] = tr\{\mathcal{F} \wedge \mathcal{F}\} \quad (46)$$

By construction, (46) is independent of the superspace metric tensor.⁵ It is also invariant under superspace gauge transformations

$$\mathcal{A} \rightarrow \mathcal{U}^\dagger \mathcal{A} \mathcal{U} + \mathcal{U}^\dagger d\mathcal{U} \quad \& \quad \mathcal{F} \rightarrow \mathcal{U}^\dagger \mathcal{F} \mathcal{U} \quad (47)$$

where

$$\mathcal{U}(y) = exp\{i\xi^a(y)\tau^a\} \quad (48)$$

Furthermore, we observe that locally (46) can be represented as the superspace exterior differential of the Parisi-Sourlas generalized Chern-Simons three-form,

$$tr\{\mathcal{F} \wedge \mathcal{F}\} = d\omega_3(\mathcal{A}) = d\left(tr\{\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\}\right) \quad (49)$$

In analogy with the ordinary space Chern-Simons three-form, the superspace $\omega_3(\mathcal{A})$ also fails to remain invariant under superspace gauge transformations. From this we conclude that the cohomology associated with (46) is nontrivial, and in particular integrals of (46) over nontrivial superspace four-cycles yield nontrivial results.

If we substitute in (46) the truncated expansions (33) and delete the θ -dependent terms we get

$$tr\{\mathcal{F} \wedge \mathcal{F}\} = tr\{F \wedge F\} + tr\{\psi \wedge \psi\} + tr\{\varphi \wedge \varphi\} + 2tr\{F \wedge \psi\} + 2tr\{F \wedge \varphi\} + 2tr\{\psi \wedge \varphi\} \quad (50)$$

Decomposing this on the basis of the superspace four-forms we find *exactly* the invariants introduced in [2], leading to Donaldson's polynomials: In this sense Donaldson's results can be understood as a manifestation of the abelian chiral anomaly in the Parisi-Sourlas superspace.

Notice that our considerations also extend to higher order topologically nontrivial polynomials. For example in a complex representation of the gauge group integrals of

⁵Here we interpret $\mathcal{A}_\alpha$ as a one-form in the 4+2 dimensional superspace; From the present point of view, the previous considerations can be seen as corresponding to the $\mathcal{A}_0 = 0$ supergauge.

$tr\{\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F}\}$ over nontrivial superspace six-cycles yield nontrivial results. It would be interesting to understand the relation between these invariants and the topology of lower dimensional manifolds. It would also be interesting to understand the role of descent equations in this context. In particular, what is the topological interpretation of *nonabelian* anomalies in the superspace? Furthermore, from [14] we learn that the conventional Chern-Simons three-form is related to Jones polynomials [15] of knot theory. Here we find that Donaldson's invariant are related to the Parisi-Sourlas generalization of this Chern-Simons three-form. Since the component field expansion of this generalization includes the conventional Chern-Simons three-form, this might provide a clue to surprising relations between chiral anomalies, Donaldson's polynomials, knot theory and conformal field theories [14].

Finally, we remark that in addition of (13) another candidate for "topological field theory" can be constructed by substituting the color magnetic field $B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a$ for the color electric field in (13). The Bianchi identity $D \cdot B = 0$ is interpreted as a reducibility relation, and the pertinent (truncated) nilpotent BRST operator is

$$\Omega = \psi^a \cdot B^a + \varphi^a D^{ab} \cdot \chi^b + \eta^a (G^a + T^a + Q^a) + \frac{1}{2} f^{abc} \eta^a \eta^b \mathcal{P}^c \quad (51)$$

Comparing the structure of (13) and (51) might then shed new light on duality relations in nonabelian gauge theories.

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