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THE DYNAMIC APERTURE PROBLEM**

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DYNAMICAL SYSTEMS METHODS IN ACCELERATOR PHYSICS: THE DYNAMIC APERTURE PROBLEM

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1. Introduction

The aim of this paper is to show how qualitative methods of the theory of non-linear Dynamical Systems are used to design and optimize the layout of magnets in a circular synchrotron particle accelerator. The dynamic aperture problem consists in the optimization of the limits of stability in the plane transversal to the machine design orbit, in order to avoid losses in the bunch of particles at injection. Here we derive a formula for the linear dimension of the stability region in the transversal plane to the accelerator, as a function of the lengths of the straight sections that surround dipoles with a parasitic sextupolar field.

The general linear theory of the alternating-gradient synchrotron accelerator has been developed by Courant and Snyder in 1958, [CS]. The tools used by these authors are essentially based on Floquet theory of linear differential equations and the resonance analysis of externally forced linear differential equations. These techniques are still important for the design of the present days machines. However, in the future, with the use of superconducting magnets, non-linear fields will be strong enough to affect the stability limits of the transversal oscillations of the bunch of particles travelling inside the vacuum chamber of the accelerator.

An important point that distinguishes externally forced linear systems from parametrically excited systems is that, in the latter amplitude growth at instability is exponential in "time", whereas in the former amplitude growth is only linear. Therefore, in particle accelerators, where the resonances are excited parametrically, an accurate knowledge of the mechanisms of resonance excitation is required. In the Courant and Snyder classical paper, [CS], this distinction is not stressed and all the theories of resonance in accelerators have been developed in the framework of small denominators, characteristic of externally forced linear systems.

In references [D] and [APD] we have developed a formalism to deal with non-linear fields strongly localized along the vacuum chamber of the accelerator. This description produces physical criteria for the optimization of the layout of magnets along the machine and for the choice of the working point of the accelerator in a suitable parameter space.

We derive here a formula for the length of a drift space that follows a superconducting dipole magnet with a sextupolar magnetic field, in order to optimize the limit of stability for oscillations of the beam of particles in the direction transversal to the design orbit of the accelerator. This is obtained through the Poincaré map associated to

the equations of motion derived in [D] and by simple topological considerations about the conservative Hénon map. The main result of this paper is summarized in Theorem 3.

2. Equations of motion and the dynamics of sextupoles

The simplest structure for a synchrotron machine is a sequence of dipole magnets, quadrupoles, and drift spaces arranged in a closed design orbit, Fig. 1. The problem we want to deal with concerns the stability of oscillations of a bunch of particles in the direction transversal to the machine design orbit and in this case the longitudinal and transversal motions can be decoupled, [CS]. In the transversal plane the equations of motion of a charged test particle of mass m and longitudinal momentum p_0 are,

$$\ddot{x} + K_x(s)x = \frac{m}{p_0^2} Q_x(x, z, s) \quad (1a)$$

$$\ddot{z} + K_z(s)z = \frac{m}{p_0^2} Q_z(x, z, s), \quad (1b)$$

where a dot over a letter represents differentiation with respect to s , $K_x(s)$ and $K_z(s)$ are periodic step functions with period L (the length of the machine closed design orbit), describing the distribution and strength of dipoles and quadrupoles. $Q_x(x, z, s)$ and $Q_z(x, z, s)$ are the non-linear force components in the x and z directions and are periodic in the variable s . For example, in the case of dipoles and quadrupoles $Q_x = 0$. For symmetric sextupoles, $Q_x = p_0^2 k_2 (x^2 - z^2)/2m$, where k_2 measures the sextupolar strength.

FIG. 1

In general dipoles bend the particles in the (x', y') plane and if we consider that the non-linear forces are symmetric in relation to the (x', y') plane we can drop the

dependence in z in equation (1a) and neglect equation (1b), (see [D] for details). Hence, for this simple case and with $Q_x = 0$, we have the simpler equation

$$\ddot{x} + K(s)x = 0$$

whose solutions are, by Floquet theorem,

$$x(s) = a\sqrt{\beta(s)}\cos(\nu_0\phi(s) + \delta)$$

where a and δ are integration constants, $\beta(s)$ is the Courant and Snyder beta function and,

$$\nu_0 = \frac{1}{2\pi} \oint \frac{dt}{\beta(t)}, \quad \phi(s) = \frac{1}{\nu_0} \int_0^s \frac{dt}{\beta(t)}. \quad (2)$$

The parameter ν_0 is the tune of the betatronic motion and $\phi(s)$ is the phase advance. For $s = L$, the length of the closed orbit, $\phi = 2\pi$.

It is easy to prove that for machines with dipoles and quadrupoles it is always possible to find a function $K(s)$ such that $\beta(s)$ is strictly positive and periodic. In such conditions, choosing the new reduced variable $u = x/\sqrt{\beta(s)}$ and ϕ for independent variable, equation (1a) reads, (dropping the z dependence),

$$\frac{d^2u}{d\phi^2} + \nu_0^2 u = \nu_0^2 \frac{m}{p_0^2} \beta^{3/2}(\phi) Q_x(u, \phi). \quad (3)$$

If we make $Q_x = 0$ in the above equation, it describes the motion of a test particle along the structure of dipoles, quadrupoles and drift spaces (straight sections) of the accelerator. If $Q_x \neq 0$ we are incorporating the effect of symmetric non-linear fields. For example, in the case of symmetric sextupoles, $Q_x(u, \phi) = p_0^2 k_2(\phi) \beta(\phi) u^2 / 2m$, where $k_2(\phi)$ gives there distribution and strengths along the design orbit.

In general, in synchrotron accelerators the tune ν_0 of the betatronic motion is greater than one and the right hand side of equation (3) has period 2π in ϕ . So, we are in presence of "superharmonic" parametric excitation and the techniques of perturbation theory do not apply. This motivates the introduction of some physical simplifications in the study of equation (3). In reference [D] we have proved the following:

Theorem 1. *Let us suppose that there are $N \geq 1$ strongly localized sextupolar fields (thin lens approximation) at phases ϕ_m , $m = 1, \dots, N$, along the machine design orbit. Thus, the right hand side of equation (3) reads:*

$$\begin{aligned} \nu_0^2 \frac{m}{p_0^2} \beta^{3/2}(\phi) Q_x(u, \phi) &= \nu_0^2 \beta^{5/2}(\phi) k_2(\phi) u^2 / 2 = \\ &= \nu_0^2 \sum_{k=-\infty}^{\infty} \sum_{m=1}^N \beta^{5/2}(\phi_m) \Delta\phi_m \frac{k_2(\phi_m)}{2} \delta(\phi - \phi_m - 2\pi k) u^2 \end{aligned}$$

where

$$\Delta\phi_m = \frac{1}{\nu_0} \int_{s_m - L_m/2}^{s_m + L_m/2} \frac{dt}{\beta(t)} \simeq \frac{L_m}{\nu_0 \beta(\phi_m)},$$

and s_m is the mean position of the localized sextupolar interaction. L_m is the interaction length. Under these conditions we have:

a) Let $u_1 = u(2\pi)$, $v_1 = \dot{u}(2\pi)/\nu_0$, $u_0 = u(0)$ and $v_0 = \dot{u}(0)/\nu_0$. Then, the Poincaré map or one-turn map of equation (3) is

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_0) & \sin(2\pi\nu_0) \\ -\sin(2\pi\nu_0) & \cos(2\pi\nu_0) \end{pmatrix} \begin{pmatrix} u_0 - \sum_{m=1}^N \alpha_m \sin(\nu_0\phi_m) u^2(\phi_m) \\ v_0 + \sum_{m=1}^N \alpha_m \cos(\nu_0\phi_m) u^2(\phi_m) \end{pmatrix}$$

where

$$u(\phi_m) = \cos(\nu_0\phi_m)u_0 + \sin(\nu_0\phi_m)v_0 + \sum_{p=1}^{m-1} \sin(\nu_0(\phi_m - \phi_p))\alpha_p u^2(\phi_p)$$

$$\alpha_m = \frac{\beta^{3/2}(\phi_m)k_2(\phi_m)L_m}{2}.$$

Moreover, the Jacobian determinant of the above map is constant and equals one.

b) If in addition for all $m = 1, \dots, N$ there exist integers $k(m) = 1, 2, \dots$, such that,

$$\nu_0\phi_m = \int_0^{s_m} \frac{dt}{\beta(t)} = k\pi,$$

the map in a) reduces to

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_0) & \sin(2\pi\nu_0) \\ -\sin(2\pi\nu_0) & \cos(2\pi\nu_0) \end{pmatrix} \begin{pmatrix} u_0 - \sum_{m=1}^N \alpha_m \sin(\nu_0\phi_m) u^2(\phi_m) \\ v_0 + \sum_{m=1}^N \alpha_m \cos(\nu_0\phi_m) u^2(\phi_m) \end{pmatrix}.$$

In the case of one sextupole, $N = 1$, the Poincaré map in Theorem 1 a) reduces to

$$T : \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_0) & \sin(2\pi\nu_0) \\ -\sin(2\pi\nu_0) & \cos(2\pi\nu_0) \end{pmatrix} \begin{pmatrix} u_0 - \alpha_1 \sin(\nu_0\phi_1) u^2(\phi_1) \\ v_0 + \alpha_1 \cos(\nu_0\phi_1) u^2(\phi_1) \end{pmatrix} \quad (4)$$

where

$$u(\phi_1) = u_0 \cos(\nu_0\phi_1) + v_0 \sin(\nu_0\phi_1),$$

which is a quadratic map of the plane. In the case of $N > 1$ sextupoles the Poincaré map has polynomial order 2^N and the coefficients of the non-linear terms grow very fast, [D]. However we show below that even with a simple map like (4) we can in fact obtain some optimizing results for the design of a particle accelerator.

We now recall the following Theorem:

Theorem 2 ([E],[H]). Let

$$H : \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_0) & \sin(2\pi\nu_0) \\ -\sin(2\pi\nu_0) & \cos(2\pi\nu_0) \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \begin{pmatrix} f(u_0, v_0) \\ g(u_0, v_0) \end{pmatrix}$$

where f and g are homogeneous polynomials of degree 2. If the Jacobian of H is non-zero and everywhere constant, then there exists a linear change of coordinates such that

H is linearly conjugated to the map

$$H' : \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_0) & \sin(2\pi\nu_0) \\ -\sin(2\pi\nu_0) & \cos(2\pi\nu_0) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 + x_0^2 \end{pmatrix} \quad (5)$$

In view of Theorem 2 we conclude that maps (4) and (5) are linearly conjugated. More precisely, there exists a linear function $h(x, y) = (u, v)$, with non-zero Jacobian determinant, such that $H' = h^{-1} \circ T \circ h$ and so the qualitative structure of the orbits in phase space of map (4) can only change if the parameter ν_0 is varied. The parameter α_1 is simply a scale factor and ϕ_1 rotates orbits in phase space (see the transformations of coordinates defined below in (7)).

An important problem in the design of the accelerator lattice concerns the length of straight sections between regions of strong non-linear fields ([D]). This can be easily seen by Theorem 1. Suppose we make all the $\phi_2, \dots, \phi_N$ decrease simultaneously to ϕ_1 or take the values $\phi_m = k(m)\pi/\nu_0$, with $m = 1, \dots, N$. In both limits the dynamics of the test particle is described by maps linearly conjugated to the Hénon map. Between these two limits, where very high order polynomial terms come into play, we can have a drastic change in the dynamics of the system, [D]. The strongest change in the phases ϕ_m can be obtained by changing the lengths of straight sections because in those situations $\beta(s) \sim s^2$. Moreover, between dipoles producing a strong non-linear parasitic sextupolar field, straight section lengths are the simplest parameters we can change to control the non-linear coupling between them, maintaining acceptable limits of stability.

Most of the accelerator lattice magnets are dipoles producing a strong sextupolar parasitic field. We want to obtain with the limits of stability around the design orbit, that is, the region of Lyapunov stability of the point $(0, 0)$ in the transversal phase space, Fig. 1. We consider a beam line as shown in Fig. 2, consisting of a sequence of smaller lines *dipole + drift space* with a thin sextupolar field in the middle of the dipole.

FIG. 2

Let μ_1 and μ_2 be the phase advances as indicated in figure 2, $2\pi\mu = \int dt/\beta(t)$, (2). With this notation the transfer map that relates the normalized position and velocity of a test particle between the beginning and the end of the accelerator small line is, by (4),

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\mu) & \sin(2\pi\mu) \\ -\sin(2\pi\mu) & \cos(2\pi\mu) \end{pmatrix} \begin{pmatrix} u_0 - \alpha_1 \sin(\pi\mu_1)u^2(\mu_1/2) \\ v_0 + \alpha_1 \cos(\pi\mu_1)u^2(\mu_1/2) \end{pmatrix} \quad (6)$$

where

$$u(\mu_1/2) = u_0 \cos(\pi\mu_1) + v_0 \sin(\pi\mu_1),$$

and $\mu = \mu_1 + \mu_2$ is the total phase advance in the beam line. By Theorem 2 this map is linearly conjugated to the Hénon map.

To obtain information about the region of stability around the point $(0,0)$ in phase space we use the results known for map (5). To be more precise we take the two sets of new variables,

$$\begin{aligned} (w, z) &= (\alpha_1 u, \alpha_1 v) \\ (x, y) &= (w \cos(\pi\mu_1) + z \sin(\pi\mu_1), -w \sin(\pi\mu_1) + z \cos(\pi\mu_1)) \end{aligned} \quad (7)$$

and a straightforward calculation shows that (6) is linearly conjugated to the Hénon map (5), ($\nu_0 = \mu$).

The Hénon map has two period one fixed points, $x_0 = (0,0)$ and $x_1 = (2 \tan(\pi\nu_0), -2 \tan^2(\pi\nu_0))$. The point x_1 is of saddle type for $\nu_0 \neq k$ and $\nu_0 \neq k + 1/2$, where k is a positive integer. For $\nu_0 = k$ and $\nu_0 = k + 1/2$, map (5) is readily integrable. It is known numerically that for all $\nu_0 \in (k, k + 1/2)$ the stable and unstable manifold of x_1 intersect transversally and so there exist a trapping region around the origin $(0,0)$ where orbits remain limited. Far from the origin the iterated points go to infinity.

For the Hénon map the trapping region around the origin can be easily computed numerically. Take the straight line with equation $y = -x \tan(\pi\nu_0)$ and test, for each ν_0 and for a sufficiently large number of iterates, if the orbits of the points $(x, -x \tan(\pi\nu_0))$ remain bounded. The maximum value that the variable x can reach with bounded iterates gives an estimate of the dimensions D of the stable region around the origin $(0,0)$. In figure 3 we represent this maximum value as a function of the parameter ν_0 .

FIG. 3

Figure 3 clearly shows how the linear dimension D of the stability region around the origin changes drastically. For $0 \leq \nu_0 \leq 0.15$ the linear dimension of the stability region along the x direction goes as (see fig. 3),

$$D = 2 \tan(\pi \nu_0) = 2\pi \nu_0 + 2\pi^3 \nu_0^3/3 + \dots \quad (8)$$

which is the x coordinate of the saddle point x_1 . In fact, if $\nu_0 \leq 0.15$, the stable (W^s) and unstable (W^u) manifolds of x_1 are almost coincident. For $\nu_0 > 0.15$ the homoclinic intersections develop too fast in the formation of horseshoes. This can be seen in figure 4 where we present the phase space for map (5) for $\nu_0 = 0.15$ and $\nu_0 = 0.2$, as well as pieces of the stable (W^s) and unstable (W^u) manifolds of x_1 .

For real accelerator lattices the phase advance μ between consecutive bending dipoles is typically in the interval $[0.02, 0.1]$ as is the case of the design lattice of the Large Hadron Collider (CERN) for protons at 450 GeV. Therefore, we can use the linear approximation for D to derive the dependence of the dimension of the stability region as a function of the accelerator parameters.

FIG. 4

As we have seen in Theorem 2 the only parameter driving the bifurcation structure of map (6) is the tune, or phase advance, ν_0 . So, we now derive an explicit formula for the phase advance μ in the beam line of figure 2. In a drift space the beta function as a function of the length s is, [S],

$$\beta(s) = \beta_0 + \dot{\beta}_0 s + \frac{4 + \dot{\beta}_0^2}{4\beta_0} s^2 \quad (9)$$

where β_0 and $\dot{\beta}_0$ are the values of $\beta(s)$ and $\dot{\beta}(s)$ at $s = 0$. Hence, the phase advance μ as a function of s becomes,

$$\mu = \frac{1}{2\pi} \int_0^s \frac{dt}{\beta(t)} = \frac{1}{2\pi} \left(\arctan\left(\frac{\dot{\beta}_0}{2} + \frac{4 + \dot{\beta}_0^2}{4\beta_0}s\right) - \arctan\left(\frac{\dot{\beta}_0}{2}\right) \right).$$

We now calculate the dynamic aperture (the linear dimension of the stability region in phase space) as a function of the straight section length for the machine line in figure 2. Denote by s_1 the length of the drift space. As, the total phase advance μ is given by $\mu = \mu_1 + \mu_2$ we have,

$$\mu = \mu_1 + \frac{1}{2\pi} \left(\arctan\left(\frac{\dot{\beta}_{02}}{2} + \frac{4 + \dot{\beta}_{02}^2}{4\beta_{02}}s_1\right) - \arctan\left(\frac{\dot{\beta}_{02}}{2}\right) \right) \quad (10)$$

where β_{02} refer to the beta function at the end of the dipole and μ_1 is the phase advance in the dipole.

As we have seen previously in (8), for $\mu \leq 0.15$ the linear dimension of the stability region is $D = 2\pi\mu$, so, the dynamic aperture of the beam line, in the Hénon map coordinates is, by (10),

$$D = D_{dipole} + \arctan\left(\frac{\dot{\beta}_{02}}{2} + \frac{4 + \dot{\beta}_{02}^2}{4\beta_{02}}s_1\right) - \arctan\left(\frac{\dot{\beta}_{02}}{2}\right) \quad (11)$$

where D_{dipole} is the dynamic aperture corresponding to the phase advance inside the dipole.

In the dipole we have, [St],

$$\beta(s) = \beta_{01} \cos^2(s/\rho) + \dot{\beta}_{01} \rho \sin(s/\rho) \cos(s/\rho) + \rho^2 \frac{4 + \dot{\beta}_{01}^2}{4\beta_{01}} \sin^2(s/\rho)$$

where ρ is the radius of curvature of the design orbit inside the dipole. Following the same argument as above we conclude that,

$$D_{dipole} = \arctan\left(\frac{\dot{\beta}_{01}}{2} + \frac{4 + \dot{\beta}_{01}^2}{4\beta_{01}}\rho \tan(L_1/\rho)\right) - \arctan\left(\frac{\dot{\beta}_{01}}{2}\right) \quad (12)$$

where L_1 is the length of the curved orbit inside the dipole and β_{01} refers to the beta function at the beginning of the dipole.

In non-normalized coordinates, $x = \sqrt{\beta_{01}}u$ and by (7), the dynamic aperture D^* of the machine line in figure (2) is,

$$D^* = \frac{\sqrt{\beta_{01}}}{\alpha_1} D \simeq \frac{\sqrt{\beta_{01}}}{\alpha_1} \left(\frac{L_1}{\beta_{01}} + \frac{s_1}{\beta_{02}} \right) = \frac{2}{\beta^{3/2}(\mu_1/2)\sqrt{\beta_{01}}k_2} + \frac{\sqrt{\beta_{01}}}{\beta_{02}\alpha_1} s_1$$

where $\alpha_1 = \beta^{3/2}k_2L_1/2$ is the sextupolar strength defined in Theorem 1 and L_1 is the dipole length; this last expression has been obtained under the hypothesis of small lengths in (11) and (12).

Therefore, we conclude that the increase of straight sections that surround dipoles originates an increase in the acceptance stability of the machine. This conclusion has been derived under the assumption that the phase advance is kept in the interval $[0, 0.15]$. The same qualitative conclusion still holds for μ in the neighbourhood of 0.5 (see figure 3), although this is in general too restrictive if we want to keep the machine as small as possible. Another possibility to increase the dynamic aperture of the beam line is to decrease the sextupolar constant α_1 (see Theorem 1) or to decrease the values of the beta function.

The conclusions of this paper can be summarized in the following theorem.

Theorem 3. *Let B be the machine line*

$$B = \text{dipole with sextupolar field} + \text{drift space}$$

and μ the phase advance inside B . Then, in the framework of the thin lens approximation (Theorem 1) and under the hypothesis that $\mu \leq 0.15$ in B , the dynamic aperture of the machine line B is,

$$D^* = \frac{\sqrt{\beta_{01}}}{\alpha_1} \left(\arctan\left(\frac{\dot{\beta}_{02}}{2} + \frac{4 + \dot{\beta}_{02}^2}{4\beta_{02}} s_1\right) + \arctan\left(\frac{\dot{\beta}_{01}}{2} + \frac{4 + \dot{\beta}_{01}^2}{4\beta_{01}} \rho \tan(L_1/\rho)\right) - \arctan\left(\frac{\dot{\beta}_{01}}{2}\right) - \arctan\left(\frac{\dot{\beta}_{02}}{2}\right) \right) \simeq \frac{2}{\beta^{3/2}(\mu_1/2)\sqrt{\beta_{01}k_2}} + \frac{\sqrt{\beta_{01}}}{\beta_{02}\alpha_1} s_1$$

where, β_{01} , $\dot{\beta}_{01}$, β_{02} and $\dot{\beta}_{02}$, are the values of the beta function and its derivative at the beginning and end of the dipole. $\beta(\mu_1/2)$ is the value of the beta function at the middle of the dipole. α_1 [resp. k_2] is the sextupolar strength as defined in Theorem 1a), L_1 is the length of the dipole, s_1 is the length of the drift space and ρ is the radius of curvature of the particle orbit inside the dipole.

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