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QUARK-GLUON TRANSPORT THEORY

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ABSTRACT

We review the present status of
formulating classical and
quantum kinetic equations for
relativistic plasmas with
Abelian and non-Abelian inter-
actions.

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1. INTRODUCTION

1.1 Historical overview and summary of present status

Although plasma physics today can look back on a long and successful history, the physics of relativistic plasmas (involving temperatures of the order of or larger than the particle masses) generally has been surprisingly little studied. Some early pioneering work in this direction, still mostly on a classical level, grew out of problems and applications in astrophysics [1] - [7]. (For a recent progress report on kinetic theory methods in cosmology see ref.[8].) A quantum mechanical framework developed from there [7,9,10,11] and was later extended and based on relativistic quantum field theory in the context of multiparticle production in collisions between elementary particles, see ref.[12] for a review. An introduction to the basic concepts leading to these latter developments can be found in the textbook on relativistic kinetic theory by *de Groot, van Leeuwen, and van Weert* [13], where also an extended list of references can be found as well as in ref.[7].

In the last few years a new field of possible applications of these methods has arisen in high-energy nuclear physics: With the beginning of experimental programs to search for quark-gluon plasma formation in relativistic nuclear collisions [14,15] the need for a detailed theoretical description of the dynamics of such a plasma is increasingly evident. Because of the short times available for equilibration in these collisions, a formalism is needed which is fundamentally capable of dealing with non-equilibrium phenomena, i.e. which is based on kinetic theory.

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However, an attempt of straightforward application of existing methods [12,13] to a plasma of quarks and gluons fails for two reasons:

- (1) The interactions between quarks and gluons are described by a non-Abelian SU(3) gauge theory. A kinetic theory of quarks and gluons has to yield gauge invariant answers, i.e. it should be formulated on a gauge covariant basis. This leads to modifications in the definition of the phase-space distribution function for quarks [16,17], compared to their analogue for, say, electrons in an interstellar magnetic field [7,9,11] or for scalar mesons produced in hadronic collisions [12,18], where such complications were either absent or could be safely neglected. Furthermore, the concept of a quantum distribution function for the gauge field itself (i.e. for gluons) did not exist at all until the advent of quark-gluon plasma physics, and a gauge covariant formulation of this problem was found only recently [19,20].
- (2) One of the most important physical aspects in relativistic nuclear collisions is the production of many new particles (using the beam energy as a source). The formulation of relativistic kinetic theory on a quantum field theoretical basis is, however, not very well established. For example, the relationship between such concepts in kinetic theory as the BBGKY hierarchy connecting single-particle and many-body distribution functions on the one hand, and the Schwinger-Dyson equations for n -point Green functions in (equilibrium or non-equilibrium) finite temperature quantum field theory on the other hand, has not yet been fully developed. How to describe non-perturbative processes like the production of quarks and gluons from an “external” color electric field (the “Schwinger mechanism” [21], cf. Chap. 5) in kinetic theory language is an open problem.

The first one of these problems has by now been more or less solved, cf. ref.[22] for an earlier review. After some early and simple considerations on how to extend the Vlasov-equation to the description of classical colored particles [16,23], a relativistic and gauge covariant quantum mechanical treatment of the quark distribution function (“Wigner function”) was developed [16,17], and an exact kinetic equation for the associated “Wigner operator” was derived [17]. Shortly afterwards also a gauge covariant Wigner operator for gluons and its equation of motion were studied [19,20]. In the semiclassical limit both equations reproduce [16,17,20,24] the classical equations from the early attempts [16,23]. For quarks this is even true for the spin structure of the theory [17,25] (see Sec. 3.4), while for the spin-1 gluons an analogous spin analysis has not yet been performed.

However, none of the problems listed under (2) above have so far been satisfactorily solved. The kinetic equations for the quark and gluon Wigner operators are each selfcontained, i.e. the equations do not couple to each other (say, through some kind of Boltzmann type collision term), but only to higher covariant derivatives of themselves and of the gluon field strength tensor $F_{\mu\nu}$. Thereby the BBGKY hierarchy is hidden, and

future research will have to unearth it from these equations, in order to cast them into a more practical form.

The question of particle production involves two steps: first the quantum kinetic equations have to be separated into a contribution from the mean (classical) color field generated by the collective color currents in the plasma and another contribution from the gluonic quantum fluctuations. Then, the latter have to be evaluated by taking ensemble expectation values. This last step sounds easier than it is: the kinetic equations involve path-ordered exponentials of gluon field operators which so far have not been converted in a general way into time-ordered products whose ensemble expectation values could be interpreted in terms of (thermal) propagators and, thus, be mapped on (thermal) quantum field theory. The only approximation scheme which has so far been applied to these equations is the semiclassical approximation [17,20] (“ Δ -expansion”, see Chap. 3) or, even more restrictive, the Abelian Dominance Approximation (see Sec. 3.5). But even there only the lowest order has been considered in the mean-field approximation. Effects of genuinely quantum field theoretical nature have so far been analyzed only in a rather preliminary way in this framework (cf. Chap. 5).

Reflecting this state of affairs, we will mostly concentrate in this review on the questions listed under (1) above and their answers. In Chapter 2 we give a short summary of the classical approach to non-Abelian kinetic theory. Although this approach is of very limited relevance for the description of a quark-gluon plasma, since the color of quarks and gluons cannot be treated classically, these equations serve as a useful check for the quantum kinetic equations in the semiclassical limit. Furthermore, from their analysis in linear response approximation a number of quite interesting results have been obtained [20,24],[26]-[31].

In Chapters 3 and 4 we proceed to quantum kinetic theory and present the definitions of and equations of motion for the quark and gluon Wigner operators. Although these equations are rather lengthy and at first sight frightening, we present them in full beauty, with all factors of $\hbar$ and c inserted, to facilitate the later discussion of different approximation schemes. In these chapters we also discuss some of the physics contained in the equations by studying their semiclassical and Abelian limits. The resulting limiting equations have been extensively studied by several groups, and some results are presented in Section 4.4, particularly their solutions in linear response approximation. The related question of color charge density oscillations (plasmons) in a quark-gluon plasma is discussed in more detail in Section 4.5, where we also explain the unresolved issue of their damping rates. Other applications will be reviewed separately by S. Mrówczyński and by A. Białas and W. Czyż in this volume.

In Chapter 5 we present the few, still fragmentary approaches to describe particle production in the transport theory of quark-gluon plasma dynamics. This leads us in

Chapter 6 to the open problems and ideas for future research in this field, by which we close this review.

In the Appendix we shortly mention an issue which is not directly related to the main subject of the review, but nevertheless of high actual interest for the dynamical description of nuclear matter before the transition into the quark-gluon plasma phase, in nuclear collisions in the several hundred MeV to few GeV per nucleon energy range: we discuss a relativistic kinetic theory for highly excited nuclear matter as described by the mean-field Walecka model.

1.2 Definitions and some basic technical tools

Before beginning with the main body of the paper, we want to summarize our conventions.

- We use the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Eventually we choose units such that $\hbar = c = 1$, but otherwise all $\hbar, c$'s are stated explicitly.

Colored objects are usually denoted in matrix notation, $O = O^a t_a$, where t_a are the $SU(3)$ generators in a given representation. The generators are taken to be Hermitean, $t_a^\dagger = t_a$, and traceless, and are normalized such that

$$[t_a, t_b] = i\hbar f_{abc} t_c .$$

Thus, for example, in the fundamental representation

$$t_a = \frac{\hbar \lambda_a}{2} \quad \text{and} \quad \text{tr}(t_a t_b) = \frac{\hbar^2}{2} \delta_{ab} ,$$

where λ_a are the Gell-Mann matrices. This normalization has the advantage that the t_a have eigenvalues $\sim \frac{\hbar}{2}$, i.e. proportional to the physical color spin of quarks. In the adjoint representation the generators are represented by Hermitean 8×8 matrices T_a with

$$(T_a)_{bc} = -i\hbar f_{abc} \quad \text{and} \quad \text{tr}(T_a T_b) = 3\hbar^2 \delta_{ab} ,$$

such that gluons have physical color eigenvalues $\sim \hbar$.

This is in analogy to spin: while the Pauli matrices have eigenvalues ± 1 , the physical quark spin is $\pm \frac{\hbar}{2}$ and is described by the vector $\vec{s} = \frac{\hbar}{2} \vec{\sigma}$, which in the Dirac theory is imbedded in the spin tensor

$$S^{\mu\nu} = \frac{\hbar}{2} \sigma^{\mu\nu} = \frac{i\hbar}{4} [\gamma^\mu, \gamma^\nu] ,$$

where we define the representation of the γ -matrices by $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$, and $\gamma_0 = \text{diag}(1, 1, -1, -1)$. The components of the spin vector satisfy the commutation relation

$$[S^i, S^j] = i\hbar \epsilon^{ijk} S^k ,$$

similar to the color commutation relations above. - In the limit $\hbar \rightarrow 0$, where spin and color spin (i.e. the expectation values of $\vec{S}$ and $\vec{t}$) are kept finite, we obtain the limit of classical angular momentum and classical color whose components commute with each other; this is indicated by the additional factor $\hbar$ on the right hand side of the commutation relations.

The covariant derivative acting on quark field operators is defined as

$$D_\mu(x) = \frac{\partial}{\partial x^\mu} - \frac{ig}{\hbar c} A_\mu(x) . \quad (1.1)$$

In the mean-field limit this generates the field strength tensor $F_{\mu\nu} = F_{\mu\nu}^a t_a$, with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \frac{g}{c} f_{abc} A_\mu^b A_\nu^c , \quad (1.2)$$

through the relation

$$F_{\mu\nu} = \frac{\hbar c}{-ig} [D_\mu, D_\nu] . \quad (1.3)$$

Note that relation (1.3) is not equivalent to (1.2) in general, if the gluon potentials in eq.(1.1) are non-commuting field operators in Fock space. The covariant derivative acting on a color octet operator $O = O^a t_a$ will be denoted by $\mathcal{D}$:

$$\mathcal{D}_\mu O \equiv [D_\mu, O] = \partial_\mu O - \frac{ig}{\hbar c} [A_\mu, O] . \quad (1.4)$$

Alternatively, we can consider O as an 8-dimensional vector $\vec{O}$ with components O^a with the covariant derivative

$$\tilde{D}_\mu \vec{O} = \partial_\mu \vec{O} - \frac{ig}{\hbar c} \mathcal{A}_\mu \vec{O} , \quad (1.5)$$

where now $\mathcal{A}_\mu = A_\mu^a T_a$ denotes the adjoint representation of the gluon vector potential, which is an 8×8 matrix. Again, eqs.(1.4) and (1.5) are only equivalent in the mean-field limit, but differ by a commutator of the field operators $\hat{A}$ and $\hat{O}$ in the quantum case. The adjoint representation of (1.4) is given by

$$\tilde{\mathcal{D}}_\mu O \equiv [\tilde{D}_\mu, O] = \partial_\mu O - \frac{ig}{\hbar c} [\mathcal{A}_\mu, O] , \quad (1.6)$$

where the commutator is both between the two 8×8 color matrices and the Fock space operators.

With these definitions the Dirac equation for the quark fields reads

$$(i\hbar\gamma^\mu D_\mu(x) - mc)\psi(x) = 0 = \bar{\psi} (i\hbar\gamma^\mu D_\mu^\dagger(x) + mc) , \quad (1.7)$$

where $D^\dagger$ is defined to act always to the left, while the Yang-Mills equation for the gluon fields is given by

$$D_\mu(x) F^{\mu\nu}(x) = [D_\mu(x), F^{\mu\nu}(x)] = -\frac{g}{c} j^\nu(x) , \quad (1.8)$$

with the quark color current operator

$$j^\nu = j_a^\nu t_a = (\bar{\psi} \gamma^\nu t_a \psi) t_a . \quad (1.9)$$

We will also use the quadratic form of Dirac's equation (remember $S^{\mu\nu} = \frac{\hbar}{2} \sigma^{\mu\nu}$):

$$(\hbar^2 D_\mu D^\mu - \frac{g}{c} S^{\mu\nu} F_{\mu\nu} + m^2 c^2) \psi(x) = 0 = \bar{\psi} (\hbar^2 D_\mu^\dagger D^{\dagger\mu} - \frac{g}{c} S^{\mu\nu} F_{\mu\nu} + m^2 c^2) . \quad (1.10)$$

A special role will be played in this paper by the *covariant translation operator* $\exp(-y \cdot D(x))$. Its action on fermion fields is given by [17]

$$e^{-y \cdot D(x)} \psi(x) = U(x, x-y) e^{-y \cdot \partial_x} \psi(x) = U(x, x-y) \psi(x-y) , \quad (1.11)$$

i.e. it shifts the argument of the fermion operator and multiplies it with a “link operator”,

$$U(x, x-y) = P \exp\left(\frac{ig}{\hbar c} \int_{x-y}^x dz^\mu A_\mu(z)\right) . \quad (1.12)$$

Here P denotes path ordering, and the path between the endpoints $a = x - y$ and $b = x$ has to be taken as the *straight line*,

$$z(s) = a + s(b-a) , \quad 0 \leq s \leq 1 , \quad (1.13)$$

for eq.(1.11) to hold [17]. The importance of the covariant translation operator resides in the fact that, while the argument of the fermion field has been shifted, the complete object (1.11) (i.e. fermion operator multiplied by the link operator) transforms under gauge transformations like a fermion still located at point x . This is due to the transformation property of U [32,33],

$$U(b, a) \rightarrow S(b) U(b, a) S^{-1}(a) . \quad (1.14)$$

Thus, a bilinear gauge invariant operator like $\bar{\psi}(x)\psi(x)$ can be point-split by this covariant translation operator without losing gauge invariance [21]:

$$\begin{aligned} \bar{\psi}(x)\psi(x) &\rightarrow \left[\bar{\psi}(x) e^{\frac{y}{2} \cdot D(x)} \right] \left[e^{-\frac{y}{2} \cdot D(x)} \psi(x) \right] \\ &= \bar{\psi}\left(x + \frac{y}{2}\right) U\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \psi\left(x - \frac{y}{2}\right) . \end{aligned} \quad (1.15)$$

A covariant translation operator can also be considered acting on color octet operators $O = O^a t_a$ by using the definition (1.4) for the covariant derivative. Then, one finds [17]

$$e^{-y \cdot \mathcal{D}(x)} O(x) = U(x, x-y) O(x-y) U(x-y, x) , \quad (1.16)$$

which under gauge transformations behaves covariantly,

$$e^{-y \cdot \mathcal{D}(x)} O(x) \rightarrow S(x) \left[e^{-y \cdot \mathcal{D}(x)} O(x) \right] S^{-1}(x) , \quad (1.17)$$

if $O(x) \rightarrow S(x)O(x)S^{-1}(x)$. The point-splitting operation for bilinears of two such operators (see Chap. 4) can be constructed similarly as in eq.(1.15) above. Furthermore, for the adjoint representation we obtain:

$$e^{-y \cdot \tilde{D}(x)} \vec{O}(x) = \tilde{U}(x, x-y) \vec{O}(x-y) , \quad (1.18)$$

with $\tilde{D}$ as in eq.(1.5) and the link operator in the adjoint representation defined by

$$\tilde{U}(x, x-y) = P \exp\left(\frac{ig}{\hbar c} \int_{x-y}^x dz^\mu \mathcal{A}_\mu(z)\right) . \quad (1.19)$$

We finally give the formula for taking derivatives of link operators, which will be needed in deriving the quantum kinetic equations of motion. - A general formula for the first order variation δU of a link operator was derived in ref.[17] and also in ref.[34] using a different technique. There δU is considered to be due to an infinitesimal variation of the path of integration, $z(s) \rightarrow z(s) + \delta(s)$, $0 \leq s \leq 1$. The general result then, of course, allows us to obtain particular derivatives with respect to the endpoints of a path, which is all that we need in the following.

Consider the limits of the path in the link operator $U(b, a)$ to be functions of some variable x , $a(x)$ and $b(x)$. Then, one obtains

$$\begin{aligned} \frac{\partial U(b, a)}{\partial x^\mu} = & \frac{ig}{\hbar c} \left\{ \frac{\partial b^\nu}{\partial x^\mu} \left[A_\nu(b) - (b-a)^\alpha \int_0^1 ds \, s \, U(b, z) F_{\alpha\nu}(z) U(z, b) \right] U(b, a) \right. \\ & \left. - U(b, a) \left[A_\nu(a) - (b-a)^\alpha \int_0^1 ds \, (s-1) \, U(a, z) F_{\alpha\nu}(z) U(z, a) \right] \frac{\partial a^\nu}{\partial x^\mu} \right\} , \quad (1.20) \end{aligned}$$

where under the integral $z = z(s)$ is given by the path (1.13). Using eq.(1.16) for the integrands, a somewhat more compact form results:

$$\begin{aligned} \frac{\partial U(b, a)}{\partial x^\mu} = & \frac{ig}{\hbar c} \left\{ \frac{\partial b^\nu}{\partial x^\mu} \left[A_\nu(b) - (b-a)^\alpha \int_0^1 ds \, s \, [e^{(s-1)(b-a) \cdot \mathcal{D}(b)} F_{\alpha\nu}(b)] \right] U(b, a) \right. \\ & \left. - U(b, a) \left[A_\nu(a) - (b-a)^\alpha \int_0^1 ds \, (s-1) \, [e^{s(b-a) \cdot \mathcal{D}(a)} F_{\alpha\nu}(a)] \right] \frac{\partial a^\nu}{\partial x^\mu} \right\} . \quad (1.21) \end{aligned}$$

Note that, up to the factors s and $(s-1)$, the integrals in eq.(1.21) are identical:

$$\begin{aligned} U(b, a) [e^{s(b-a) \cdot \mathcal{D}(a)} F_{\alpha\nu}(a)] &= [e^{(s-1)(b-a) \cdot \mathcal{D}(b)} F_{\alpha\nu}(b)] U(b, a) \\ &= U(b, z) F_{\alpha\nu}(z) U(z, a) . \quad (1.22) \end{aligned}$$

Such manipulations are helpful for the derivations underlying Chapters 3 and 4: eq.(1.22) may always be used in such a way as to make the argument of $F_{\alpha\nu}$ and of the covariant derivative operator equal to x , the space-time coordinate of the Wigner operator.

2. CLASSICAL KINETIC THEORY FOR COLORED PARTICLES

2.1 Classical equations of motion for particles with color and spin

Let us consider the quark color generators $\hat{Q}_a \equiv -t_a = -\frac{\hbar}{2}\lambda_a$ as Heisenberg operators satisfying Heisenberg's equations of motion,

$$\dot{\hat{Q}}_a = \frac{i}{\hbar}[\mathcal{H}, \hat{Q}_a] . \quad (2.1)$$

If the theory is to be formulated in a relativistically invariant way, the time derivative in eq.(2.1) should be with respect to proper time, and the "Hamiltonian" $\mathcal{H}$ generating the proper-time evolution is to be taken as the "quadratic Dirac Hamiltonian" [25],

$$\mathcal{H} \equiv -\frac{1}{2m} \left[\left(i\hbar\partial_\mu + \frac{g}{c}A_\mu \right)^2 + \frac{g}{2c}S^{\mu\nu}F_{\mu\nu} - m^2c^2 \right] , \quad (2.2)$$

which is a Lorentz scalar (cf. Sec. 1.2). In the classical c-number limit eq.(2.1) leads to the equation of motion [25],

$$m \frac{dQ_a}{d\tau} = -\frac{g}{c}f_{abc} \left(p^\mu A_\mu^b - \frac{1}{mc}p^\alpha \tilde{F}_{\alpha\beta}^b S^\beta \right) Q^c , \quad (2.3)$$

where now Q_a are the c-number (i.e. commuting) components of an 8-component classical color vector $\vec{Q}$ describing the coupling of a classical colored particle to the eight color potentials A_μ^a . In eq.(2.3) $\tilde{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$ is the dual field strength tensor, and

$$S^\beta = -\frac{1}{2mc}\epsilon^{\beta\nu\lambda\rho}p_\nu S_{\lambda\rho} \quad (2.4)$$

is the normalized ($p_\mu S^\mu = 0$, $S^\mu S_\mu = -\vec{s}^2$) classical spin 4-vector contained in the spin tensor defined in Sec. 1.2. Note that in the second term in (2.3),

$$-\frac{1}{mc}p^\alpha \tilde{F}_{\alpha\beta} S^\beta = \frac{1}{2}S^{\mu\nu}F_{\mu\nu} , \quad (2.5)$$

which is the ordinary interaction between spin and magnetic field reducing to $\vec{s} \cdot \vec{B}$ in the particle's rest frame. Eq.(2.3) conserves the SU(3) Casimir invariants, i.e. the length $Q^a Q_a$ of $\vec{Q}$ and the cubic invariant $d_{abc}Q^a Q^b Q^c$ (where d_{abc} are the symmetric structure constants of SU(3)). Thus, the equation describes precession of the classical color vector of the particle due to two effects: direct interaction (color exchange) with an external color field A_μ^a , and coupling of the particle's spin to the color magnetic field. In the case of vanishing particle spin this equation was derived by Wong [35].

If the spin couples to the color magnetic field, it will similarly start to precess, and its equation of motion is given by the c-number limit of $\dot{S}^\mu = \frac{i}{\hbar}[\mathcal{H}, S^\mu]$, namely [25]

$$m \frac{dS^\mu}{d\tau} = \frac{g}{c}Q^a \left[F_a^{\mu\nu} S_\nu + \frac{1}{(mc)^3} (p^\mu S^\nu - p^\nu S^\mu) (D_\nu \tilde{F}_{\alpha\beta})_a p^\alpha S^\beta \right] . \quad (2.6)$$

If one neglects the second term due to inhomogeneities of the external field and non-Abelian effects, this equation is recognized as the BMT equation [36] for a spinning particle with Landé g -factor 2. Had we started from the Yang-Mills Hamiltonian rather than the Dirac Hamiltonian, we would have obtained for the spin-1 gluons a different g -factor, but otherwise the same equation. So with respect to color, we cannot distinguish in the structure of the equations between quarks and gluons: by going to the classical limit (i.e. effectively to very high-dimensional representations of the color generators) we have *lost the difference in color between quarks and gluons*. However, *in their spin aspects they still remain different*: The Landé factor distinguishes quarks from gluons in their coupling to the magnetic-field. Since we are not going to work out the spin aspects of the resulting classical kinetic theory in any detail, we will not further elaborate on this point.

Finally, we need an equation of motion for the momentum of the particle. It is given by

$$m \frac{dp^\mu}{d\tau} = \frac{g}{c} Q^a \left[F_a^{\mu\nu} p_\nu - \frac{1}{mc} (D^\mu \tilde{F}_{\alpha\beta})_a p^\alpha S^\beta \right] = \frac{g}{c} Q^a \left[F_a^{\mu\nu} p_\nu + \frac{1}{2} D^\mu (S^{\alpha\beta} F_{\alpha\beta})_a \right] . \quad (2.7)$$

Again neglecting the effects of field inhomogeneities, we recognize the (colored version of) the relativistic Lorentz force law. The second term here describes the possible gain in energy-momentum due to the space-time variation of the spin magnetic interaction energy in an inhomogeneous color magnetic field.

2.2 Classical kinetic equations for the 1-particle distribution functions in a quark-gluon plasma

Now we are ready to write down a classical kinetic equation of the Vlasov type for the single-particle phase space distribution for colored and spinning particles. Since the particles' momentum p^μ , the color Q_a , and the spin S_μ all are dynamical variables (i.e. evolve in time under the influence of an external or intrinsic mean color field), phase space has to be spanned by the 20 coordinates x^μ, p^μ, Q^a and S^μ . Only in the absence of classical color and spin it reduces to the conventional 8-dimensional phase space (x^μ, p^μ) which, by using the mass-shell constraint between p^0 and $\vec{p}$ for the classical particles, is further reduced to the well-known dimensions $(\vec{x}, \vec{p}, t)$. In our larger 20-dimensional phase space the integration measure is given by $d\Sigma_\mu dP dQ dS$, where $d\Sigma_\mu$ is the surface element for some space-like hypersurface Σ , and

$$\begin{aligned} dP &= 2 \theta(p_0) \delta(p^2 - m^2) \frac{d^4 p}{(2\pi\hbar)^3} = \frac{d^3 p}{(2\pi\hbar)^3 E} \Big|_{p_0=E=\sqrt{\vec{p}^2+m^2}} \\ dQ &= \delta(Q^a Q_a - q^2) \delta(d_{abc} Q^a Q^b Q^c - \vec{q}^3) d^8 Q \\ dS &= \delta(p_\mu S^\mu) \delta(S^\mu S_\mu + \vec{s}^2) d^8 S , \end{aligned} \quad (2.8)$$

with δ -functions fixing the mass-shell and normalization constraints for p^μ , Q^a , and S^μ .

The probability to find a classical particle at a given point in this phase space is given by the 1-particle distribution function $f(x, p, Q, S)$. It has to be a Lorentz scalar and gauge invariant. Since under (infinitesimal) gauge transformations,

$$\begin{aligned} A_\mu^a(x) &\rightarrow A_\mu^a(x) + \partial_\mu \epsilon^a(x) - \frac{g}{\hbar c} f^{abc} \epsilon^b(x) A_\mu^c(x) , \\ F_{\mu\nu}^a(x) &\rightarrow F_{\mu\nu}^a(x) - \frac{g}{\hbar c} f^{abc} \epsilon^b(x) F_{\mu\nu}^c(x) , \\ Q^a &\rightarrow Q^a - \frac{g}{\hbar c} f^{abc} \epsilon^b(x) Q^c , \end{aligned} \quad (2.9)$$

also the equations (2.3), (2.6), and (2.7) are invariant (i.e. a gauge transformation on the color fields can be absorbed by an x -dependent transformation of the classical color coordinates Q which leaves the integration measure dQ invariant), a gauge and Lorentz invariant expression for the time evolution of $f(x, p, Q, S)$ is given by

$$m \frac{df}{d\tau} \equiv \left[p^\mu \frac{\partial f}{\partial x^\mu} + m \dot{p}^\mu \frac{\partial f}{\partial p^\mu} + m \dot{Q}^a \frac{\partial f}{\partial Q^a} + m \dot{S}^\mu \frac{\partial f}{\partial S^\mu} \right] = C(x, p, Q, S) , \quad (2.10)$$

where C is a collision term describing short-range 2-body collisions. Inserting eqs.(2.3) and (2.7) and leaving all spin effect aside (to keep the expressions manageable), we obtain the following, gauge invariant equation for the 1-particle distribution function of a plasma of classical colored particles [16,23]:

$$p^\mu \left[\partial_\mu - \frac{g}{c} Q_a F_{\mu\nu}^a(x) \partial_p^\nu - \frac{g}{c} f_{abc} A_\mu^b(x) Q^c \partial_Q^a \right] f(x, p, Q) = C(x, p, Q) . \quad (2.11)$$

If there are antiparticles involved, their distribution function $\bar{f}(x, p, Q)$ obeys a similar equation, with Q^a replaced by $-Q^a$ (i.e. the second term changes sign). These equations have to be closed by an equation for the mean color field A_μ which, of course, is the Yang-Mills equation

$$(D_\mu F^{\mu\nu})_a(x) = -\frac{g}{c} j_a^\nu(x) = \frac{g}{c} \int p^\nu Q_a [f_q(x, p, Q) - \bar{f}_{\bar{q}}(x, p, Q) + f_g(x, p, Q)] dP dQ , \quad (2.12)$$

where f_q , $\bar{f}_{\bar{q}}$, and f_g are the distribution functions for quarks, antiquarks, and gluons respectively.

Equations (2.11,2.12) together form the basis of a relativistic kinetic description for a plasma of colored particles. The mean field terms on the left hand side of eq.(2.11) generalize those known from the usual Vlasov equation for electromagnetic plasmas; however, in addition to the drift in momentum induced by the electric and magnetic fields (non-relativistically the combination $\vec{E} + \vec{v}/c \times \vec{B}$ occurs as the coefficient of the momentum derivative of f), there are now also drift terms in the color sector of phase space, due to the non-Abelian interaction between the colored particles and the potentials A_μ .

The collision term on the right hand side of eq.(2.11) couples the 1-particle distribution function to two-body correlations. So actually eqs.(2.11,2.12) generally do not close; closure can be obtained, however, by factorizing the two-body correlations into products of single-particle distribution functions (Boltzmann approximation). Without this approximation further kinetic equations are needed describing the evolution of the 2-body distribution function which then again couples to 3-body correlations, and so on. This BBGKY hierarchy [37] of coupled equations has not been constructed yet for the non-Abelian case; in principle it should emerge from the quantum mechanical formulation of the following chapters in the classical limit, but this has so far not been shown explicitly.

2.3 Color moment equations

From the kinetic equations (2.11) one can construct several infinite hierarchies of moment equations, by forming moments involving powers of the color vector Q^a , of the momentum vector p^μ , or both. The color moment equations prove useful later when comparing with the quantum mechanical formulation, since it turns out that the lowest color moments of $f(x, p, Q)$ can be identified with the classical limit of the color components of the Wigner function. The two lowest moments of the momentum operator formed with these color moments then lead to equations of motion for macroscopic entities, namely the space-time densities of energy-momentum, baryon number and color current, i.e. they yield a chromohydrodynamic description of the plasma [23,38]. Cutting off this hierarchy after the lowest two orders, of course, implies that one is throwing away information: one obtains a macroscopic rather than microscopic description of the plasma dynamics. It has been attempted to salvage with increasing accuracy the information contained in the kinetic description by taking into account higher and higher moments of the kinetic equations [39], cutting off the hierarchy at larger and larger orders in a semiclassically controlled way. Although these attempts are not based on our gauge invariant formulation of kinetic theory and eventually lead to very complicated equations, too, this approach may still in the future prove to be a practical alternative to directly solving the high-dimensional color kinetic equations.

For later reference we will now shortly review these color moment and chromohydrodynamical equations [23]. We define the color singlet, octet, etc. distribution functions as the following moments of $f(x, p, Q)$:

$$f(x, p) = \int f(x, p, Q) dQ ,$$

$$f_a(x, p) = \int Q_a f(x, p, Q) dQ ,$$

$$f_{ab}(x, p) = \int Q_a Q_b f(x, p, Q) dQ, \quad \text{etc.} \quad (2.13)$$

For these one obtains from eq.(2.11), by taking appropriate color moments,

$$\begin{aligned} p^\mu \partial_\mu f(x, p) &= \frac{g}{c} p^\mu F_{\mu\nu}^a(x) \partial_p^\nu f_a(x, p) + \int C(x, p, Q) dQ, \\ p^\mu \left[\partial_\mu \delta_{ac} + \frac{g}{c} f_{amc} A_\mu^m(x) \right] f_c(x, p) \\ &= \frac{g}{c} p^\mu F_{\mu\nu}^b(x) \partial_p^\nu f_{ab}(x, p) + \int Q_a C(x, p, Q) dQ, \\ p^\mu \left[\partial_\mu \delta_{ac} \delta_{bd} + \frac{g}{c} (\delta_{ac} f_{bmd} + \delta_{bd} f_{amc}) A_\mu^m(x) \right] f_{cd}(x, p) \\ &= \frac{g}{c} p^\mu F_{\mu\nu}^c(x) \partial_p^\nu f_{abc}(x, p) + \int Q_a Q_b C(x, p, Q) dQ, \quad \text{etc.} \end{aligned} \quad (2.14)$$

Classically, all these moments are independent. Quantum mechanically, the color charges Q_a do not commute, and the color algebra between them [40] allows in the case of quarks (where $Q_a \leftrightarrow -\frac{\lambda_a}{2}$) to express the second color moment f_{ab} in terms of f and f_a [16]:

$$f_{ab} = \frac{\delta_{ab}}{6} f - \frac{1}{2} d_{abc} f_c. \quad (2.15)$$

Hence for quarks the color hierarchy (2.14) can be truncated by hand by imposing eq.(2.15) even on the classical level and rewriting the second of eqs.(2.14) as

$$\begin{aligned} p^\mu \left[\delta_{ac} \partial_\mu + \frac{g}{c} f_{amc} A_\mu^m(x) + \frac{g}{2c} d_{amc} F_{\mu\nu}^m(x) \partial_p^\nu \right] f_c(x, p) \\ = \frac{g}{6c} p^\mu F_{\mu\nu}^a(x) \partial_p^\nu f(x, p) + \int Q_a C(x, p, Q) dQ. \end{aligned} \quad (2.16)$$

In the following chapters we will show that the first eq.(2.14) and eq.(2.16) are exactly reproduced in the semiclassical limit of the Wigner function formulation.

For gluons which are in the adjoint representation ($Q_a \leftrightarrow T_a$ with $(T_a)_{bc} = i f_{bac}$) eq.(2.15) is not applicable, and the color hierarchy can only be closed after the second order (see Chap. 4).

2.4 Chromo-hydrodynamical equations for a classical colored fluid

We will close this chapter with a short summary of the lowest momentum moments, i.e. the chromo-hydrodynamical equations. Baryon number, energy-momentum, and color conservation in 2-body collisions result in the vanishing of the following moments of the

collision term $C(x, p, Q)$ in eq.(2.11) [23]:

$$\begin{aligned} \int [C_q - \bar{C}_{\bar{q}}] dP dQ &= 0 & (\text{baryon number conservation}) , \\ \int Q_a [C_q - \bar{C}_{\bar{q}} + C_g] dP dQ &= 0 & (\text{color conservation}) , \\ \int p^\mu [C_q + \bar{C}_{\bar{q}} + C_g] dP dQ &= 0 & (\text{energy-momentum conservation}) . \end{aligned} \quad (2.17)$$

Here, in obvious notation, C_q , $\bar{C}_{\bar{q}}$ and C_g are the collision terms in the quark, antiquark, and gluon version of eq.(2.11).

Let us now define the baryon current,

$$b_\mu(x) = \int p_\mu (f_q(x, p, Q) - \bar{f}_{\bar{q}}(x, p, Q)) dP dQ = \int p_\mu (f_q(x, p) - \bar{f}_{\bar{q}}(x, p)) dP , \quad (2.18)$$

using in the second equality the color singlet moment $f(x, p)$ from eq.(2.13). The color current is given by

$$j_\nu^a(x) = - \int p_\nu (f_q^a(x, p) - \bar{f}_{\bar{q}}^a(x, p) + f_g^a(x, p)) dP , \quad (2.19)$$

which follows from eq.(2.12) using the color octet moment as defined in eq.(2.13). Finally, the energy-momentum tensor for quark-gluon matter is defined by

$$T_{mat}^{\mu\nu}(x) = \int p^\mu p^\nu (f_q(x, p) + \bar{f}_{\bar{q}}(x, p) + f_g(x, p)) dP . \quad (2.20)$$

Then, it is straightforward [23] to derive from eqs.(2.11,2.17) the following generalized “chromo-hydrodynamical” equations:

$$\partial_\mu b^\mu(x) = 0 , \quad (2.21)$$

$$(D_\mu j^\mu)_a(x) = \partial_\mu j_a^\mu(x) + \frac{g}{c} f_{abc} A_\mu^b(x) j_c^\mu(x) = 0 , \quad (2.22)$$

$$\partial_\mu T_{mat}^{\mu\nu}(x) = \frac{g}{c} j_\mu^a(x) F_a^{\mu\nu}(x) = -\partial_\mu T_{field}^{\mu\nu}(x) , \quad (2.23)$$

supplemented by the Yang-Mills equation,

$$(D_\mu F^{\mu\nu})_a(x) = \partial_\mu F_a^{\mu\nu}(x) + \frac{g}{c} f_{abc} A_\mu^b(x) F_c^{\mu\nu}(x) = -\frac{g}{c} j_a^\nu(x) . \quad (2.24)$$

In the second equality in eq.(2.23) the definition of the energy-momentum tensor for the mean field $F_{\mu\nu}$,

$$T_{field}^{\mu\nu} = F_a^{\mu\alpha} F_\alpha^{a\nu} + \frac{1}{4} g^{\mu\nu} (F_a^{\alpha\beta} F_{\alpha\beta}^a) , \quad (2.25)$$

was used together with eq.(2.24).

Equations (2.21-2.23) still have to be supplemented by a decomposition of the currents and the energy-momentum tensor into local densities and a local hydrodynamic flow velocity. This is carried out in detail in refs.[23,26], where it is shown how the usual ideal fluid decomposition gives rise to the equations of ideal chromo-hydrodynamics, while an expansion around local equilibrium leads to equations for viscous colored fluids and defines dissipative effects through transport coefficients for heat and color conduction and for shear and bulk viscosity.

An important step in understanding the structure of the quantum kinetic equations studied in the following two chapters will be to derive analogous equations to (2.17-2.24) from the quark and gluon Wigner functions. The most interesting and most difficult part will consist in extracting information on the collision terms where, as already mentioned, much work remains to be done.

3. QUARK TRANSPORT EQUATIONS

3.1 Preliminaries

Before we embark upon the derivation of quantum transport equations for the quark and gluon Wigner operators, it seems worthwhile to recall how one would derive transport equations for a classical plasma and a non-relativistic field theory respectively. It will serve us as an illustration of how to employ the relevant particle/field equations of motion in order to obtain those for the phase space distributions. Thus, for the interesting cases of quarks and gluons later on we will only briefly indicate the derivations and refer the reader to the original references for all details [17,19,20,41].

Classical Transport Theory

A classical plasma is characterized by the phase space density, $f(x, p, t) = \sum_i \delta(x - x_i(t))\delta(p - p_i(t))$, depending on three-vectors of position and momentum, the corresponding particle coordinates, and implicitly also on time.

Note that throughout this Sec. 3.1 x, p denote the *three - vectors* of position and momentum, in distinction to the four-vector notation used elsewhere in this paper.

The classical trajectories $(x_i(t), p_i(t))$ obey

$$\dot{x}_i = p_i/m, \quad \dot{p}_i = F_{ext}(x_i) + F_2(x_i), \quad (3.1)$$

where F_{ext} denotes a possible external force and $F_2(y) = \int d^3x d^3p \nabla_x V(y - x) f(x, p, t)$ is the selfconsistent force, which is a functional of the phase space density. From eq.(3.1) we see that

$$\dot{f} = \sum_i (\dot{x}_i \nabla_{x_i} + \dot{p}_i \nabla_{p_i}) \delta(x - x_i(t)) \delta(p - p_i(t)) = -\left(\frac{p}{m} \cdot \nabla_x + F(x) \cdot \nabla_p\right) f(x, p, t). \quad (3.2)$$

Taking the ensemble average of this Klimontovich equation leads to the Vlasov-Boltzmann equation:

$$(md/dt + p \cdot \nabla_x + m\langle F(x) \rangle \cdot \nabla_p) \langle f(x, p, t) \rangle = C(\langle f \rangle), \quad (3.3)$$

where the mean force is given by $F_{ext} + F_2$ with f replaced by its ensemble average, $\langle f \rangle$, and where the collision term can be extracted [42] from the correlation term

$$C(f) = -m \int d^3x' d^3p' \nabla_{x'} V(x - x') \nabla_p (\langle f(x, p, t) f(x', p', t) \rangle - \langle f(x, p, t) \rangle \langle f(x', p', t) \rangle). \quad (3.4)$$

Neglecting two-body correlations in the plasma is equivalent to neglecting the collision term. In that case eq.(3.3) reduces to the Vlasov equation which applies only to "collisionless" plasmas. What makes the collision term particularly difficult to calculate in QED is the infinite Coulomb cross section. Medium polarization effects that screen long-range fields must then be taken into account. In addition a physical assumption must be made regarding different relaxation time scales for correlation functions (the BBGKY hierarchy). This procedure leads in the quasi-linear approximation to the Balescu-Lenard form for the collision term involving the dielectric function.

Quantum Transport Theory

The quantum mechanical analogue of the phase space density is the Wigner function [43]. We know that $\psi^\dagger(x)\psi(x)$ is the density of particles in coordinate space. Also $\psi^\dagger(p)\psi(p)$ is the density in momentum space. A natural candidate for the density in (x, p) phase space then is

$$W(x, p) = \psi^\dagger(x)\delta(p - \hat{p})\psi(x) , \quad (3.5)$$

where $\hat{p}$ is the momentum operator. The delta function acts as a projector onto momentum space and is defined by the Fourier transform as

$$W(x, p) = \int \frac{d^3y}{(2\pi\hbar)^3} e^{ipy/\hbar} \psi^\dagger(x) e^{-iy\hat{p}/\hbar} \psi(x) . \quad (3.6)$$

Recall that $\exp(iy\hat{p}/\hbar)$ generates a translation by y when acting to the right or $-y$ when acting to the left. In order to make W hermitian we will make it act halfway to the left and halfway to the right by representing $\hat{p}$ as $\frac{1}{2}i\hbar(\partial_x - \partial_x^\dagger)$. In this way we recover the familiar expression for the Wigner function,

$$W(x, p) = \int \frac{d^3y}{(2\pi\hbar)^3} e^{ipy/\hbar} \psi^\dagger(x + \frac{1}{2}y) \psi(x - \frac{1}{2}y) , \quad (3.7)$$

in terms of a mixed Fourier transform. The second-quantized Wigner operator, $\hat{W}$, is then obtained by replacing $\psi(x)$ by the Heisenberg field operator $\hat{\psi}(x)$. The advantage of second quantization is of course that it allows us to pass from a single-particle to a many-body quantum theory. The ensemble average of $\hat{W}$ is what corresponds to the classical phase space density.

The equation of motion for $\hat{W}$ obviously follows from the Schrödinger equation [43], $[-i\hbar\partial_t - \hbar^2\nabla^2/2m + \hat{U}(x)]\hat{\psi}(x) = 0$. To proceed we compute $p \cdot \nabla \hat{W}(x, p)$. Within the integration, p can be replaced by $i\hbar\nabla_y$ after integrating by parts. Then, with $\nabla_y \cdot \nabla_x = \frac{1}{2}(\nabla_{x+y/2}^2 - \nabla_{x-y/2}^2)$, we obtain, using the Schrödinger equation,

$$p \cdot \nabla_x \hat{W} = m \int \frac{d^3y}{(2\pi\hbar)^3} e^{ipy/\hbar} (-\partial_t + \frac{i}{\hbar} [\hat{U}(x + \frac{1}{2}y) - \hat{U}(x - \frac{1}{2}y)]) \hat{\psi}^\dagger(x + \frac{1}{2}y) \hat{\psi}(x - \frac{1}{2}y) . \quad (3.8)$$

Expanding in a Taylor series, $U(x + \frac{1}{2}y) = \exp(\frac{1}{2}y \nabla_x)U(x)$, we can pull the exponential out of the integral by replacing y by $-i\hbar \nabla_p$ term by term in the series. With this trick, the right hand side of eq.(3.8) can be written formally as

$$m(-\partial_t + \frac{2}{\hbar}[\sin(\frac{1}{2}\hbar\Delta)\hat{U}])\hat{W} ,$$

where we introduced [17] the “triangle” operator, $\Delta \equiv \nabla_x \cdot \nabla_p$ with ∇_x only acting on the potential and ∇_p only acting on the Wigner operator. In this way we finally obtain the non-relativistic quantum transport equation:

$$(m\partial_t + p \cdot \nabla_x - \nabla_x \hat{U} \cdot \nabla_p)\hat{W}(x, p) = Q(\hbar\Delta)\hat{U}(x)\hat{W}(x, p) , \quad (3.9)$$

where the particular “quantum” operator is given by

$$Q(\hbar\Delta) = \frac{2}{\hbar} \sin(\frac{1}{2}\hbar\Delta) - \Delta = -\frac{\hbar^2}{24}\Delta^3 + O(\hbar^4) . \quad (3.10)$$

A crucial step now consists here in going from this operator equation, which still contains all of the Schrödinger field theory, over to an equation for the Wigner function. For this purpose one has to take a suitable ensemble average depending on the physical situation under investigation (e.g. vacuum, colliding nuclei, thermal equilibrium). This feature always has to be kept in mind in the following, where it will be specified in more detail when needed.

Then, formally in the limit $\hbar \rightarrow 0$, we recover the Vlasov-Boltzmann equation. The Vlasov (“mean-field”) part follows from approximating the ensemble average of $\hat{U}\hat{W}$ on the l.h.s. by $\langle \hat{U} \rangle \langle \hat{W} \rangle$. The collision term is contained in the correlation function $C(x, p) = \Delta(\langle \hat{U}\hat{W} \rangle - \langle \hat{U} \rangle \langle \hat{W} \rangle)$ just as in the classical case. The terms on the right hand side of eq.(3.9) correspond to genuine quantum corrections. It is important, however, to remember that quantum corrections also occur in the collision term, C , since correlation functions are affected by quantum fluctuations as well as the mean-field dynamics. In ref.[44] it was shown that quantum corrections influence the collision terms if the density is so high that the two-body scattering rate approaches $\hbar/\Omega$, where Ω is the characteristic single-particle energy in the plasma.

The importance of quantum corrections in the collisionless domain depends on the magnitude of the dimensionless ratio $\hbar\Delta \sim \hbar/(\Delta R_U \Delta P_W)$, where ΔR_U is the characteristic *spatial* scale of variation of the potential and where ΔP_W is the characteristic *momentum* scale over which $W(x, p)$ varies appreciably. Classical transport theory thus applies only for relatively slowly varying potentials and phase space densities.

3.2 The relativistic quark Wigner operator

Assuming that a quark-gluon plasma eventually can be generated in ultra-relativistic nuclear collisions (apart from its transient existence in the early universe) [14,15], we encounter the problem to determine its formation time and describe its evolution towards chemical and thermal equilibrium. For that purpose it is necessary to derive transport equations for quarks and gluons that can be applied even if equilibrium is never achieved. While there has been a considerable amount of work on deriving relativistic Abelian transport equations in the recent past [7],[9]-[13],[18,41,45], work on gauge covariant non-Abelian transport equations [16,34,46] has only even more recently been begun. In the following we present and study the constraint and proper transport equations for the relativistic gauge covariant Wigner operator for fermions interacting via $SU(N)$ gauge fields as developed in ref.[17].

Some of the more formal questions to be answered are: What restrictions are imposed by gauge invariance or covariance respectively? How does spin enter into the transport equations? Is there a meaningful semiclassical limit of transport theory?

Usually, the operator that is expected to have the closest connection with the classical distribution functions is the Wigner operator [7,12,13]

$$\hat{W}(x, p) \equiv \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \bar{\psi}(x + \frac{1}{2}y) \otimes \psi(x - \frac{1}{2}y) = \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \bar{\psi}(x) e^{\frac{1}{2}y \cdot \partial_x^\dagger} \otimes e^{-\frac{1}{2}y \cdot \partial_x} \psi(x), \quad (3.11)$$

where $\partial_x^\dagger \equiv \overleftarrow{\partial}/\partial x^\mu$ and $\partial_x \equiv \overrightarrow{\partial}/\partial x^\mu$ are the generators of translations acting to the left and to the right respectively. Unfortunately, the above definition cannot be correct for a gauge theory, since $\hat{W}$ does not transform covariantly under a gauge transformation. A gauge covariant definition can, however, be constructed by substituting the covariant derivative, $D^\mu \equiv \partial^\mu - igA^\mu$, and its adjoint in place of ∂^μ and its adjoint. Applying this minimal substitution rule leads to the following definition of the relativistic gauge covariant Wigner operator for spin-1/2 particles:

$$\hat{W}(x, p) \equiv \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \bar{\psi}(x) e^{\frac{1}{2}y \cdot D_x^\dagger} \otimes e^{-\frac{1}{2}y \cdot D_x} \psi(x). \quad (3.12)$$

The tensor product in eq.(3.12) implies that $\hat{W}$ is a matrix in spinor (4×4 components) as well as in color space ($N \times N$ components). Explicitly, concerning the color indices, $W_{ij} \sim \bar{\psi}_j \psi_i$ (and similarly for the spin indices), which will be used together with ordinary matrix multiplication. Under a local gauge transformation, $S(x) \equiv \exp i\theta_a(x)t_a$, $\psi(x) \longrightarrow S(x)\psi(x)$ and $D^\mu \rightarrow S(x)D^\mu S^{-1}$, and the Wigner operator, eq.(3.12), transforms covariantly:

$$\hat{W}(x, p) \longrightarrow S(x)\hat{W}(x, p)S^{-1}(x). \quad (3.13)$$

We shall see below how this definition of $\hat{W}$ is related to that proposed in ref.[16]. Note that the covariant derivative is the operator that corresponds to the ordinary *kinetic* momentum, $\hat{\pi}^\mu \equiv P^\mu + gA^\mu = iD^\mu$, where $P^\mu = i\partial_x^\mu$ is the canonical momentum conjugate to the position coordinate. The on-shell condition $\hat{\pi} \cdot \hat{\pi} = m^2$ applies to the kinetic rather than the conjugate momentum in the classical limit. Therefore, it is natural that the covariant rather than the ordinary derivative appears in the definition of the Wigner function in eq.(3.12).

Recalling that the field strength tensor in general is an $N \times N$ matrix in color space (cf. Sec. 1.2), $F_{\mu\nu} \equiv [D_\mu, D_\nu]/(-ig)$, which obeys the field equation $[D_\mu, F^{\mu\nu}] = -gj^\nu$, we note that the quark color current operator, eq.(1.7), can now be expressed as

$$j^\mu = \int d^4p \, t_a \, \text{Tr} \, \gamma^\mu t_a \hat{W}(x, p) , \quad (3.14)$$

where the trace refers to spinor and color indices. In the following we set $\hbar = c = 1$, but reinstate ordinary units whenever appropriate.

Under a local gauge transformation S not only D_μ and $F^{\mu\nu}$ transform covariantly but also j^μ , i.e., $j \rightarrow S(x)jS^{-1}(x)$, because $t_a \cdot \text{Tr}(t_a S \hat{W} S^{-1}) = S t_a S^{-1} \cdot \text{Tr}(t_a \hat{W})$. Similarly, other observables of the quark field can be expressed in terms of $\hat{W}$ [13,17].

Next, we return to some technical features which become particularly important in the context of our non-Abelian transport theory. Using eqs.(1.11-1.13), we can replace the covariant translation operators in eq.(3.12) (i.e. the exponential of a covariant derivative) in terms of link operators U . Thus, we obtain the gauge covariant Wigner operator in the form

$$\hat{W}(x, p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \bar{\psi}(x + \frac{1}{2}y) U(x + \frac{1}{2}y, x) \otimes U(x, x - \frac{1}{2}y) \psi(x - \frac{1}{2}y) . \quad (3.15)$$

The advantage of the definition, eq.(3.12), is that the path defining $U(b, a)$ automatically turns out to be the straight line, cf. eqs.(1.11-1.13). Starting with eq.(3.15) instead raises the question of the choice of the path. A unique definition of the path is important, however, since the value of $U(b, a)$ depends on the path except in the trivial case, when $A^\mu = S\partial^\mu S^{-1}$ is a pure gauge field with $F^{\mu\nu} = 0$. Therefore, although $\hat{W}$ as given in eq.(3.15) transforms covariantly for any choice of path, owing to the peculiar transformation property of link operators [32,33], see eq.(1.14), the path ambiguity must be removed. This is achieved by requiring that the variable p in $\hat{W}(x, p)$ corresponds to the kinetic momentum and the kinetic momentum operator in general. That requirement is what led to the definition of $\hat{W}$ given in eq.(3.12). (Note, however, that in the classical limit to be discussed below the path ambiguity drops out of the equations of motion for the corresponding Wigner function [16].) It was shown in ref.[17] that for any other choice of the path eq.(1.11) does not hold, and hence the interpretation of the phase space coordinate p as the kinetic momentum could not hold either.

For the derivation of the equation of motion for $\hat{W}$ in the following section one needs derivatives of a link operator with respect to the endpoints of the associated path. The relevant formulae were collected in eqs.(1.20,1.21) in Sec. 1.2. Higher order derivatives can be calculated by repeated application of these formulae. We also refer the reader to ref.[17] for more details about link operators.

3.3 Derivation of quark transport equations

The Wigner operator as given by eqs.(3.12,3.15) is related to a gauge covariant density matrix operator,

$$\hat{\rho}(x + \frac{y}{2}, x - \frac{y}{2}) \equiv \bar{\psi}(x + \frac{y}{2})U(\frac{y}{2}, x) \otimes U(x, x - \frac{y}{2})\psi(x - \frac{y}{2}) , \quad (3.16)$$

via a Fourier transform,

$$\hat{W}(x, p) = \int \frac{d^4 y}{(2\pi)^4} e^{-ipy} \hat{\rho}(x + \frac{y}{2}, x - \frac{y}{2}) . \quad (3.17)$$

The equation of motion for the density matrix follows from Dirac's equation with

$$\bar{\psi}(x_2)U(x_2, x) \otimes U(x, x_1) (i\gamma^\mu D_\mu(x_1) - m) \psi(x_1) = 0 , \quad (3.18)$$

where $x_1 \equiv x - \frac{1}{2}y$ and $x_2 \equiv x + \frac{1}{2}y$. This will become the so-called *constraint equation* to be interpreted later on. A similar one could be obtained, of course, by applying the adjoint Dirac's equation to the left. However, we will find it convenient to consider also a linear combination of the quadratic Dirac equation and its adjoint instead:

$$\begin{aligned} & \bar{\psi}(x_2)U(x_2, x) \otimes U(x, x_1) \left(D_\mu(x_1)D^\mu(x_1) - \frac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu}(x_1) \right) \psi(x_1) \\ & - \bar{\psi}(x_2) \left(D_\mu^\dagger(x_2)D^\mu(x_2) - \frac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu}(x_2) \right) U(x_2, x) \otimes U(x, x_1)\psi(x_1) = 0 . \end{aligned} \quad (3.19)$$

While the equation resulting from eq.(3.19) is not independent of the constraint equation deduced from eq.(3.18), we will see that it corresponds more closely to the form of a proper *transport equation* that we expect for the Wigner operator. In order to convert eq.(3.18) and eq.(3.19) into equations for the Wigner operator, the tedious task of moving the derivative operators to the left or right in these equations (i.e. commuting them with link operators) has to be performed.

For completeness we define the product of any operator $\hat{O}$ times $\hat{\rho}$ by

$$\begin{aligned} \hat{O}\hat{\rho}(x_2, x_1) & \equiv \bar{\psi}(x_2)U(x_2, x) \otimes \hat{O}U(x, x_1)\psi(x_1) , \\ \hat{\rho}(x_2, x_1)\hat{O} & \equiv \bar{\psi}(x_2)U(x_2, x)\hat{O} \otimes U(x, x_1)\psi(x_1) . \end{aligned} \quad (3.20)$$

We remark here that this definition is quite essential for keeping track of the operator ordering, since for interacting quantum fields e.g. $[(F_{\mu\nu})_{ab}, U_{cd}] \neq 0$. For the special case of a purely classical external field ordinary matrix multiplication rules suffice, since we must only keep track of the sequence of color indices, while the order of matrix elements is irrelevant. In general those matrices are field operators and their ordering is important.

The Constraint Linear Equation

Starting from eq.(3.18) and following quite similar steps as were illustrated for the simple Schrödinger field theory case in Sec. 3.1, the final result for the quark Wigner operator equation can be derived [17]:

$$\begin{aligned} & \left(\gamma^\mu p_\mu - m + \frac{1}{2} i \gamma^\mu \mathcal{D}_\mu(x) \right) \hat{W}(x, p) \\ &= \frac{1}{2} i g \partial_p^\nu \gamma^\mu \left(\int_0^1 ds \frac{1}{2} (1+s) [e^{-\frac{1}{2} i s \partial_r \cdot \mathcal{D}(x)} F_{\nu\mu}(x)] \hat{W}(x, p) \right. \\ & \quad \left. + \hat{W}(x, p) \int_0^1 ds \frac{1}{2} (1-s) [e^{\frac{1}{2} i s \partial_r \cdot \mathcal{D}(x)} F_{\nu\mu}(x)] \right) , \end{aligned} \quad (3.21)$$

where ∂_p always acts on $\hat{W}$ and $\mathcal{D}$ within [...]. This is a *gauge covariant operator equation*. Its semiclassical limit together with the proper transport equation will be discussed later on.

Constant Abelian Field Case

First, however, in order to gain some understanding of the structure of the above equation, we consider the special case of a *constant Abelian field* $F^{\mu\nu}$. In that case $\mathcal{D}_\lambda F_{\mu\nu} = 0$, and thus eq.(3.21) reduces to

$$(\gamma \cdot K - m) \hat{W}(x, p) = 0 , \quad (3.22)$$

where

$$K^\mu \equiv p^\mu + \frac{1}{2} i (\partial_x^\mu + g F^\mu{}_\nu \partial_p^\nu) . \quad (3.23)$$

Multiplying eq.(3.22) by $(\gamma \cdot K + m)$ gives rise to a *complex* equation:

$$\left(p^2 - m^2 - \frac{1}{4} (\partial_x + g F \cdot \partial_p)^2 + i(p \cdot \partial_x + g p \cdot F \cdot \partial_p) + \frac{1}{2} g \sigma^{\mu\nu} F_{\mu\nu} \right) \hat{W}(x, p) = 0 . \quad (3.24)$$

We can isolate the Hermitian or anti-Hermitian part of the equation by adding or subtracting its adjoint. This leads to the following *two* equations:

$$\left(p^2 - m^2 - \frac{1}{4}(\partial_x + gF \cdot \partial_p)^2 \right) \hat{W}(x, p) + \frac{1}{4}gF_{\mu\nu}\{\sigma^{\mu\nu}, \hat{W}(x, p)\} = 0 \quad , \quad (3.25)$$

$$(p \cdot \partial_x + gp \cdot F \cdot \partial_p) \hat{W}(x, p) - \frac{1}{4}igF_{\mu\nu}[\sigma^{\mu\nu}, \hat{W}(x, p)] = 0 \quad , \quad (3.26)$$

where we used the property $\hat{W}^\dagger = \gamma^0 \hat{W} \gamma^0$.

The first equation generalizes the classical mass-shell condition and may be called a proper *constraint equation*. For a study of its consequences in a gauge-non-covariant formulation of the usual Wigner function approach see ref.[13]. The second equation generalizes the Vlasov equation to include spin-dependent effects, and thus may be called a proper *transport equation*. From eqs.(3.25, 3.26) one sees directly that p plays the role of a kinetic momentum variable, as we had intended by the construction of eq.(3.12).

We remark that the Abelian transport theory for QED with arbitrary external fields was studied thoroughly in ref.[41]. Particular attention was paid to the resulting spin structure of the equations. The decomposition into scalar, pseudo-scalar, vector etc. components, which was developed there, can be applied in the present case as well. This allows important simplifications to be made systematically, which seems important for initial numerical studies of the present transport theory (see also the Appendix for the corresponding nuclear matter problem).

The Proper Quark Transport Equation

We conclude from the preceding subsection that eq.(3.21) contains both a constraint and a transport equation. In the general case it is, however, not so simple to uncover the transport equation from it. In effect, the transport equation followed by applying the Dirac operator twice. Thus, to obtain the general transport equation directly, it is more convenient to start with the quadratic Dirac equation in the form of eq.(3.19).

Using the same technique and steps as above, which were employed in the derivation of the general constraint equation (3.21), we now turn to the much more tedious derivation of a proper transport equation for the Wigner operator.

Starting from eq.(3.19) one again moves the derivatives out of the respective pairs of field operators, $\bar{\psi}(x_2) \cdots \partial_{x_1, x_2} \cdots \psi(x_1) = \partial_{x_1, x_2} \bar{\psi}(x_2) \cdots \psi(x_1) + \text{corrections}$ etc., until one finally obtains an equation which can be rewritten in terms of the density matrix $\hat{\rho}$, eq.(3.16), and, finally, in terms of the Wigner operator. The main difficulty is that about an order of magnitude more terms arise due to the necessity of calculating second derivatives of the link operators. The result of the lengthy calculation is [17]:

$$\begin{aligned}
p \cdot \mathcal{D}(x) \hat{W}(x, p) = & \\
& - \frac{1}{2} \frac{g}{c} p^\mu \partial_p^\nu \int_0^1 ds \left\{ [e^{-s\Delta} F_{\mu\nu}(x)] \hat{W}(x, p) + \hat{W}(x, p) [e^{s\Delta} F_{\mu\nu}(x)] \right\} \\
& + \frac{1}{2} i \frac{1}{\hbar} \frac{g}{c} \left\{ S^{\mu\nu} [e^{-\Delta} F_{\mu\nu}(x)] \hat{W}(x, p) - \hat{W}(x, p) [e^{\Delta} F_{\mu\nu}(x)] S^{\mu\nu} \right\} \\
& + \frac{1}{4} i \hbar \frac{g}{c} \partial_p^\nu \int_0^1 ds s \left\{ [e^{-s\Delta} F_{\nu\mu}(x)] \mathcal{D}^\mu(x) \hat{W}(x, p) - \mathcal{D}^\mu(x) \hat{W}(x, p) [e^{s\Delta} F_{\nu\mu}(x)] \right\} \\
& - \frac{1}{8} i \hbar \left(\frac{g}{c} \right)^2 \partial_p^\mu \partial_p^\nu \int_0^1 ds s \int_0^1 d\tilde{s} \\
& \quad \left\{ [e^{-s\Delta} F_{\mu\eta}(x)] \left([e^{-\tilde{s}\Delta} F_\nu^\eta(x)] \hat{W}(x, p) + \hat{W}(x, p) [e^{\tilde{s}\Delta} F_\nu^\eta(x)] \right) \right. \\
& \quad \left. - \left([e^{-\tilde{s}\Delta} F_\nu^\eta(x)] \hat{W}(x, p) + \hat{W}(x, p) [e^{\tilde{s}\Delta} F_\nu^\eta(x)] \right) [e^{s\Delta} F_{\mu\eta}(x)] \right\} ,
\end{aligned} \tag{3.27}$$

where the *triangle operator* is defined as $\Delta \equiv \frac{1}{2} i \hbar \partial_p \cdot \mathcal{D}(x)$ and $S_{\mu\nu} = \frac{\hbar}{2} \sigma_{\mu\nu}$ as before; [...] delimits where the $\mathcal{D}$ -derivative from Δ acts and ∂_p always acts on $\hat{W}$. Eq.(3.27) presents the exact *quantum transport equation for the QCD Wigner operator* defined in eq.(3.12). It is a *gauge covariant operator equation*. Note that the first two lines in eq.(3.27) have a structure similar to that of the classical Abelian Vlasov equation. We also remark that, in order to simplify the right hand side of this equation, we reparametrized paths [47] as compared to ref.[17].

In the next section we show how a classical non-Abelian Vlasov type equation can be derived from eq.(3.27) including a spin-dependent term and how to calculate quantum corrections systematically.

3.4 The semiclassical limit

In units where all quantities are measured in terms of a length scale L , $\hbar$ does not appear in the equations. However, we see from eqs.(3.21, 3.27) that $F^{\mu\nu}$ is always acted on by an operator of the form

$$\hat{O}(\Delta) = \int ds f(s) e^{g(s)\Delta} = \sum_{n=0}^{\infty} a_n \Delta^n , \tag{3.28}$$

where a_n are dimensionless constants and $\Delta \sim \partial_p \cdot \mathcal{D}$. If we choose ordinary classical units again for p in terms of some momentum scale, while retaining length units for x , then

obviously Δ has to be replaced by

$$\Delta \longrightarrow \hbar \Delta . \quad (3.29)$$

In the particular semiclassical limit to be discussed here, we therefore retain only the leading powers of Δ in eq.(3.28).

Carrying out this "triangle-expansion" in the constraint equation (3.21) leads to

$$\begin{aligned} (\gamma^\mu p_\mu - m) \hat{W}(x, p) = \\ - \frac{1}{2} i \gamma^\mu \mathcal{D}_\mu(x) \hat{W}(x, p) - \frac{1}{2} i g \partial_p^\mu \gamma^\nu (F_{\mu\nu}(x) \hat{W}(x, p) - \frac{1}{4} [F_{\mu\nu}(x), \hat{W}(x, p)]) \\ + O(\partial_p \cdot \Delta F \cdot \gamma \hat{W}) . \end{aligned} \quad (3.30)$$

In the case of an *external Abelian gauge potential* eq.(3.30) reduces to

$$\begin{aligned} (\gamma^\mu p_\mu - m) \hat{W}(x, p) = \\ - \frac{1}{2} i \gamma_\mu \partial_x^\mu \hat{W}(x, p) - \frac{1}{2} i g \partial_p^\mu \gamma^\nu F_{\mu\nu}(x) \hat{W}(x, p) + O(\partial_p \cdot \Delta F \cdot \gamma \hat{W}) , \end{aligned} \quad (3.31)$$

which for constant field strengths coincides with eq.(3.22).

We see that the on-shell condition, $p^2 = m^2$, can hold for quarks only, when the *covariant derivative* of $\hat{W}$ is small compared to p (in the reference frame of a specified ensemble) and the *field strength* is also small. By small we mean that the ensemble averaged terms in eq.(3.30) satisfy:

$$\langle p \hat{W} \rangle \gg \langle \mathcal{D} \hat{W} \rangle , \quad (3.32)$$

$$\langle p \hat{W} \rangle \gg \langle g F \partial_p \hat{W} \rangle . \quad (3.33)$$

These conditions constrain the possible ensembles for which a simple semiclassical picture may hold. Whether such ensembles exist and correspond to interesting physical situations cannot, of course, be guaranteed ahead of time. Note that for non-Abelian fields, the smallness of the covariant derivative means not only that the spatial gradients in the system are sufficiently small, but also that A and $\hat{W}$ approximately *commute* (!), i.e.

$$\langle p \hat{W} \rangle \gg \langle [g A, \hat{W}] \rangle . \quad (3.34)$$

When the above conditions are not satisfied, then significant off-shell corrections must be applied to the transport theory of quarks.

For covariant constant fields , satisfying

$$\mathcal{D}_\lambda F_{\mu\nu} = 0 , \quad (3.35)$$

the higher order corrections in ΔF vanish. However, in general $O(\Delta^n F)$ corrections must be calculated. Only for slowly varying fields, in the sense that

$$\langle F\hat{W} \rangle \gg \langle DF \partial_p \hat{W} \rangle , \quad (3.36)$$

can an expansion in powers of Δ possibly converge rapidly.

In the general case of strong or rapidly varying fields the full quantum equation, eq.(3.21), must be solved. This would be equivalent to solving the field equations, i.e. hopeless at this time. Only under the rather restrictive conditions, eqs.(3.32, 3.33, 3.36), can we expect that the transport theory for quarks reduces to a simpler, more manageable form.

Turning now to the transport equation, eq.(3.27), we obtain by the Δ expansion the *semiclassical transport equation for the QCD Wigner operator*:

$$\begin{aligned} p \cdot \mathcal{D}(x) \hat{W}(x, p) + \frac{1}{2} \frac{g}{c} \left\{ p^\mu F_{\mu\nu}(x), \partial_p^\nu \hat{W}(x, p) \right\} - \frac{1}{2} i \frac{g}{\hbar c} \left[S^{\mu\nu} F_{\mu\nu}(x), \hat{W}(x, p) \right] \\ = \frac{1}{8} i \hbar \frac{g}{c} \partial_p^\mu \left[F_{\mu\nu}(x), (\mathcal{D}^\nu(x) \hat{W}(x, p)) \right] - \frac{1}{16} i \hbar \left(\frac{g}{c} \right)^2 \partial_p^\mu \partial_p^\nu \left[F_\mu^\lambda(x) F_{\nu\lambda}(x), \hat{W}(x, p) \right] \\ + O(\Delta F \hat{W}) \end{aligned} \quad (3.37)$$

For example, the lowest order corrections to the Vlasov and spin terms on the l.h.s. of eq.(3.37) appearing among the $O(\Delta F \hat{W})$ terms are given by

$$- \frac{1}{8} i \hbar \frac{g}{c} p^\mu \partial_p^\nu \partial_p^\lambda \left[(\mathcal{D}_\lambda(x) F_{\nu\mu}(x)), \hat{W}(x, p) \right] + \frac{1}{4} \frac{g}{c} \left\{ (\mathcal{D}_\nu(x) S^{\alpha\beta} F_{\alpha\beta}(x)), \partial_p^\nu \hat{W}(x, p) \right\} \quad (3.38)$$

Eq.(3.37) clearly generalizes Vlasov's equation in the following sense: i) it is still an operator equation; ii) the first and second term on the l.h.s. of eq.(3.37) present the usual combination of phase space variables and derivatives, however, modified by (anti)commutators, which can be simplified for the external Abelian field problem (cf. Sec. 3.3); iii) explicit spin-dependent corrections arise; iv) quantum corrections can be systematically calculated via expansion of eq.(3.27) in powers of the Δ operator.

Note that, however, the "triangle expansion" as applied to the full quantum transport equation is not completely identical to an expansion in powers of $\hbar$! The structure of eq.(3.27) implies terms of three different orders in $\hbar$ at a given order of expansion in Δ . Nevertheless, as we discussed above, it is the assumed smallness of products of typical length and momentum scales as compared to a fixed, physical $\hbar$ and of certain commutators that justifies this expansion.

On the other hand, the classical kinetic equations of Chap. 2 originate from eq.(3.27) by a "serious" $\hbar$ -expansion, i.e. by dropping in eq.(3.37) all terms of order $\hbar$ (although they stem from the zeroth order of the Δ -expansion), but keeping the second term of (3.38)

(although it is of order Δ^1). The dangerous-looking third term of eq.(3.37) is actually only of order $\hbar^0$ by the definition of the spin tensor, $S \sim \hbar\sigma$ (cf. Sec. 1.2). - If the second term in eq.(3.37) is combined with the second term of (3.38), it exhibits a Vlasov mean-field term (corresponding to the generalized Lorentz force law) as in eq.(2.7).

We remark here that the classical transport for spinless quarks given in ref.[16] corresponds to the sum of the first two terms on the l.h.s. of eq.(3.37) being set equal to zero and decomposing them in color space. Furthermore, adding the corrections from (3.38) to the r.h.s. of eq.(3.37) and dropping all $S^{\mu\nu}$ -dependent terms reproduces the result for scalar quarks obtained in ref.[34] apart from a sign error there.

As a final limiting form, one may consider the transport equation which follows from eq.(3.27) for the case of an *external Abelian gauge potential* [17]. The resulting proper transport equation has the form of a Vlasov equation for the Wigner operator including spin-dependent quantum corrections. It generalizes the result obtained by Remler [10] for the case of scalar QED to spinor QED and reduces to eq.(3.26) for constant field tensors. Furthermore, it can be shown [41] that the resulting equation allows a derivation of the Bargmann-Michel-Telegdi equation [36] (eq.(2.6) without color indices), which describes classically a relativistic spinning particle interacting with an external electromagnetic field. Somewhat related considerations for quarks interacting with classical color fields can be found in ref.[48]. For an interesting study (using a stochastic process) of the significance of the higher gradients, which appear in the semiclassical limit, cf. eq.(3.37), see ref.[49].

The expressions *classical* or *semiclassical* employed in this section mostly refer to the approximate expansion of operators in powers of Δ acting on F . To relate the operator equations to transport equations, expectation values of the equations with respect to a suitable ensemble must be calculated as we discuss in the next section.

3.5 The Abelian dominance approximation

The constraint and transport operator equations are completely equivalent to the original Dirac field equation, eq.(1.7). They provide, however, a systematic means to study the semiclassical limit of QCD as expressed in terms of the Wigner function $\langle \hat{W} \rangle$. The choice of the ensemble over which $\hat{W}$ is to be averaged is, of course, dictated by the particular physical situation under study. At present it is not known which field configurations are most relevant in actual physical processes such as nuclear collisions. This uncertainty is due to the unsolved confinement problem in QCD. To make further progress it is necessary to make an *Ansatz*, i.e. a bold and unjustified guess, about the characteristics of that ensemble. Following ref.[17], we proceed in the spirit of the MIT Bag model and the color flux tube models [50,51] and assume that it is sensible to talk about quarks, as if they

obeyed some effective plasma transport equation at least in some small volume, which provides a cut-off on the strong long-range interactions.

Above we observed that the transport equations can only be simplified if rather restrictive conditions, eqs.(3.32, 3.33, 3.36), apply for the relevant physical ensemble. As a final exercise we now show that at least in the context of the color flux tube models the reduction can be carried further. - We refer the reader to the articles by S. Mrówczyński and by A. Białas and W. Czyż in this volume for more information on phenomenological applications of the quark mean-color-field equations presently discussed as well as of the corresponding gluon transport equations given later on.

Here we consider an ensemble that corresponds to a quark plasma in a finite flux tube in which quarks are subject to a covariant constant or at most slowly varying mean field, $\langle F^{\mu\nu}(x) \rangle$. There exists a gauge where $\langle F^{\mu\nu}(x) \rangle$ is diagonal. Since every diagonal traceless $N \times N$ matrix can be expanded in terms of the $N - 1$ commuting generators, h_i , of $SU(N)$ defined by [50]

$$h_j \equiv (2j(j+1))^{-\frac{1}{2}} \text{diag}(1, \dots, 1, -j, 0, \dots, 0) , \quad (3.39)$$

with $-j$ appearing in the $j + 1$ column, we can always write

$$\langle F_{\mu\nu}(x) \rangle \equiv S(x) \vec{F}_{\mu\nu}(x) \cdot \vec{h} S^{-1}(x) , \quad (3.40)$$

where $S(x)$ is a particular gauge transformation. (For further details on the Cartan-Weyl basis related to this Abelian dominance approximation cf. Sec. 4.3.)

If we now make the *Ansatz* that the ensemble in the flux tube is such that $\hat{W}$ is diagonal in the *same gauge* where $\langle F_{\mu\nu}(x) \rangle$ is diagonal, then we can express

$$\langle \hat{W} \rangle \equiv S(x) \left(\sum_{j=1}^{N-1} W^j h_j + W^0 \mathbf{1} \right) S^{-1}(x) , \quad (3.41)$$

i.e. decompose $\langle \hat{W} \rangle$ in terms of N Wigner *functions* depending on x, p . We suppress presently irrelevant spinor indices, but a similar decomposition in spinor space in terms of the Clifford algebra can be performed [7,41]. Note that the property $\hat{W}^\dagger = \gamma^0 \hat{W} \gamma^0$ insures that the N diagonal elements in color space $\langle \hat{W} \rangle^{ii}$ are real numbers after taking appropriate traces in spinor space.

Eq.(3.41) is a strong model assumption. It is, however, plausible that for slowly varying fields it is satisfied if the initial conditions for the plasma are assumed to correspond to $\langle \hat{W} \rangle = 0$, as in flux tube models [31,50,51].

If eq.(3.41) holds, then it is obvious that the most convenient gauge to work in is the $S(x)$ gauge. In that gauge by assumption

$$\langle \hat{W} \rangle_{ij} = (\vec{W} \cdot \vec{h} + W^0 \mathbf{1})_{ij}$$

$$= \delta_{ij}(\vec{W} \cdot \vec{\epsilon}_j + W^0) \equiv \delta_{ij}f_j, \quad (3.42)$$

with the “charges” $\vec{\epsilon}_j$ given by

$$\vec{\epsilon}_j \equiv (\vec{h})_{jj} = ((h_1)_{jj}, \dots, (h_{N-1})_{jj}). \quad (3.43)$$

These are just the elementary weight vectors of $SU(N)$. Eq.(3.42) provides the model dependent relation between the Wigner operator and the classical quark distribution functions.

The semiclassical transport equations for the $f_j(x, p)$ in this model are obtained by taking the expectation value of eq.(3.37) in this ensemble. Using eqs.(3.40-3.43), the color structure of the semiclassical transport equation, eq.(3.37), is simplified considerably:

$$\begin{aligned} & (p \cdot \partial_x - g\vec{\epsilon}_j \cdot \vec{F}_{\mu\nu} p^\nu \partial_p^\mu) f_j(x, p) \\ &= \frac{1}{4} i g \vec{\epsilon}_j \cdot \vec{F}_{\mu\nu} [\sigma^{\mu\nu}, f_j(x, p)] + \tilde{C}_j(x, p), \end{aligned} \quad (3.44)$$

where $\tilde{C}_j$ represents correlation terms of the form

$$\tilde{C}_j(x, p) = \frac{1}{2} g p^\mu \partial_p^\nu \left(\langle \{F_{\nu\mu}, \hat{W}(x, p)\} \rangle - 2 \langle F_{\nu\mu} \rangle \langle \hat{W}(x, p) \rangle \right)_j + \dots \quad (3.45)$$

It is known [13] that the Boltzmann collision terms are contained in such correlations. However, for the quark-gluon plasma the derivation of Debye screened collision terms still has to be performed. We stop here with having shown how the Vlasov terms arise in a particular phenomenological model from the underlying quantum theory.

Note that all the non-Abelian commutator terms dropped out for the model *Ansatz*, eq.(3.41). We observe an effective coupling constant $g\vec{\epsilon}_j$ entering the above set of N Abelian-like equations for the components f_j of $\langle \hat{W} \rangle$ (cf. eq.(3.42)). In the absence of correlations the equations would simply decouple. This seems to be as close as one can get to the classical Vlasov equation starting from the Wigner operator defined in eq.(3.12). It is valid for fermions with positive or negative energy. For $p^0 < 0$ we obtain the equation for antiquarks by replacing p^μ by $-p^\mu$ everywhere, which effectively reverses the sign of g in eq.(3.44) as expected. We see the explicit spin-dependence as well. The important quantum corrections, especially pair production, are buried in the uncalculated correlation terms. However, first attempts have been made to calculate quark and gluon pair production by external color fields, which we represent in Chapter 5.

Similar conclusions can be reached for the electron-positron plasma [41], in which case the transport equation is obtained by replacing $\vec{\epsilon}_j \rightarrow 1$ and the above matrix transport equation becomes a single Vlasov equation including corrections.

4. GLUON TRANSPORT EQUATIONS

The gauge field was treated basically as an operator in the previous chapter, but earlier work and the above study of quark transport in particular were aiming at a mean-field description of the system. Thus, quarks are considered to interact with an external field or with each other via a self-consistently generated mean field (Hartree approximation). However, for non-Abelian gauge theories and especially in applications to the quark-gluon plasma this approximation is insufficient. Even if there were no quarks, one encounters the possibility here that gluons are spontaneously created from a classical long-range background field. Thus, there are gluons in addition to quarks which interact with that field and help to ultimately neutralize it [31,50,51].

Therefore, a proper kinetic description for the gluons was introduced in ref.[19], which allows to study the fluctuating part of the field separately from its mean-field behavior. Following refs.[19,20] below, we define a *gauge covariant Wigner operator for the gluon field* and derive its quantum transport equation. We discuss an appropriate semiclassical limit and show how a non-Abelian Vlasov type equation arises for gluon fluctuations as well as for quarks and, surprisingly, with a very similar structure.

Previous development of transport theory in this direction was hindered by the fact that no gauge covariant gluon (or photon) distribution function was known. Since gauge covariance, however, was a strong guiding principle in the derivation of the transport theory for quarks and provided insight in how to approximate the operator equations semiclassically, one keeps it rather than to study an object like the gluon number operator in a specified gauge. This motivated the particular definition of a gauge covariant gluon Wigner operator in eq.(4.1) below. Finally, to demonstrate the consistency of the resulting semiclassical equations, we present the linear response analysis from ref.[20] calculating the permeability function of a collisionless gluon plasma - the corresponding dispersion relations and static Debye screening length agree with those from perturbative one-loop QCD calculations for a hot equilibrium plasma. Similarly, the chromomagnetic behavior will be shortly reconsidered.

In analogy to the quark Wigner operator one defines the *gauge covariant Wigner operator for spin-1 gluons* [19] in terms of the field operators,

$$\hat{\Gamma}_{\mu\nu}(x, p) \equiv \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} [e^{\frac{1}{2} y \cdot \mathcal{D}(x)} F_{\mu}^{\lambda}(x)] \otimes [e^{-\frac{1}{2} y \cdot \mathcal{D}(x)} F_{\lambda\nu}(x)] , \quad (4.1)$$

with the covariant derivative of a second-rank tensor as defined in eq.(1.4). Under local gauge transformations F , $\mathcal{D}F$, and so $\hat{\Gamma}$ transform covariantly. In eq.(4.1) we suppressed four color indices of $\hat{\Gamma}$, $\hat{\Gamma}_{ijkl} \sim [\dots]_{ij}[\dots]_{kl}$ ($i, j, k, l = 1, \dots, N$), but explicitly indicated its

tensor structure. $\hat{\Gamma}$ is closely related to the energy-momentum tensor of the field,

$$\hat{T}_{\mu\nu}(x) \equiv \text{Tr} \left(F_\mu^\lambda F_{\lambda\nu} + \frac{1}{4} g_{\mu\nu} F_{\lambda\tau} F^{\lambda\tau} \right) = \text{Tr} \int d^4p \left(\hat{\Gamma}_{\mu\nu}(x, p) - \frac{1}{4} g_{\mu\nu} \hat{\Gamma}_\lambda^\lambda(x, p) \right) , \quad (4.2)$$

where the trace refers to color indices, $\text{Tr} A \otimes B \equiv A_{ij} B_{ji}$. Eq.(4.2) provides the connection between $\hat{\Gamma}$ and observables of the gauge field. Similarly as for the fermion Wigner operator [17,41] one may write $\text{Tr} \hat{\Gamma}_{\mu\nu}(x, p) = F_\mu^\lambda(x) \delta^4(p - i\mathcal{D}(x)) F_{\lambda\nu}(x)$, which resembles a classical phase-space distribution with $i\mathcal{D}$ appearing as the operator associated with the gluon kinetic momentum p .

Using eq.(1.16), with link operators U as before, cf. eq.(1.12) and the subsequent discussion in that section, we obtain from eq.(4.1)

$$\hat{\Gamma}_{\mu\nu}(x, p) \equiv \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} U(x, x_2) F_\mu^\lambda(x_2) U(x_2, x) \otimes U(x, x_1) F_{\lambda\nu}(x_1) U(x_1, x) , \quad (4.3)$$

where $x_2 \equiv x + \frac{1}{2}y$, $x_1 \equiv x - \frac{1}{2}y$. Note that $\hat{\Gamma}_{\mu\nu}^\dagger = \hat{\Gamma}_{\nu\mu}$, since the link operators are unitary. The implicit appearance of straight line paths in eq.(4.3) is related to the requirement that p have the physical interpretation of kinetic momentum as in the quark case before.

4.1 Derivation of the Wigner operator equation

The relevant properties of link operators were studied in detail above and were quite essential in the derivation of transport equations for the quark Wigner operator. The same strategy of how to derive a transport equation given an underlying field operator equation and using a rather definite set of formal manipulations was applied to the gluon case in refs.[19,20]. Keeping these preparatory remarks in mind, we sketch the derivation of the quantum transport equation for $\hat{\Gamma}$ proceeding in several steps in parallel to Chap.3:

1. We want to calculate $p \cdot \tilde{\mathcal{D}} \hat{\Gamma}_{\mu\nu}$, where $\tilde{\mathcal{D}}$ here is defined by ($\hat{\Gamma} \equiv A \otimes B$)

$$\tilde{\mathcal{D}}(A \otimes B) \equiv (\mathcal{D}A) \otimes B + A \otimes (\mathcal{D}B) , \quad (4.4)$$

since this is the gauge covariant generalization of $p \cdot \partial_x \hat{\Gamma}$, which would arise in the absence of interactions.

2. Representing $\hat{\Gamma}$ as in eq.(4.3), we carry $p \cdot \tilde{\mathcal{D}}$ into the integrand and eventually convert $p \rightarrow -i\partial_y$ (by partial integration) and $y \rightarrow i\partial_p$, if we want to pull ∂_p out of the integral.

3. We calculate all necessary derivatives of link operators which occur in the integrand by repeatedly applying the derivative formulae, eqs.(1.20-1.22). These are straightforward though tedious manipulations. To help keeping track of the ordering of terms, it is useful to define left and right commutators,

$$[\hat{\mathcal{O}}, A \otimes B]_L \equiv [\hat{\mathcal{O}}, A] \otimes B , \quad [A \otimes B, \hat{\mathcal{O}}]_R \equiv A \otimes [B, \hat{\mathcal{O}}] , \quad (4.5)$$

where ordinary (color) matrix multiplication applies.

4. By applying eq.(1.16) together with the group property of link operators, $U(a, b)U(b, c) = U(a, c)$, for b along the straight-line path between a and c , we obtain the result [19,20]:

$$\begin{aligned}
& p \cdot \tilde{\mathcal{D}}(x) \hat{\Gamma}_{\mu\nu}(x, p) = \\
& + \frac{1}{2} \frac{g}{c} p^\sigma \partial_p^\tau \int_0^1 ds \left\{ [e^{s\Delta} F_{\sigma\tau}, \hat{\Gamma}_{\mu\nu}]_L + [\hat{\Gamma}_{\mu\nu}, e^{-s\Delta} F_{\sigma\tau}]_R \right\} \\
& + \frac{1}{4} i\hbar \frac{g}{c} \partial_p^\sigma \int_0^1 ds s \left\{ [e^{s\Delta} F_{\tau\sigma}, \hat{\mathcal{D}}^\sigma \hat{\Gamma}_{\mu\nu}]_L - [\hat{\mathcal{D}}^\sigma \hat{\Gamma}_{\mu\nu}, e^{-s\Delta} F_{\tau\sigma}]_R \right\} \\
& + \frac{1}{8} i\hbar \left(\frac{g}{c}\right)^2 \partial_p^\sigma \partial_p^\tau \int_0^1 ds s \int_0^1 d\tilde{s} \\
& \quad \left\{ [e^{s\Delta} F_{\sigma\eta}, [e^{\tilde{s}\Delta} F_\tau^\eta, \hat{\Gamma}_{\mu\nu}]_L + [\hat{\Gamma}_{\mu\nu}, e^{-\tilde{s}\Delta} F_\tau^\eta]_R]_L \right. \\
& \quad \left. - [e^{\tilde{s}\Delta} F_\tau^\eta, \hat{\Gamma}_{\mu\nu}]_L + [\hat{\Gamma}_{\mu\nu}, e^{-\tilde{s}\Delta} F_\tau^\eta]_R, e^{-s\Delta} F_{\sigma\eta}]_R \right\} \\
& + \frac{g}{c} \left\{ [e^\Delta F_{\mu\lambda}, \hat{\Gamma}_\nu^\lambda]_L - [\hat{\Gamma}_{\mu\lambda}, e^{-\Delta} F_\nu^\lambda]_R \right\} \\
& + \frac{1}{2} i\hbar \frac{g}{c} \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-ip \cdot y/\hbar} \left\{ [e^{\frac{1}{2}y \cdot \mathcal{D}} (\mathcal{D}_\mu j^\lambda - \mathcal{D}^\lambda j_\mu)] \otimes [e^{-\frac{1}{2}y \cdot \mathcal{D}} F_{\lambda\nu}] \right. \\
& \quad \left. - [e^{\frac{1}{2}y \cdot \mathcal{D}} F_\mu^\lambda] \otimes [e^{-\frac{1}{2}y \cdot \mathcal{D}} (\mathcal{D}_\lambda j_\nu - \mathcal{D}_\nu j_\lambda)] \right\} , \quad (4.6)
\end{aligned}$$

where $\Delta \equiv \frac{1}{2} i\hbar \partial_p \cdot \mathcal{D}$, $\mathcal{D} \equiv \mathcal{D}(x)$, $F \equiv F(x)$, $\hat{\Gamma} \equiv \hat{\Gamma}(x, p)$, and where $\mathcal{D}$ and ∂_p (from Δ) always act on the F immediately following it and on $\hat{\Gamma}$ respectively. j denotes the quark color current operator, eq.(3.14). As a final manipulation the quadratic Yang-Mills equation was employed here to evaluate terms involving $\mathcal{D}^2 F$ [19,20] giving rise to the terms of the last three lines, and we again reparametrized paths to streamline the right hand side as in the quark transport eq.(3.27). Eq.(4.6) constitutes the exact *gauge covariant quantum transport equation for the QCD gluon Wigner operator* as defined in eq.(4.1).

Furthermore, we remark that by changing $p \rightarrow -p$ (and ∂_p, Δ correspondingly) in eq.(4.6) the structural similarity to the proper quark transport equation (3.27) becomes most obvious; such a change will be incorporated in a slight redefinition of the gluon Wigner operator in the adjoint representation, cf. eqs.(4.7,4.9) below, which fully exhibits this similarity.

Note that besides the proper transport equation for $\hat{\Gamma}$ one expects a second one describing a generalized mass-shell constraint as for quarks, see e.g. eq.(3.25). In principle, it can be derived by considering $p^2\hat{\Gamma}$ and following similar steps as 1. - 4. above. However, this has not been done so far, since the main interest was to derive semiclassical on-shell transport equations first of all (cf. Sec. 4.2).

To conclude this section, we present the proper transport equation for the gluon Wigner operator as it is obtained in the *adjoint representation* [24]. Here the following definition of $\hat{\Gamma}$ seems most appropriate:

$$\hat{\Gamma}_{\mu\nu}(x, p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \left[e^{-\frac{y}{2} \cdot \vec{D}(x)} \vec{F}_\mu^\lambda(x) \right] \left[e^{\frac{y}{2} \cdot \vec{D}(x)} \vec{F}_{\lambda\nu}(x) \right]^\dagger, \quad (4.7)$$

which is an 8×8 -matrix in color space, expressed as the dyadic product of an 8-component color vector and its adjoint (i.e. transposed and Hermitean conjugate). Note that the gluon potentials A_μ^a and fields $F_{\mu\nu}^a$ as operators in Fock space are Hermitean. Using the identity (1.18) we can rewrite $\hat{\Gamma}$ from eq.(4.7),

$$\hat{\Gamma}_{\mu\nu}^{ab}(x, p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \left[\tilde{U}\left(x, x - \frac{y}{2}\right) \vec{F}_\mu^\lambda\left(x - \frac{y}{2}\right) \right]^a \left[\vec{F}_{\lambda\nu}^\dagger\left(x + \frac{y}{2}\right) \tilde{U}\left(x + \frac{y}{2}, x\right) \right]^b, \quad (4.8)$$

where we have explicitly indicated the color indices. The equation of motion for this gluon Wigner operator reads:

$$\begin{aligned} p \cdot \vec{D}(x) \hat{\Gamma}_{\mu\nu} = & \\ & - \frac{1}{2} \frac{g}{c} p^\sigma \partial_p^\tau \int_0^1 ds \left\{ [e^{-s\Delta} \mathcal{F}_{\sigma\tau}] \hat{\Gamma}_{\mu\nu} + \hat{\Gamma}_{\mu\nu} [e^{s\Delta} \mathcal{F}_{\sigma\tau}] \right\} \\ & - \frac{g}{c} \left\{ [e^{-\Delta} \mathcal{F}_{\mu\lambda}] \hat{\Gamma}_\nu^\lambda - \hat{\Gamma}_\mu^\lambda [e^{\Delta} \mathcal{F}_{\lambda\nu}] \right\} \\ & + \frac{1}{4} i \hbar \frac{g}{c} \partial_p^\tau \int_0^1 ds s \left\{ [e^{-s\Delta} \mathcal{F}_{\tau\sigma}] \vec{D}^\sigma \hat{\Gamma}_{\mu\nu} - \vec{D}^\sigma \hat{\Gamma}_{\mu\nu} [e^{s\Delta} \mathcal{F}_{\tau\sigma}] \right\} \\ & - \frac{1}{8} i \hbar \left(\frac{g}{c} \right)^2 \partial_p^\sigma \partial_p^\tau \int_0^1 ds s \int_0^1 d\tilde{s} \\ & \quad \left\{ [e^{-s\Delta} \mathcal{F}_{\sigma\eta}] ([e^{-\tilde{s}\Delta} \mathcal{F}_\tau^\eta] \hat{\Gamma}_{\mu\nu} + \hat{\Gamma}_{\mu\nu} [e^{\tilde{s}\Delta} \mathcal{F}_\tau^\eta]) \right. \\ & \quad \left. - ([e^{-\tilde{s}\Delta} \mathcal{F}_\tau^\eta] \hat{\Gamma}_{\mu\nu} + \hat{\Gamma}_{\mu\nu} [e^{\tilde{s}\Delta} \mathcal{F}_\tau^\eta]) [e^{s\Delta} \mathcal{F}_{\sigma\eta}] \right\} \\ & - \frac{1}{2} i \hbar \frac{g}{c} \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-i p \cdot y / \hbar} \left\{ \left[e^{-\frac{y}{2} \cdot \vec{D}} (\vec{D}_\mu \vec{j}^\lambda - \vec{D}^\lambda \vec{j}_\mu) \right] \left[e^{\frac{y}{2} \cdot \vec{D}} \vec{F}_{\lambda\nu} \right]^\dagger \right. \\ & \quad \left. - \left[e^{-\frac{y}{2} \cdot \vec{D}} \vec{F}_\mu^\lambda \right] \left[e^{\frac{y}{2} \cdot \vec{D}} (\vec{D}_\lambda \vec{j}_\nu - \vec{D}_\nu \vec{j}_\lambda) \right]^\dagger \right\}. \quad (4.9) \end{aligned}$$

Here the triangle operator Δ has the same meaning as in the case of the quark Wigner operator equation, but the covariant derivative in Δ has now to be taken in the adjoint representation as well, cf. eq.(1.6),

$$\Delta = \frac{i\hbar}{2} \partial_p \cdot \tilde{\mathcal{D}}(x) . \quad (4.10)$$

Again, the momentum derivative acts on the Wigner operator, while the covariant spatial derivative acts only on the field strength operator immediately following it.

Note the striking similarity between the above form of the gluon Wigner operator equation and the proper quark transport equation, eq.(3.27). Compared to eq.(4.6), all L and R commutators of color 3×3 matrices have been replaced by matrix multiplication of 8×8 matrices. Note, however, that L and R commutators also implied commutators between the field operators in Fock space which vanished completely from eq.(4.9). They have been absorbed by the different ordering between the fields due to the covariant derivatives $\mathcal{D}$, eq.(1.4), and $\tilde{\mathcal{D}}$, eq.(1.5), which enter the definitions in eq.(4.1) and eq.(4.7) respectively.

In addition, there are also crucial differences to be observed: the appearance of “spin”-terms (i.e. terms in the second line on the r.h.s. of each equation), which is well understood in the spin-1/2 case [17,41] but much less so for the spin-1 gluons (cf. Sects. 4.2, 4.3); then, there appears the explicit coupling of the gluon Wigner operator to the quark color current in eq.(4.9), whereas in the quark case the coupling to gluon fields is completely “spread out” into the variety of different terms.

4.2 The proper semiclassical gluon transport equation

Up to now the study of the gluon Wigner operator was completely general and exact. However, in order to proceed we turn to the semiclassical limit now trying to extract a proper transport equation for gluon fluctuations. As we have seen in the quark case, the *semiclassical limit of transport equations* is described by the lowest order terms of an expansion in powers of $\Delta \sim \hbar \partial_p \cdot \mathcal{D}$ and by replacing the Wigner operator by an ensemble average in the end.

However, for gluon fields there is an important intermediate step [19,20]. One wants to separate $\hat{\Gamma}$ into the interesting *incoherent fluctuation part* $\hat{G}$ plus $\bar{\Gamma}$ corresponding to the external or self-consistent mean field $\bar{F}$ representing the *coherent* part of the field,

$$G_{\mu\nu} \equiv \langle \hat{\Gamma}_{\mu\nu} - \bar{\Gamma}_{\mu\nu} \rangle , \quad a_\mu \equiv A_\mu - \langle A_\mu \rangle \equiv A_\mu - \bar{A}_\mu . \quad (4.11)$$

Here a_μ denotes the quantum fluctuations around the classical potential $\bar{A}_\mu$ and $\bar{\Gamma}_{\mu\nu}$ is defined by eq.(4.1) with $F \rightarrow \bar{F}$ (barred quantities henceforth refer to $\bar{A}$). It is important

to realize that $\langle \hat{\Gamma} \rangle \neq \bar{\Gamma}$ and therefore $G \equiv \langle \hat{G} \rangle \neq 0$. In the absence of fluctuations the classical mean field is, of course, given by

$$[\bar{D}_\mu, \bar{F}^{\mu\nu}] = -g\bar{j}^\nu, \quad (4.12)$$

where $\bar{j}^\nu \equiv \langle j^\nu \rangle$ is the ensemble averaged quark current, cf. eq.(3.14).

To derive the analog of Vlasov's equation for gluons from eq.(4.6), i.e. an equation for $G \equiv \langle \hat{G} \rangle$, one first observes that formally $\bar{\Gamma}$ also obeys eq.(4.6) involving the corresponding barred quantities. Next, we keep only the zeroth order terms from an expansion in powers of Δ of both the equations for $\hat{\Gamma}$ and $\bar{\Gamma}$. We insert eqs.(4.11) into the resulting equation for $\hat{\Gamma}$ and keep only the zeroth order terms in a_μ ; this is equivalent to *linearizing in the fluctuations a_μ and neglecting correlations* when an ensemble average is taken with $\langle a_\mu \rangle = 0$. Thus for example $\langle A\hat{G} \rangle \approx \bar{A}G$. Subtracting the mean-field equation for $\bar{\Gamma}$ from the equation for $\hat{\Gamma}$ one obtains the semiclassical gauge covariant *transport equation for gluon fluctuations* [19,20]:

$$\begin{aligned} p \cdot \bar{D} G_{\mu\nu} = & \frac{1}{2} \frac{g}{c} p^\sigma \partial_p^\tau \{ [\bar{F}_{\sigma\tau}, G_{\mu\nu}]_L + [G_{\mu\nu}, \bar{F}_{\sigma\tau}]_R \} \\ & + \frac{1}{8} i \hbar \frac{g}{c} \partial_p^\tau \{ [\bar{F}_{\tau\sigma}, \bar{D}^\sigma G_{\mu\nu}]_L - [\bar{D}^\sigma G_{\mu\nu}, \bar{F}_{\tau\sigma}]_R \} \\ & + \frac{1}{16} i \hbar \left(\frac{g}{c} \right)^2 \partial_p^\sigma \partial_p^\tau \{ [\bar{F}_{\eta\sigma}, [\bar{F}_\tau^\eta, G_{\mu\nu}]_L + [G_{\mu\nu}, \bar{F}_\tau^\eta]_R]_L \\ & \quad - [[\bar{F}_\tau^\eta, G_{\mu\nu}]_L + [G_{\mu\nu}, \bar{F}_\tau^\eta]_R, \bar{F}_{\eta\sigma}]_R \} \\ & + \frac{g}{c} \{ [\bar{F}_{\mu\lambda}, G_\nu^\lambda]_L - [G_{\mu\lambda}, \bar{F}_\nu^\lambda]_R \}. \end{aligned} \quad (4.13)$$

Here again, as in the quark case, $\hbar$ has been assumed to be small compared to $\Delta x_F \Delta p_G$, where Δx_F and Δp_G are typical length and momentum scales over which $\bar{F}_{\mu\nu}$ and G respectively vary appreciably in the plasma. More precisely, we used an expansion in powers of $\Delta \sim \hbar \partial_p \cdot \mathcal{D}$ and, therefore, corrections to the above equation in powers of Δ can principally be calculated from the general transport equation for $\hat{\Gamma}$, eq.(4.6). In a strict $\hbar$ -expansion the second to forth lines of eq.(4.13) would be dropped in the lowest order approximation; in the Abelian Dominance Approximation discussed below this occurs automatically.

Furthermore, in deriving eq.(4.13) one neglected correlations except those represented by G itself, cf. (4.11). For example, we set $\langle \hat{\Gamma} F \rangle \approx \bar{\Gamma} \bar{F}$. Thus, eq.(4.13) describes the plasma in the collisionless regime, since formally collision terms arise from the consideration of correlations [13]. There is no explicit coupling to the quark current due to the neglect of correlations, e.g. $\langle A j \rangle \approx \bar{A} \bar{j}$. This latter approximation, however, can perhaps be most easily circumvented by introducing a semi-phenomenological source term to

describe Bremsstrahlung-like gluons (cf. Chap. 5 for such a treatment of the somewhat related problem of particle production by a mean color field).

Note that eq.(4.13) transforms covariantly under local gauge transformations of the classical field provided that G does so. This is kept as an additional working assumption, which eventually should be justified by a consistent quantization of the gluon field and is presently being studied by one of us (H.-Th. E.).

We study and solve eq.(4.13) in the Abelian dominance and linear response approximation respectively in the next sections. Furthermore, it should be noted that this equation can again be rewritten in an eventually more convenient form by choosing the adjoint representation for the $SU(N)$ generators instead of the fundamental one, i.e. starting from eq.(4.9) (cf. [47]). This will be demonstrated in ref.[26], where eq.(4.13) together with the corresponding semiclassical quark transport equation from Chap.3 will be applied to describe a variety of idealized physical situations.

Next, we remark that the first line of eq.(4.13) resembles a relativistic Vlasov equation [13]. As could be expected, gluon fluctuations cause a color current which interacts with the mean field through a Lorentz force term. To determine this current heuristically, we observe that the $\hat{\Gamma}$ -equation in its original form [19], cf. eq.(4.6), implies an interesting local conservation law [20] (notation as therein),

$$\begin{aligned} 0 &= \int d^4p \{ \tilde{\mathcal{D}}_\sigma p^\sigma + \frac{1}{2} i\hbar [e^{\hbar\Delta} \mathcal{D}_\sigma] [e^{\hbar\Delta} \mathcal{D}^\sigma] - [e^{-\hbar\Delta} \mathcal{D}_\sigma]_L [e^{-\hbar\Delta} \mathcal{D}^\sigma]_L \} \hat{\Gamma}_{\mu\nu} \\ &= \tilde{\mathcal{D}}_\sigma \int d^4p \{ p^\sigma + \frac{1}{2} i\hbar (\mathcal{D}^\sigma - \mathcal{D}_L^\sigma) \} \hat{\Gamma}_{\mu\nu} + \hbar^2 (\text{first order in } \Delta) , \end{aligned} \quad (4.14)$$

defining a covariantly conserved local quantity. Its precise meaning is not yet fully understood. Anyway, subtracting the corresponding mean field equation, handling correlations as before, and neglecting $O(\hbar)$ corrections in the second of eqs.(4.14), one obtains $0 = \tilde{\mathcal{D}}_\sigma \int d^4p p^\sigma G_{\mu\nu}$. Thus, one is led to associate the integrand $p^\sigma G_{\mu\nu}$ with a phase-space distribution of the fluctuation current. However, additional color and Lorentz indices in eqs.(4.14) above indicate that it presents a more general object. Comparison with the Vlasov term in eq.(4.13) implies the way to contract the color indices,

$$t_a J_{\mu\nu}^{a\sigma} \equiv -t_a p^\sigma \text{Tr} \{ [t^a, G_{\mu\nu}]_L + [G_{\mu\nu}, t^a]_R \} , \quad (4.15)$$

where t_a are $SU(N)$ generators, and from which a proper four-current can be extracted as shown below.

4.3 The Abelian dominance approximation

Similar as in the case of quarks we now assume that the *classical field* $\bar{F}$ and *potential* $\bar{A}$ both can be diagonalized in color space. Then, one chooses a gauge which rotates

$\bar{F}$ into the Abelian subalgebra of $SU(N)$. This is always possible for fields of the form $\bar{F}_{\mu\nu}(x) = f_{\mu\nu}(x) \cdot n^i t_i$, particularly for covariant constant fields, and $\bar{A}$ becomes diagonal in the same (global) gauge. We keep this Abelian dominance approximation [17,20,50,52] also for fields which vary slowly in color space.

It is convenient to work in the Cartan-Weyl basis of $SU(N)$ [50,52] which consists of $N-1$ Abelian generators h_j and $N(N-1)$ non-Abelian generators e_{ij} ($i, j = 1, \dots, N$; $i \neq j$) satisfying

$$[h_i, h_j] = 0, \quad [h_i, e_{jk}] = (\vec{\eta}_{jk})_i e_{jk}, \quad [e_{ij}, e_{jk}] = \frac{1}{\sqrt{2}} e_{ik} \text{ for } i \neq j \neq k, \quad (4.16)$$

where the h_j are defined by $h_j \equiv (2j(j+1))^{-\frac{1}{2}} \text{diag}(1, \dots, 1, -j, 0, \dots, 0)$, with $-j$ appearing in the $j+1$ column, and where

$$\vec{\eta}_{ij} \equiv \vec{\epsilon}_i - \vec{\epsilon}_j, \quad \vec{\epsilon}_i \equiv (\vec{h})_{ii} = ((h_1)_{ii}, \dots, (h_{N-1})_{ii}), \quad (4.17)$$

are adjoint representation weight vectors $\vec{\eta}$ of $SU(N)$ as expressed in terms of elementary weight vectors $\vec{\epsilon}$.

The Abelian dominance approximation is implemented by replacing $\bar{F} \rightarrow \bar{F} \cdot \vec{h}$, $\bar{A} \rightarrow \bar{A} \cdot \vec{h}$, and with an Ansatz for G (henceforth summing only over upper and lower color indices),

$$G_{\mu\nu}(x, p) \equiv G_{\mu\nu}^{ij}(x, p) e_{ji} \otimes e_{ij} + G_{\mu\nu}^j(x, p) h_j \otimes h_j, \quad (4.18)$$

which corresponds to $N-1$ “neutral” plus $N(N-1)$ “charged” components (cf. eqs.(4.19), (4.20), and (4.25) below), i.e. $N^2 - 1$ color degrees of freedom. Then, making use of the properties of the Cartan-Weyl basis one decomposes the transport eq.(4.13). The resulting Vlasov type equations are [20]:

$$p \cdot \partial_x G_{\mu\nu}^j = 0, \quad (4.19)$$

$$(p \cdot \partial_x - g p^\sigma \partial_p^\tau \bar{F}_{\tau\sigma} \cdot \vec{\eta}^{ij}) G_{\mu\nu}^{ij} = -g \vec{\eta}^{ij} \cdot (\bar{F}_\mu^\sigma G_{\sigma\nu}^{ij} - G_{\mu\sigma}^{ij} \bar{F}^\sigma_\nu), \quad (4.20)$$

where the terms on the r.h.s. of (4.20) are due to the last line in (4.13). We observe the effective coupling $\sim g \vec{\eta}$ for the charged gluons similar to an effective coupling $\sim g \vec{\epsilon}$ derived for quarks in Chap.3. Note that the neutral gluons presently are described by a free transport equation and completely decouple from the system.

Additional terms from eq.(4.13) do not enter the transport equations, which is entirely due to the Abelian dominance approximation. Otherwise they would be affected by typically non-Abelian commutator terms.

Generally $\Gamma_{\mu\nu}$ and thus $G_{\mu\nu}$ consist of traceless symmetric and antisymmetric parts which, since $\Gamma_{\mu\nu} = \Gamma_{\nu\mu}^\dagger$, correspond to Hermitian and anti-Hermitian parts respectively.

Additionally there is a trace part, which does not contribute to the $T_{\mu\nu}$ -tensor, eq.(4.2); its equations follow trivially from eqs.(4.19,4.20) by contraction with $g^{\mu\nu}$. One tentatively sets

$$G_{\mu\nu}^j(x, p) = p_\mu p_\nu f^j(x, p) , \quad (4.21)$$

$$G_{\mu\nu}^{ij}(x, p) = p_\mu p_\nu f^{ij}(x, p) , \quad (4.22)$$

which on-shell contribute only to the traceless symmetric part of $G_{\mu\nu}$. Note that on-shell $p_\mu p_\nu$ commutes with $p \cdot \partial_x$, but it does not at all commute with $F_{\tau\sigma} p^\sigma \partial_p^\tau$ in eq.(4.20). However, the additional terms precisely cancel the r.h.s. of that equation. Thus [19,20],

$$p \cdot \partial_x f^j = 0 , \quad (4.23)$$

$$(p \cdot \partial_x - g p^\sigma \partial_p^\tau \vec{F}_{\tau\sigma} \cdot \vec{\eta}^{ij}) f^{ij} = 0 , \quad (4.24)$$

and, therefore, the Ansätze (4.21,4.22) in terms of real scalar distributions are consistent with the above transport equations. They are motivated by the relation of G to the energy-momentum tensor and by the form of the zeroth order Wigner function. For a thermalized isotropic ensemble one obtains from (4.1):

$$G_{\mu\nu}^{(g=0)} = 2(2\pi)^{-3} p_\mu p_\nu \delta(p^2) \left\{ \theta(p^0) n_B(p^0) + \theta(-p^0) n_B(-p^0) + 1 \right\} \\ \cdot \sum \{ e_{ji} \otimes e_{ij} + h_j \otimes h_j \} , \quad (4.25)$$

where $n_B(p^0) \equiv (e^{p^0/T} - 1)^{-1}$. Apart from the $p_\mu p_\nu$ - and color-factors $G_{\mu\nu}^{(g=0)}$ equals two times (for two gluon polarizations) the Wigner function for a massless scalar particle [20,53]. Of course, in what sense can (4.21,4.22) be considered as complete? This question still deserves further study as well as the antisymmetric part of $G_{\mu\nu}$. We do not see any explicitly spin dependent terms here in analogy with the quark transport equations. They may be present, however, on the r.h.s. of (4.20). Generally, a spin decomposition for an object like the gluon Wigner operator/function corresponding to a spin-1 vector boson field has not yet been performed in analogy to the spin-1/2 fermion case [41].

To complete the above set of transport equations the color current J^λ of gluon fluctuations discussed in connection with (4.13-4.15) has to be identified. We set $J^\lambda \equiv \int d^4 p t_a J_{\mu\nu}^{a\lambda} g^{\mu\nu} / p^2$. Then, one obtains [20]:

$$J^\lambda(x) = \int d^4 p p^\lambda \vec{h} \cdot \vec{\eta}_{ij} f^{ij}(x, p) \equiv \vec{J}^\lambda \cdot \vec{h} , \quad (4.26)$$

which is diagonal; only a more general $G_{\mu\nu}$ than in (4.18) would give off-diagonal contributions here. Eq.(4.24) implies $\partial_x \cdot J = 0$, as it should be. In the presence of fluctuations J^λ

enters the r.h.s. of the mean field eq.(4.12), where for consistency also the quark current is required to be diagonal, i.e. $\vec{j} \rightarrow \vec{j} \cdot \vec{h}$. Thus, eq.(4.12) is replaced by

$$\partial_\mu \vec{F}^{\mu\nu} = -g(\vec{J}^\nu + \vec{j}^\nu) , \quad (4.27)$$

which determines the $N - 1$ Abelian field components.

Eqs.(4.19-4.24,4.26,4.27) together form the final set of equations for the study of gluon fluctuations under the influence of a selfconsistent or external classical color field in Abelian dominance approximation. Of course, the current $\vec{j}$ has to be determined in the quark sector of transport theory developed in Chap. 3 or be treated as an external source. We refer the reader to refs.[26,31] concerning applications of these equations, except for the linear response study represented in the following section.

4.4 The linear response analysis

With the number of approximations involved one would like to confirm that the semiclassical transport theory developed above well describes known features of the quark-gluon plasma. For that purpose we study the response of (charged) gluons to small perturbations around a thermalized equilibrium state.

To calculate the electric permeability in linear response approximation one observes that, apart from the color structure (4.24) and (4.27) are identical with ordinary Vlasov and inhomogeneous Maxwell equations respectively. Therefore, we follow mutatis mutandis the linear response analysis of an electron plasma given in ref.[54] as applied to the present case in ref.[20]. - A similar analysis starting from the classical color kinetic equations of Sec. 2.2 was done in ref.[27] yielding the same results as below. - Let δf denote a perturbation of the homogeneous equilibrium distribution function f_E , then the perturbed distribution f is

$$f^{ij}(x, p) = f_E^{ij}(p) + \delta f^{ij}(x, p) . \quad (4.28)$$

With no preferred direction in color space, $f_E^{ij} = f_E$, one verifies that in equilibrium there is no fluctuation current, $\vec{J}_E^\lambda = 0$, since $\sum_{ij} \vec{\eta}_{ij} = 0$. Fourier transforming and applying retarded boundary conditions (damping in the infinite past) [54], we solve (4.24) for δf^{ij} in the linearized approximation,

$$\delta f^{ij}(k, p) = -g \frac{p^\sigma \partial_p^\tau \vec{F}_{\tau\sigma}(k) \cdot \vec{\eta}^{ij}}{i(k \cdot p + i\epsilon p^0)} f_E(p) , \quad (4.29)$$

where $\epsilon \rightarrow 0^+$. Introducing a color polarization $\vec{P}$ similar to the usual one [54],

$$ik \cdot \vec{P}(k) = g\delta \vec{J}^0(k) , \quad -ik^0 \vec{P}(k) = -g\delta \vec{J}(k) , \quad (4.30)$$

where $\delta \vec{J}^\lambda$ is the fluctuation current due to the perturbation δf^{ij} according to (4.26), one obtains the color electric induction $\vec{D}$,

$$\begin{aligned}\vec{D}(k) &= \vec{E}(k) + \frac{2Ng^2}{k^0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \frac{\vec{E}(k) \cdot \nabla_p n_B(p)}{k^0 p - \mathbf{k} \cdot \mathbf{p} + i\epsilon p} \\ &\equiv \epsilon(k) \cdot \vec{E}(k) ,\end{aligned}\quad (4.31)$$

where $p \equiv |\mathbf{p}|$, n_B denotes the Bose distribution, and a vacuum contribution has been omitted in the course of the derivation [20]. From eq.(4.31) we read off the color electric permeability matrix ϵ , which is diagonal in the color indices (suppressed in the following). Decomposing ϵ into transverse and longitudinal parts, $\epsilon^{kl} \equiv \epsilon_T(\delta^{kl} - k^k k^l/k^2) + \epsilon_L k^k k^l/k^2$ with $\epsilon_{(T,L)} \equiv \epsilon_{(T,L)}(\omega, \mathbf{k})$, one obtains ($k \equiv |\mathbf{k}|$):

$$\epsilon_L(\omega, \mathbf{k}) = 1 + \frac{m_{el}^2}{k^2} \left\{ 1 - \frac{1}{2} \frac{\omega}{k} \left[\ln \left| \frac{k+\omega}{k-\omega} \right| - i\pi\theta(k-\omega) \right] \right\} , \quad (4.32)$$

and,

$$\epsilon_T(\omega, \mathbf{k}) = 1 - \frac{m_{el}^2}{2k^2} \left\{ 1 - \frac{1}{2} \left(\frac{\omega}{k} - \frac{k}{\omega} \right) \left[\ln \left| \frac{k+\omega}{k-\omega} \right| - i\pi\theta(k-\omega) \right] \right\} , \quad (4.33)$$

with the electric mass defined by $m_{el}^2 \equiv \frac{1}{3} Ng^2 T^2$. Note that the function in curly brackets in eq.(4.32) is well known from classical kinetic theory and that there is no imaginary part (absence of Landau damping, cf. the dispersion relations below and Fig. 1) for $\omega > k$ [54].

That one encounters the electric mass here, which is related to the Debye length λ_D , $m_{el}^2 = 1/\lambda_D^2$, follows from the relation between the static limit of ϵ_L and the electric potential ϕ of a colored point test charge in the plasma (cf. [20,54]),

$$\phi(\mathbf{k}) = \frac{const}{k^2 \epsilon_L(0, \mathbf{k})} = \frac{const}{k^2 + m_{el}^2} . \quad (4.34)$$

After a Fourier transformation eq.(4.34) gives the usual Debye-screened potential with an exponential decay length λ_D . The gluon electric mass calculated here outside the framework of perturbative high-temperature QCD agrees with the gauge invariant value found there (see refs.[55] and references therein).

Furthermore, using eqs.(4.32,4.33) we can relate the dispersion relations for longitudinal and transverse plasma oscillations [54], which follow from eq.(4.27) in the presence of the plasma as in electrodynamics,

$$\epsilon_L(\omega, \mathbf{k}) = 0 , \quad \epsilon_T(\omega, \mathbf{k}) - k^2/\omega^2 = 0 , \quad (4.35)$$

directly to those determined for example by the poles of the gluon propagator in temporal axial gauge [55]. In terms of the polarization tensor $\Pi_{\mu\nu}(\omega, \mathbf{k})$ we obtain:

$$\begin{aligned}\text{L: } 0 &= 1 - \Pi_{00}/k^2 = \epsilon_L(\omega, \mathbf{k}) , \\ \text{T: } 0 &= \omega^2 - k^2 - \frac{1}{2}(\Pi_{ii} - \omega^2 \Pi_{00}/k^2) = \omega^2 \epsilon_T(\omega, \mathbf{k}) - k^2 ,\end{aligned}\quad (4.36)$$

where the second equality in both cases applies for $\Pi_{\mu\nu}$ evaluated to $O(g^2)$ and to leading order in the high-temperature expansion. Note that eqs.(4.36) also imply equality of those imaginary parts describing Landau damping according to eqs.(4.32,4.33). We cannot expect to learn more about damping here, since we have not yet derived any collision terms entering eq.(4.13) or eqs.(4.23,4.24). In thermal QCD such effects come in at next-to-leading order in the high-temperature expansion [55]. To leading order in this expansion the solutions of the dispersion relations (4.35) determining the collective modes are identical to those from high-temperature QCD [56]. In the long-wavelength limit one obtains the longitudinal and transverse modes,

$$k \ll \omega_0 : \quad \omega_L^2 = \omega_0^2 + \frac{3}{5}k^2, \quad \omega_T^2 = \omega_0^2 + \frac{6}{5}k^2, \quad (4.37)$$

with the well-known gluon plasma frequency $\omega_0^2 \equiv \frac{1}{3}m_{el}^2$ [55]. For the short-wavelength oscillations the result is:

$$k \gg \omega_0 : \quad \omega_L^2 = k^2 \left(1 + 4\exp(-2 - 2k^2/3\omega_0^2)\right), \quad \omega_T^2 = k^2 + \frac{3}{2}\omega_0^2. \quad (4.38)$$

Note that including N_f flavors of massless quarks at zero chemical potential simply changes the plasma frequency to $\omega_0^2 = \frac{1}{9}(N + \frac{1}{2}N_f)g^2T^2$. A numerical evaluation of the dispersion relations is shown in Fig. 1 [27].

To conclude this section, we consider the magnetic behavior of the charged gluons. Performing a linear response analysis as above, one solves the spatial part of eq.(4.27) by introducing a magnetization $\vec{M}^{ij}$. The field equation thus becomes

$$k_i \vec{F}_{eff}^{ij}(k) \equiv k_i \left(\vec{F}^{ij}(k) + \vec{M}^{ij}(k) \right) = 0, \quad (4.39)$$

with $\vec{M}^{ij}$ satisfying

$$k_i \vec{M}^{ij}(k) = -k_0 \vec{D}^j(k). \quad (4.40)$$

Finally, proceeding similarly as in the electric field case one obtains the *effective magnetic field* [20]:

$$\vec{F}_{eff}^{ij}(\omega, \mathbf{k}) = \left\{1 - \frac{\omega^2}{k^2} \epsilon_T(\omega, \mathbf{k})\right\} \vec{F}^{ij}(\omega, \mathbf{k}). \quad (4.41)$$

In the static limit $\vec{F}_{eff}^{ij}(0, \mathbf{k}) = \vec{F}^{ij}(0, \mathbf{k})$ and there is *no screening* of static magnetic fields in the present approximation. This agrees with the result of a recent 1-loop calculation in high-T QCD, where the absence of magnetic screening and a QED-like behavior respectively were observed including $g^2 \log \frac{T}{k}$ -corrections (also for longitudinal colormagnetic fields) [57]. Furthermore, eqs.(4.39,4.41) imply the same dispersion relation as for the transverse electric modes in (4.35). Consequently, there exist *transverse magnetic oscillations* with the same plasma frequency, $\omega_0^2 = \frac{1}{9}Ng^2T^2$, as the electric ones [20]. They have not been studied in perturbative high-T QCD in a similar way as the screening case

in ref.[57], but the result given above confirms the relation between transverse electric and magnetic oscillations expected on general grounds [56].

Fig. 1 . The dispersion relation for the longitudinal (L) and transverse (T) colored plasma modes at a temperature of 200 MeV and a baryon chemical potential $\mu=0$ (solid lines) and $\mu=300$ MeV (dashed lines). The calculation was done for finite quark masses ($m_{u,d}=10$ MeV, $m_s=150$ MeV) and $\alpha_s=0.3$. The shaded area indicates the region of Landau damping.

It should be remarked that the linear response analysis is valid only in the limit of sufficiently small fields [54]. Otherwise the assumption of an isotropic equilibrium distribution entering eq.(4.31) cannot be justified. Further studies of linear response and particularly of classical instabilities as described by the semiclassical transport equations will be reported in Chap.6 and ref.[26] respectively.

In conclusion, we may state that in the high-temperature limit we find satisfactory agreement between standard results from perturbative QCD and the description of simple collective gluon properties in the semiclassical transport theory. These results provide an important piece of circumstantial evidence relating transport theory at $O(\hbar)$ to one-loop perturbative QCD. Of course, the main point in formulating transport equations for the quark-gluon plasma is that they are not limited to describe close-to-equilibrium phenomena (as implied by linear response).

4.5. Damping rates for colored plasma oscillations

As we saw in the previous section, the semiclassical kinetic equations (3.44,4.20,4.24) without collision terms yield completely real dispersion relations [20,24,27], i.e. the corresponding collective modes are undamped. If the collision terms are approximated by a relaxation time Ansatz [24], $C \sim (f - f_{eq})/\tau_0$, where f_{eq} are the equilibrium Bose and Fermi distributions for quarks and gluons respectively (cf. eq.(4.25)), the modes are damped away in a time τ_0 . The usual imaginary part due to *collisionless* (mean-field or Landau) damping is only present in the region of spacelike momenta [54,58] and thus does not affect the above modes which have always $\omega^2 > k^2$ (cf. Fig. 1).

The results for ω_T and ω_L in the previous subsection are independent of the gauge, as a result of using a gauge covariant kinetic framework as a starting point and an approximation scheme which to this order in the perturbation preserves gauge invariance. The above dispersion relations agree with the leading term from a high-temperature expansion for the poles of the one-loop finite temperature QCD gluon propagator, which is also known to be gauge invariant [56,59,60].

Of course, we know physical mechanisms which can contribute an imaginary part to the dispersion relation: pair decay of the perturbation into massless quark-antiquark or gluon pairs, or higher order scattering processes. Depending on the sign of this imaginary part, the plasma oscillations will be either damped or unstable. To find damping in the dispersion relations or calculate τ_0 in the relaxation time Ansatz requires an approximation to the quantum kinetic equations which keeps 2-body correlations and, thus, generates nonvanishing collision terms. This has not yet been achieved in a systematic way, but we will now discuss several ad hoc approaches to deal with this problem, and the difficulties to which they lead.

In the framework of finite temperature QCD, it can be attempted to extract the imaginary part residing in the non-leading terms of a high-temperature expansion for the poles of the response function [55],[59]-[64]. Unfortunately, the relevant QCD response function which defines plasma oscillations has so far not been written down in a generally accepted, gauge invariant way [65]. As a result its imaginary part and, thus, the plasmon damping rate is plagued by gauge dependence. It does not show up, however, in the leading term of a high-temperature expansion, but crucially affects the next-to-leading term. The different results obtained up to now do not even agree in sign!

The calculation which is perhaps closest in spirit to the linear response analysis of Sec. 4.4 and which leads to a result for the damping rate which has a very close formal relation to a Boltzmann-Nordheim type collision term in the kinetic equations is given in [55]. After it was found that the poles of the finite temperature *gluon propagator* and, in particular, their imaginary parts exhibit gauge dependence as soon as they are studied

beyond the leading term in a high-temperature expansion [59,60,64], the authors of ref.[55] reinvestigated the linear response approach to perturbations around thermal equilibrium in QCD. It was suspected that the problem of gauge dependence reflected an incomplete analysis and that there are generally more contributions to the response function than just the gluon propagator. A first such attempt had been performed by Kajantie and Kapusta [55], where the QCD plasma was subjected to a pulse of an external electric field, and the oscillatory response of the plasma was studied. Choosing an external electric *field* rather than an external *potential or current* was motivated by the realization that a coupling of the type

$$H_{ext} = \frac{1}{2} \int \text{tr } F_{\mu\nu} \mathcal{F}^{\mu\nu} d^3x = \int (\vec{E}_a \cdot \vec{\mathcal{E}}_a - \vec{B}_a \cdot \vec{\mathcal{B}}_a) d^3x , \quad (4.42)$$

(where $\mathcal{F}^{\mu\nu}$ is the external perturbation) is invariant under simultaneous gauge transformations of $F_{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$, while the more conventional coupling

$$H_{ext} = - \int j_\mu^a \mathcal{A}_a^\mu d^3x \quad (4.43)$$

(with either j_μ or $\mathcal{A}_\mu$ being the external source) cannot be consistently made gauge invariant [66]. - In ref.[65], however, also the above Ansatz, eq.(4.42), and the present scheme of linear response calculations was criticized, since the resulting response functions still show dubious gauge transformation properties.

While the calculation of Kajantie and Kapusta was performed in temporal axial gauge (TAG), $A_0^a = 0$, and was subsequently criticized for its lack of rigor in dealing with the unphysical poles in the propagator which are introduced by this gauge condition, in [55] this calculation was supplemented by an analysis in Coulomb gauge which is free of these problems. - In a recent paper it was shown [67] that the formal “tricks” applied in refs.[55] to the axial gauge poles in TAG may after all have produced the correct result; the TAG result for the longitudinal gluon polarization operator was verified by using a regulated version of static axial gauge $\partial_0 A_0 = 0$, taking all limits very carefully. - Here we will concentrate on the Coulomb gauge calculation to avoid all unnecessary discussion about technicalities.

The complication with Coulomb gauge (compared to TAG) is that the electric field is not just the time-derivative of the vector potential, but also contains non-Abelian terms which are nonlinear in A_μ ; therefore, the electric response function contains in addition to a propagator-like term $\sim \partial_x \partial_{x'} \langle A_\mu(x) A_\nu(x') \rangle$ also terms involving three and four A -operators, i.e. contributions from three- and four-point functions. Fortunately, the four-gluon term does not contribute at order g^2 , i.e. at the one-loop level; the three-gluon contribution turns out to be completely *real* and serves only to regularize an infrared divergence in the real part of the gluon propagator contribution to the electric response function. Thus, the *imaginary* part of the electric field response function, which determines

the damping rate, is given purely by the imaginary part of the thermal gluon propagator and is identical with the expression obtained in TAG. It is due to the gluon loop with two *physical*, i.e. transversal gluons going around the loop.

This remarkable fact permits a physical interpretation for the imaginary part and the damping in terms of the decay of the massive ($\omega^2 > k^2$) electric perturbation into a pair of massless ($\omega^2 = k^2$) and transverse (i.e. physical) gluons. Indeed, following Weldon [68], the imaginary part of the response function can be written [55] as an integral over phase space of the corresponding decay matrix element multiplied with the appropriate thermal distribution functions for the in- and outgoing particles (quarks and gluons), including Bose-Einstein enhancement and Fermi-Dirac suppression on the final states:

$$\begin{aligned} \text{Im}R_{L/T} = & -\frac{1}{16\pi k} \left\{ \theta(\omega - k) \int_{\omega_-}^{\omega_+} dp |\mathcal{M}_{L/T \rightarrow q\bar{q}}|^2 [(1 - n_F)(1 - n'_F) - n_F n'_F] \right. \\ & - 2\theta(k - \omega) \int_{k_+}^{\infty} dp |\mathcal{M}_{L/T+q \rightarrow q}|^2 [n_F''(1 - n_F) - n_F(1 - n_F'')] \\ & + \theta(\omega - k) \int_{\omega_-}^{\omega_+} dp |\mathcal{M}_{L/T \rightarrow gg}|^2 [(1 + n_B)(1 + n'_B) - n_B n'_B] \\ & \left. - 2\theta(k - \omega) \int_{k_+}^{\infty} dp |\mathcal{M}_{L/T+g \rightarrow g}|^2 [n_B''(1 + n_B) - n_B(1 + n_B'')] \right\}, \quad (4.44) \end{aligned}$$

with $R_{L/T} = R_{L/T}(k, \omega)$. This expression bears very strong resemblance to a Boltzmann-Nordheim collision term in the kinetic equations, including the effects of quantum statistics on the occupation of the final states. This demonstrates that generally there must exist quite close relationships between non-equilibrium phenomena in finite temperature quantum field theory and kinetic theory. In eq.(4.44) n_F and n_B denote the Fermi and Bose distributions, primes and double primes indicate dependence on $p' = \omega - p$, $p'' = p - \omega$ respectively, and $\omega_{\pm} = (\omega \pm k)/2$, $k_{\pm} = (k \pm \omega)/2$.

The matrix elements (in the order of appearance) correspond to the decay of a longitudinal/transverse perturbation into a massless quark-antiquark pair (Fig. 2a), the scattering of the perturbation by on-shell quarks from the thermal quark and antiquark distributions in the plasma (Fig. 2b), the decay into a pair of massless, transverse gluons (Fig. 2c), and the scattering off massless, transverse gluons from the thermal gluon distribution (Fig. 2d). For each such process, its inverse contributes with a minus sign, enabling us to interpret the two terms from each process as a loss and a gain term respectively. The relevant matrix elements are:

$$|\mathcal{M}_{L \rightarrow q\bar{q}}|^2 = |\mathcal{M}_{L+q \rightarrow q}|^2 = 4N_f g^2 p^2 \sin^2(\vec{p}, \vec{k}), \quad (4.45)$$

$$|\mathcal{M}_{T \rightarrow q\bar{q}}|^2 = -|\mathcal{M}_{T+q \rightarrow q}|^2 = 2N_f g^2 \left(\frac{\omega^2 - k^2}{2} - p^2 \sin^2(\vec{p}, \vec{k}) \right), \quad (4.46)$$

$$\begin{aligned}
|\mathcal{M}_{L \rightarrow gg}|^2 &= |\mathcal{M}_{L+g \rightarrow g}|^2 \\
&= Ng^2 \frac{k^2}{2} \left(1 + \cos^2(\vec{p}, \vec{q})\right) \left|1 - \frac{p^2}{\omega^2 - k^2} \sin^2(\vec{p}, \vec{k})\right|, \quad (4.47)
\end{aligned}$$

$$\begin{aligned}
|\mathcal{M}_{T \rightarrow gg}|^2 &= |\mathcal{M}_{T+g \rightarrow g}|^2 \\
&= Ng^2 \left[2(p^2 + q^2)(1 - \cos^2(\vec{p}, \vec{q})) + p^2 \sin^2(\vec{p}, \vec{k})(1 + \cos^2(\vec{p}, \vec{q}))\right] \quad (4.48)
\end{aligned}$$

Here $\vec{k}$ is the momentum of the electric perturbation (i.e. plasmon), and $\vec{p}$ and $\vec{q} = -\vec{k} - \vec{p}$ are the momenta of in- or out-going quarks, antiquarks, or gluons respectively.

Fig. 2 . Processes leading to the damping of colored plasma oscillations [55].

Corresponding to this intuitive interpretation in terms of physical processes, the result for the damping rate is positive, i.e. the plasma oscillations are *damped* at a rate $\gamma = \frac{Ng^2T}{24\pi}$ [55]. The close relationship of this damping rate, calculated in thermal QCD, with a Boltzmann-Nordheim collision term appears to be of a general nature. It will be a focus of attention in our further studies of collision terms in the quantum transport equations which were presented in the last two chapters. There does not seem to be a similarly intuitive interpretation of the damping rate in terms of a physical decay or scattering process for the more “elegant” calculations based on the QCD effective action [61,62] which yield negative values for γ , nor for Nadkarni’s calculation [63] using Cornwall’s “gauge invariant propagator” [69], which also yields instability at one-loop order.

The value for the damping rate given above for a quark-gluon plasma with a temperature of 200 MeV and a coupling constant $\alpha_s \sim 1$ leads to damping of the oscillations after 3-5 periods, corresponding to an equilibration time scale for the color perturbation of 1-2 fm/c. This is not extremely short, but perhaps short enough to render some qualitative credibility to dynamical calculations based on the *assumption* of local color equilibrium.

One has, of course, to be aware of the fact that a determination of γ at the one-loop level cannot be fully consistent, since there are most probably corrections from two-loop

and higher order diagrams which contribute to the term $\sim g^2 T$ [55,59,60]. Physically speaking, the decay of a massive perturbation in the plasma into two bare transverse gluons with $\omega = |\vec{k}|$ should be replaced by a decay into dressed gluons with a medium-induced effective mass and a modified dispersion relation $\omega(k)$. This can, of course, entirely change the value of γ , although not its sign. These corrections (which most probably will not even be calculable in perturbation theory), however, cannot be invoked to explain the different results obtained by the various groups at the one-loop level: we think that these discrepancies must mean that in each case a different question has been asked and therefore a different answer was obtained. How to ask the same question in all of these approaches, i.e. how to subject the system in each case to the same physical perturbation and calculate the same physical response, this problem has to be solved before we will be able to understand the existing and apparently contradicting results.

5. PARTICLE PRODUCTION BY MEAN COLOR FIELDS

In Sec. 4.4 we saw a first application of semiclassical gluon transport theory as given in eqs.(4.19-4.24,4.26,4.27). Here we will outline, how, besides the Vlasov/Lorentz-force term, a mean color field enters the semiclassical equations through a particle source term and a vacuum polarization current. The physical picture invoking this process, i.e. the color string or flux tube model, has been widely discussed in refs.[31,50,51,70] and in references therein. It should, however, be noted that, apart from further formal development of the presented quark-gluon transport theory, the main interest in particle production by mean fields stems from the fact that it might provide a mechanism to initialize a quark-gluon plasma in ultrarelativistic heavy-ion collisions.

As explained in Chaps. 3 and 4, the semiclassical transport equations are basically obtained through an expansion in powers of $\Delta \sim \hbar D \cdot \partial_p$ of their full quantum versions. Additionally one neglects correlations (except those from $G_{\mu\nu}$, cf. Sec. 4.2). Therefore, it has been no surprise that for example only classical Landau damping was observed so far, cf. eqs.(4.32, 4.33). Furthermore, eqs.(4.36) are valid only for the leading terms of a high-T expansion. How do we learn about the non-leading (non-Landau imaginary) parts of these equations? As was explained in Sec. 4.5 for perturbative QCD, an important contribution here is due to vacuum polarization, i.e. decay of the classical mean field/collective mode via pair production.

With this twofold motivation to include mean field particle production we shortly discuss previous approaches. Based on Schwinger's non-perturbative calculation of vacuum polarization in a classical field [21,32], various forms of source terms have been proposed in refs.[31,50,51,70]. They are, however, not quite satisfactory: (i) They only apply for fields which are constant in space-time, where certainly finite-size effects must be considered for heavy-ion collision plasmas [71]. It is unclear and even highly improbable that the field will be arranged nicely and even stay longitudinally oriented. (ii) They do not know about the longitudinal momentum dependence of the produced particle spectrum with a correspondingly uncertain polarization current, which is usually argued away by assuming longitudinal boost invariance and that pairs are produced at rest. (iii) They involve a non-analytic function of the coupling constant and, therefore, vacuum polarization will not stand on the same footing with future perturbative collision terms. In conclusion, there are at least three arguments in favor of supplementing these approaches by a perturbative treatment. Then, the above listed difficulties do not arise, as we illustrate here for the case of quarks following the outline in ref.[20]. Finally, we also shortly represent the non-perturbative approach for quarks and gluons based on ref.[50].

5.1 The perturbative transport theory approach

The semiclassical quark transport equation and color current respectively are (in Abelian dominance approximation, see Chap. 3):

$$\left(p \cdot \partial_x - g \vec{\epsilon}^j \cdot \vec{F}_{\mu\nu} p^\nu \partial_p^\mu \right) f^j = \frac{1}{4} i g \vec{\epsilon}^j \cdot \vec{F}_{\mu\nu} [\sigma^{\mu\nu}, f^j] + S^j, \quad (5.1)$$

$$\vec{j}^\lambda = \int d^4p \vec{\epsilon}_j \text{Tr} \gamma^\lambda f^j + \vec{j}_{vac}^\lambda, \quad (5.2)$$

where the (4x4 spin-matrix) Wigner function f^j describes quarks and antiquarks. One now has to derive the source term S^j and induced vacuum current $\vec{j}_{vac}^\lambda$ due to the external or self-consistently generated mean color field $\vec{F}_{\mu\nu}$ acting on the polarizable vacuum.

Note that presently we introduce (like in all the references cited above) a quantum effect par excellence, i.e. mean-field particle production, as an additional term into the otherwise classical or semiclassical (as in Chapters 3 or 4) transport equation. Similarly one would introduce e.g. particle sources resulting from a chemical reaction or any external device, whereas a fully consistent approach so far has not been developed. This is, of course, related to the fact that the relevant correlation functions (collision terms) in the case of interacting fields have not been studied. On the other hand, in the mean-field case selfconsistency would be automatically achieved by “simply” solving for example the full quantum transport equation given below (eventually coupled back through a classical Yang-Mills equation to determine the mean field) and, thus, neglecting gluon fluctuations. In ref.[72] the mean-field limit has been studied in this way for a scalar field coupled to an ordinary constant electric field, however, neglecting the backreaction of the produced particles on the field. Using a WKB approximation to solve the full transport equation, essentially the result for the total rate of particle production known from the Schwinger approach was reproduced.

Using a perturbative approach previously applied in ϕ^4 -models [53], we begin with the Fourier-transformed vacuum Wigner function

$$\begin{aligned} f_{vac}^j(q, p) &\equiv \int d^4x d^4y e^{iq \cdot x} e^{-ip \cdot y} \langle 0 | \bar{\psi}^j(x + \frac{1}{2}y) \psi^j(x - \frac{1}{2}y) | 0 \rangle \\ &= -(2\pi)^5 \delta^4(q) \delta(p^2 - m^2) \theta(-p^0) (\gamma \cdot p + m). \end{aligned} \quad (5.3)$$

Then, one has to solve a full *quantum* transport equation for the quark Wigner function [20],

$$\left(\gamma \cdot (p + \frac{1}{2}q) - m \right) f^j(q, p) = -g \int \frac{d^4q'}{(2\pi)^4} \vec{\epsilon}^j \cdot \vec{A}_\mu(q') \gamma^\mu f^j(q - q', p - \frac{1}{2}q'), \quad (5.4)$$

$$= ig \int \frac{d^4q'}{(2\pi)^4} \int_0^1 ds (s-1) \vec{\epsilon}^j \cdot \vec{F}_{\mu\nu}^j(q') \gamma^\mu \partial_p^\nu \exp(sq' \cdot \partial_p) f^j(q - q', p - \frac{1}{2}q'), \quad (5.5)$$

which both follow from Dirac's equation and can be used alternatively together with appropriate definitions of f^j . Eq.(5.4) holds for the usual *gauge dependent* Wigner function [7,13], cf. eq.(3.11). Eq.(5.5) holds for the *gauge covariant* Wigner function studied in Chap. 3, cf. eq.(3.12); it is invariant with respect to the remaining $U(1)^{N-1}$ gauge freedom in Abelian dominance approximation and analogous to the Abelian plasma equation [41].

One solves the transport equation (5.4) iteratively starting with f_{vac}^j , which induces a perturbative expansion of the solution in powers of g , and subject to the *constraint* $f_j^\dagger(q, p) = \gamma^0 f_j(-q, p) \gamma^0$ implied by the definition of fermion Wigner functions. This gives [20] the first order induced current $\bar{j}_{vac}^{\lambda(1)}$ after subtracting the $O(g^0)$ divergent vacuum contribution,

$$\bar{j}_{vac}^{\lambda(1)}(q) = \int d^4p \, \bar{\epsilon}_j \, Tr \, \gamma^\lambda f_{(1)}^j(q, p) = -\frac{1}{2} ig \Pi_T^{(1)}(q^2) q_\mu F^{\lambda\mu}(q) + \dots, \quad (5.6)$$

where $\Pi_T^{(1)}$ denotes the transversal part of the first order QED polarization tensor and well-known gauge invariance breaking terms [32,73] are indicated. As an interesting aspect of this result note that one could possibly calculate, using a solution to (5.5) instead of (5.4), vacuum polarization in an arbitrary external field in a gauge invariant way, which is still problematic even in QED.

Using reduction formulae, the finite second order gauge invariant source term is obtained from the second order solution to (5.4) [20]:

$$S_{(2)}^j(q, p) = (2\pi)^4 \delta^4(q) \{ \theta(p^0) \mathcal{N}_{(2)}^j(p) + \theta(-p^0) \mathcal{N}_{(2)}^j(-p) \}, \quad (5.7)$$

with the momentum spectrum of produced particles, $\mathcal{N} \equiv 2\omega dN/d^3p$,

$$\begin{aligned} \mathcal{N}_{(2)}^j(p) &= \frac{1}{2} (2\pi)^{-3} \sum_r \bar{u}^r(p) (\gamma \cdot p - m) f_{(2)}^j(0, p) (\gamma \cdot p - m) u^r(p) \\ &= \frac{g^2}{4\pi^2} \int \frac{d^4q}{(2\pi)^4} \theta(q^0 - p^0) \delta((q - p)^2 - m^2) \\ &\quad \cdot \{ |\vec{\epsilon}^j \cdot \vec{E}(q)|^2 - |\vec{\epsilon}^j \cdot \vec{B}(q)|^2 - |\vec{\epsilon}^j \cdot \vec{A}(q) \cdot (q - 2p)|^2 \}, \end{aligned} \quad (5.8)$$

depending on the color fields. The antiparticle contribution in (5.7) arises similarly. Calculating the total vacuum decay rate from (5.7,5.8) reproduces the known QED result [32] apart from the color factors.

These results can easily be generalized to include initial on-shell (anti-)quark distributions other than the vacuum. Furthermore, note that the induced vacuum current (5.6), being linear in $F_{\mu\nu}$, will affect a *linear* response analysis as in the Sec.4.4, whereas the source term (5.7), being essentially quadratic in field strengths, would be neglected. Finally, by comparison with the above calculation one observes that the gluon quantum

transport equation, eq.(4.6), is, as it stands, not suited to perform a similar calculation of gluon production.

We now turn to a different approach [50] which reduces the *constant external field* problem for quarks and gluons in such a way that it allows to apply non-perturbative results similar to the ones obtained by Schwinger for the QED case [21].

5.2 The non-perturbative Schwinger mechanism

Abelian Case

In string models for multiparticle production [51],[74]-[77] one imagines that multiple soft gluon exchange leaves both projectile and target in color non-singlet states. Because of confinement though, the resulting color electric fields extending between the receding projectile and target fragments are assumed to be confined to a narrow color flux tubes of some finite transverse area $A_\perp \sim 1\text{fm}^2$. The field is furthermore assumed to be a constant along the beam direction with a strength $E \sim gQ/A_\perp$, where Q is the net “charge” on the projectile. In *Abelian* models [51,74,75,77] Q is treated simply as an integer and the pair production rate per unit volume, $w_{1/2}(\sigma, m)$, for $q\bar{q}$ pairs of mass, m , is taken from the well-known Schwinger formula [32,21,73]

$$w_{1/2}(\sigma, m) = \theta(\sigma) \frac{\sigma}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_{m^2}^{\infty} dE_\perp^2 \exp(-n\pi E_\perp^2/\sigma) \approx \theta(\sigma) \frac{\sigma^2}{24\pi} , \quad (5.9)$$

with $\theta(x) = 0(1)$ for $x < (>)0$, and where the approximation holds for $\pi m^2/\sigma \ll 1$. In the above expression $\sigma = gE$ is the effective string tension, which phenomenologically is taken to be of the order of 1 GeV/fm.

Using eq.(5.9) one can estimate the rate with which the external “color” field is neutralized due to $q\bar{q}$ production as in ref.[51]. First note that conservation of energy requires that in order to produce a pair, each with transverse energy $m_\perp$ and zero longitudinal momentum, that pair must separate by at least a distance $r_c \sim 2m_\perp/\sigma$. The minimum volume for pair production is thus $\delta V \sim 2A_\perp m_\perp/\sigma$. Once a pair has been produced in this finite volume element the field between them is reduced by $\delta E = g/A_\perp$ if we assume that all flux lines are confined to the original flux tube. With this crucial model assumption one then estimates [51]

$$\frac{dE(t)}{dt} \approx -\frac{g}{A_\perp} w_{1/2} \delta V \propto \sigma \langle m_\perp \rangle = -aE(t)^{3/2} , \quad (5.10)$$

where a is a positive constant. The power 3/2 follows from dimensional considerations, since locally $E(t)$ is the only dimensioned quantity that can set the time scale (also $\langle m_\perp \rangle \propto E^{1/2}$ for the same reason). Writing $E(t) = gQ(t)/A_\perp$, the solution of (5.10) is simply

$$Q(t) = Q_0/(1 + t/\tau_{1/4}(Q_0))^2 , \quad (5.11)$$

with

$$\tau_{1/4} = \tau_1 Q_0^{-1/2} , \quad (5.12)$$

in terms of the characteristic time $\tau_1 \sim 1 \text{ fm}/c$ required to neutralize 3/4 of the field strength in an elementary string with unit charge. The most important feature of the above solution is that it shows that the neutralization time of strong ($Q_0 \gg 1$) Abelian fields *decreases* as $Q_0^{-1/2}$. This is good news from the point of view of creating a quark gluon plasma. The faster the fields neutralize the more time will there be for the quarks and gluons to come into chemical and thermal equilibrium and thus to produce interesting signatures [14,15] of this new state of matter.

Non-Abelian SU(N) Case

Encouraged by the above Abelian analysis we turn next to investigate non-Abelian aspects of the problem. In particular, how do we treat the non-Abelian character of the charges of both quarks and gluons? How do quarks and gluons compete in the neutralization process? How close is the resulting quark-gluon plasma to local equilibrium conditions at time $\tau_{1/4}$? The answer to these questions was obtained by calculating the pair production rates for SU(N) in the one-loop $O(\hbar)$ approximation in ref.[50].

We start once more with the Heisenberg field equations for quarks and gluons,

$$(i\gamma^\mu D_\mu - m_f)\psi_f = 0 , \quad (5.13)$$

$$[D^\mu, F_{\mu\nu}] = -gj_\nu , \quad (5.14)$$

where $j_\nu = \sum_f (\bar{\psi}_f \gamma_\nu t_a \psi_f) t_a$ and f labels the quark flavors.

A covariant-constant field [78], which satisfies eq.(5.14) in the source free region, is of the form

$$F_{\mu\nu} = \langle F_{\mu\nu} \rangle U(x) n_a t_a U^\dagger(x) , \quad (5.15)$$

where $\langle F_{\mu\nu} \rangle$ is independent of x_μ , and n_a is an $N^2 - 1$ dimensional color vector. The unitary matrix $U(x) = \exp(i\theta_a(x)t_a)$ implements arbitrary local gauge transformations. The covariant derivative is given by $D_\mu = U(x)(\partial_\mu + \frac{1}{2}ig\langle F_{\mu\nu} \rangle x^\nu n_a t_a)U^\dagger(x)$.

Since $n_a t_a$ is Hermitian, there exists an x_μ independent matrix $V \in \text{SU}(N)$ such that $V n_a t_a V^\dagger$ is diagonal. We can thus transform the external field into diagonal form by making a gauge transformation $G = V^\dagger U^\dagger(x)$ under which $D_\mu \rightarrow G D_\mu G^\dagger$. Finally, one may now again use most conveniently the Cartan-Weyl basis of SU(N) as described in Sec. 4.3.

In the gauge which diagonalizes $\langle A_\mu \rangle$, the external field can be expressed in terms of $N - 1$ Abelian components, $\vec{H}^\mu = (H_1^\mu, \dots, H_{N-1}^\mu)$, as

$$\langle A^\mu \rangle = \sum_{i=1}^N H_i^\mu h_i \equiv \vec{H}^\mu \cdot \vec{h} , \quad (5.16)$$

where $\vec{h} \equiv (h_1, \dots, h_{N-1})$. To take quantum fluctuations around this external field into account, the gluon field operator can then be written as

$$A_\mu = \langle A_\mu \rangle + B_\mu = \vec{H}_\mu \cdot \vec{h} + B_\mu , \quad (5.17)$$

where B_μ represents the quantum fluctuations.

The physical significance of $\vec{\epsilon}_i$ can be seen from eq.(5.13) by considering the equation of motion for the tranformed quark field, $\psi' \equiv G^\dagger \psi$. Eq.(5.13) then reduces to the set of equations

$$(\gamma_\mu(i\partial^\mu + g\vec{\epsilon}_c \cdot \vec{H}^\mu) - m_f)\psi'_c = O(B\psi') . \quad (5.18)$$

The approximation of neglecting higher order quantum fluctuations beyond the one-loop order is equivalent to neglecting the $O(B\psi')$ terms on the right hand side of eq.(5.18). Thus, in the one-loop approximation, the equations for the N quarks (of each flavor) in the prime basis decouple and reduce to Abelian type equations where $\vec{H}^\mu$ plays the role of an effective electromagnetic field that couples to quarks with effective "charges" $g\vec{\epsilon}_c$. Since we already know the pair creation rate, $w_{\frac{1}{2}}(eF^{\mu\nu}; m)$, of fermions in an external Abelian field $F^{\mu\nu}$, eq.(5.9), we can immediately write down the pair creation rate per unit volume of ψ'_c quarks of flavor f as [50]

$$w_{q_c, f} = w_{\frac{1}{2}}(g\vec{\epsilon}_c \cdot \vec{F}^{\mu\nu}; m_f) , \quad (5.19)$$

where $\vec{F}^{\mu\nu} = \partial^\mu \vec{H}^\nu - \partial^\nu \vec{H}^\mu$ is a constant $N - 1$ dimensional vector.

Turning next to gluons, the equations of motion for B^μ in the one one-loop approximation are obtained by linearizing eq.(5.14) in B^μ . It is most convenient to expand B^μ in the Cartan-Weyl basis as

$$B^\mu \equiv B_a^\mu t_a = \vec{C}^\mu \cdot \vec{h} + \sum_{i \neq j=1}^N W_{ij}^\mu e_{ij} . \quad (5.20)$$

Inserting eq.(5.17) into eq.(5.14) and using the Cartan-Weyl expansion, eq.(5.20), for B^μ leads to the following equations of motion for the fluctuations $\vec{C}^\mu$ and W_{mn}^μ in the (linearized) one-loop approximation

$$\partial_\mu(\partial^\mu \vec{C}^\nu - \partial^\nu \vec{C}^\mu) = 0 , \quad (5.21)$$

$$(D_{mn})_\mu(D_{mn}^\mu W_{mn}^\nu - D_{mn}^\nu W_{mn}^\mu) - (W_{mn})_\mu[D_{mn}^\mu, D_{mn}^\nu] = 0 , \quad (5.22)$$

where the effective covariant derivative D_{mn}^μ is given by

$$D_{mn}^\mu = \partial^\mu - ig\vec{\eta}_{mn} \cdot \vec{H}^\mu . \quad (5.23)$$

We therefore see that the Abelian fluctuations, $\vec{C}^\mu$, obey free field equations where as the non-Abelian fluctuations, W_{mn}^μ , obey Abelian vector field equations in the external field, $\vec{H}^\mu$, with an anomalous magnetic moment coupling [79]. Note that $[D_{mn}^\mu, D_{mn}^\nu] = ig\vec{\eta}_{mn} \cdot \vec{F}^{\mu\nu}$. Obviously these equations decouple in the present approximation. The effective “charge” of the W_{mn}^μ gluon is given by $g\vec{\eta}_{mn}$. Pair production in SU(N) covariant constant fields is thus equivalent to $N(N-1)/2$ different SU(2) problems. Therefore, the pair creation rate per unit volume of $W_{mn}W_{nm}$ gluon pairs can be calculated from the known [78,79] rate, $w_1(gF^{\mu\nu})$, of vector mesons for SU(2) covariant constant fields as

$$w_{gmn} = w_1(g\vec{\eta}_{mn} \cdot \vec{F}^{\mu\nu}) , \quad (5.24)$$

where $\vec{F}^{\mu\nu}$ is the same external covariant constant SU(N) field as in eq.(5.19) and the spin-1 rate is given by [80]

$$w_1(\sigma) = \theta(\sigma) \frac{\sigma}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^\infty dp_\perp^2 \exp(-n\pi p_\perp^2/\sigma) = \frac{1}{2} w_{\frac{1}{2}}(\sigma; 0) . \quad (5.25)$$

The case of particular interest in phenomenological applications [74,76] corresponds to constant color electric fields created between interacting partons in high-energy collisions. For that case $\vec{F}^{30} = -\vec{F}^{03} = \vec{E} = \vec{Q}E_0$, where $E_0 = g/A_\perp$ for a flux tube of transverse area $A_\perp$ and $\pm\vec{Q}$ are the effective color charges of the projectile and target. The elementary $q\bar{q}$ string corresponds in this picture to $\vec{Q} = \vec{\epsilon}_c$. The adjoint $g\bar{g}$ string corresponds to $\vec{Q} = \vec{\eta}_{ij}$. Note that $\vec{\eta} \cdot \vec{\eta}/(\vec{\epsilon} \cdot \vec{\epsilon}) = 2N/(N-1)$.

In high energy nuclear collisions [76] or very high energy hadronic collisions, multiple soft gluon exchange may lead to large effective charges. If $\mathcal{N}$ gluons are exchanged with random charges, then $\langle Q^2 \rangle = \mathcal{N}(1 + N^{-1})^{-1}$ since only $N(N-1)$ of the $N^2 - 1$ gluons are “charged” with $\vec{\eta}_{ij} \cdot \vec{\eta}_{ij} = 1$. Of course, such color electric fields are unstable against pair production, as we saw above. If one assumes that the transverse area $A_\perp$ remains the same as for the elementary flux tube [50], the pair production rate per unit volume of massless quark-antiquark and gluon pairs is given from eqs.(5.19,5.24) by

$$w_q = \frac{N_f g^2}{24\pi} \sum_{c=1}^N (\vec{\epsilon}_c \cdot \vec{E})^2 = \frac{Q^2}{2} N_f w_0 , \quad w_g = \frac{g^2}{48\pi} \sum_{i>j=1}^N (\vec{\eta}_{ij} \cdot \vec{E})^2 = \frac{Q^2}{4} N w_0 , \quad (5.26)$$

where $w_0 = (gE_0)^2/(24\pi)$ and N_f is the number of quark flavors such that $\pi m_f^2/\sigma \ll 1$.

One can now answer another one of the initial questions. The ratio of quarks to gluons just after the color field is neutralized is $(q/g)_{neut} \approx 2N_f/2N = 1$ for $N = N_f = 3$.

In comparison, local equilibrium at zero chemical potential is characterized by $(q/g)_{\text{equil}} = (\frac{3}{4} N N_f)/(N^2 - 1) = 0.84$. Thus, this ratio is surprisingly close to equilibrium in spite of the fact that non-equilibrium tunneling was involved. Note however that no “neutral” gluons are produced via pair production.

Generalizing the discussion from the Abelian case, one can estimate the rate of neutralization for strong covariant constant fields ($Q^2 \gg \eta^2 = 1$) in the non-Abelian case as well, see refs.[22,50] for details. Thus, the characteristic time required to neutralize $3/4$ of the initial color field is obtained,

$$\tau_{1/4}(Q_0) = 0.36 \tau_1 Q_0^{-1/2} , \quad (5.27)$$

where Q_0 denotes the length of the Cartan charge vector $\vec{Q}(t=0)$ chosen along one of the weight or root vectors for simplicity. Again we observe the characteristic $Q_0^{-1/2}$ behavior for the decrease of the neutralization time with the initial charge as in the Abelian case. The factor 0.36 represents the combined effect of both $q\bar{q}$ and $g\bar{g}$ production and significantly shortens the neutralization time over the Abelian case where only $q\bar{q}$ contribute.

There are several interesting phenomenological consequences of the above results in connection with ultra-relativistic nuclear collisions [50,31]. First, most of the quarks and gluons produced in the neutralization of strong color fields may appear at proper times an order of magnitude smaller than in elementary pp or e^+e^- collisions (if multiply charged strings in the above sense do exist at all). Obviously shorter times imply that plasmas with higher initial energy densities can be produced [51] and chances for local equilibration are greater. On the other hand, the equilibration time due to ordinary kinetic phenomena could be long compared to the neutralization time [81]. Fortunately, the color neutralization mechanism leads to initial conditions that are not far from local equilibrium. First, as we noted, in strong fields the neutralization mechanism leads to production of quarks and gluons at comparable rates (for SU(3)). Furthermore, u,d,s quarks are produced with nearly the same abundance, since their masses become irrelevant. The chemical composition of this non-equilibrium plasma, therefore, is not far from that in local equilibrium. Secondly, the distributions of transverse momenta from eqs.(5.9 ,5.25) are approximately exponential as in local equilibrium although gluons materialize with $\sim 30\%$ larger transverse momentum than quarks due to their larger effective charge. Thus, one concludes that the non-equilibrium quantum tunneling dynamics and the specific non-Abelian features of gluon pair production may play an important role in creating plasma initial conditions close to local equilibrium at very early times in the collision.

6. SUMMARY AND OUTLOOK

In this review we have shown how to set up a kinetic theory for the quark-gluon plasma, based on QCD as a quantum field theory. Wigner operators generating the quantum mechanical phase space distribution functions for quarks and gluons have been defined in a way which ensures special relativistic and gauge covariance of the theory. Exact equations of motion for these Wigner operators were derived, and it was seen that the equations for quarks and for gluons are structurally very similar (Chaps. 3 and 4).

Differences occur in the color and spin sectors: The quark Wigner operator is in the fundamental representation of $SU(3)$, i.e. it is a color 3×3 matrix which can be decomposed into color singlet and color octet components. The gluon Wigner operator is best defined in the adjoint representation, i.e. it is a color 8×8 matrix, whose irreducible components under color $SU(3)$ transformations include a color singlet, a symmetric and an antisymmetric color octet, two color decuplets, and a 27-plet. In both cases the equations of motion couple all irreducible components among each other [82].

In the spin sector, the quark Wigner operator can be decomposed [41] into the 16 basis elements of the Dirac-Clifford algebra, i.e. it contains scalar, pseudoscalar, vector, axial vector, and tensor components. The pseudoscalar and axial vector components couple to each other, but decouple from the rest for suitable initial conditions (see the Appendix for related questions in nuclear matter transport theory). For gluons the decomposition of the irreducible spin components is still an open question; in this case the spin structure is coupled to the Lorentz-group structure of the Wigner operator and so far has not been analyzed in detail. As a consequence, no fully specified expression for the gluon color current in terms of the gluon Wigner operator, which presumably also contains the gluon spin contribution, has been found yet.

The equations of motion for the Wigner operators are very complicated and still contain all the quantum mechanical information carried by the quark and gluon fields. To obtain an irreversible and statistical description, the Wigner operator has to be appropriately averaged, i.e. ensemble expectation values have to be taken and eventually strongly oscillating ("Zitterbewegung"-) components have to be integrated out [13]. This step has not yet been performed in a systematic and controlled way. The present analysis restricts itself to a semiclassical mean-field (Hartree) approximation: in the case of quarks, gluons affect their dynamics only through the mean color field; in the case of gluons themselves, the Wigner function is separated into a mean-field and a fluctuating part, and the dynamics of the fluctuations is governed by the mean field in a way very similar to the quark case, with all higher order correlations between gluonic fluctuations and the mean field being neglected.

In this semiclassical mean-field limit, the resulting equations of motion for the quark

and gluon Wigner function very closely resemble the Vlasov equation for a classical electromagnetic plasma, except for the additional color and spin structure. However, even the latter can be understood in a purely classical model (where spin and color are macroscopic quantities), as shown in Chap. 2. The quantum kinetic theory in its semiclassical mean-field limit exactly maps on this classical description (see Chap. 3).

The semiclassical mean-field limit is a sufficiently powerful approximation for a study of collective colored plasma oscillations and their dispersion relations, which can be achieved in a rather straightforward way using a linear response analysis (see Sec. 4.4). One reproduces the well-known results from finite temperature QCD, at least the gauge invariant leading terms of a high-temperature expansion in thermal QCD. Missing, however, are any imaginary parts in the dispersion relation which would lead to damping and equilibration, due to the neglect of 2-body correlations ("collision terms") in this approximation.

Much future attention will therefore have to focus on the systematic evaluation of correlations between quarks and gluons in order to derive the collision terms (cf. ref.[83] for an early attempt). We have discussed some fragmentary and not yet very systematic approaches to this problem in Sec. 4.5 and Chap. 5, but pointed out the problems of gauge dependence which one often encounters in these *ad hoc* attempts. There we also explained difficulties related to the derivation of source terms for typical quantum processes like quark and gluon pair production by the mean color electric field in the plasma. Much remains to be done here.

Another point that is still not completely clear is the question, to what extent the approximation schemes mentioned in this review are of a purely perturbative nature, and how one would include nonperturbative phenomena, e.g. in the collision terms. We discussed the semiclassical expansion (" Δ -expansion"), which looks like an expansion in powers of $\hbar$ (see Sec. 3.4), and the Abelian Dominance Approximation. - Note, however, that the Δ -expansion with each order in $\Delta \sim \hbar \partial_p \cdot \mathcal{D}$ automatically brings in an additional power of g through the non-Abelian commutator residing in the covariant derivative. Thus, apart from the considerations justifying the Δ -expansion in Secs. 3.4 and 4.2, it may alternatively be viewed as the expansion in powers of g which keeps covariance at every order. - Due to the basically non-perturbative nature of QCD (perhaps even in the quark-gluon plasma phase [63]), the development of powerful non-perturbative methods to deal with the quantum kinetic equations would turn them into an even more useful practical tool for the dynamical description (of the early stages) of relativistic nuclear collisions.

However, even if such practical goals still appear to lie some time in the future, the conceptual features of the quantum kinetic approach are nevertheless very interesting. Once carried through to the end, it will hopefully yield a set of translation rules between finite temperature field theory concepts and those of transport theory near equilibrium,

thereby casting some of the abstract notions of thermal field theory into the more intuitively understandable language of classical plasma physics. The non-equilibrium features of the kinetic approach in turn will then provide a valuable guiding line for the non-equilibrium generalization of thermal field theory. This will have important conceptual and practical applications also in other fields of theoretical physics besides the study of highly relativistic nuclear collisions and of matter under extreme conditions of (energy) density and pressure in general. We mention for example the dynamical evolution of the early universe, where non-equilibrium particle production mechanisms play an important role in understanding baryon number generation and reheating of the universe after the initial inflationary phase.

Appendix: NUCLEAR MATTER TRANSPORT EQUATIONS

The investigation of nuclear matter properties at extreme densities and temperatures is one of the main motivations for studying high-energy heavy-ion collisions. A central problem is to find a link between the observed fragment distributions and the underlying nuclear equation of state also at energies where one does not yet expect the underlying parton degrees of freedom to play an essential role [84]. Furthermore, the role of a hadronic matter background in reactions leading to quark-gluon plasma formation is not at all understood and deserves kinetic studies of its own right.

The results we present in the following are based on ref.[85], where also a more detailed discussion of their phenomenological consequences can be found. Resulting semi-classical transport equations were applied e.g. in refs.[86] and, furthermore, collision terms were recently derived in ref.[87] based the one-boson-exchange approximation (we refer the reader to these works for further references on this field, which is currently studied intensely by several groups). As will become obvious below, the one essential simplification encountered when studying as an example the Walecka model [88] of hadronic interactions (in the mean field approximation) is that it is not a gauge theory like QCD. A prominent disadvantage of this effective theory, however, are its disturbingly large coupling constants obtained through phenomenological fits. This in turn raises doubts about the validity of developing perturbative nuclear matter transport theory beyond the mean-field limit.

Anyway, we presently describe a selfconsistent relativistic quantum transport theory for interacting baryons and mesons. As the basis of such a theory the Walecka model (“quantum” hadrodynamics, QHD) is taken, in which nucleons are coupled to pions, scalar, vector, and ρ -mesons [88]. In QHD the model parameters are constrained by static nuclear matter properties. Unfortunately, this does not completely determine all of them, and much freedom remains in the form of the equation of state at high energy densities. The hope is that nuclear collisions will provide further constraints on those parameters and hence on the sought-after equation of state.

As shown in ref.[85] the transport theory based on the Walecka model connects a number of features which are important in the analysis of nuclear collisions. In particular it naturally leads to momentum dependent forces that were expected [89] and shown in refs.[90] to influence significantly the collective flow properties and the pion yields.

A consistent description of nuclear matter at low excitation energies is provided by the QHD Lagrangian:

$$\begin{aligned}\mathcal{L} = & \bar{\psi} [i\gamma^\mu D_\mu - m(x)] \psi \\ & + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) - \frac{1}{4} w^{\mu\nu} w_{\mu\nu} + \frac{1}{2} m_v^2 v_\mu v^\mu\end{aligned}$$

$$-\frac{1}{4}\vec{b}_{\mu\nu} \cdot \vec{b}^{\mu\nu} + \frac{1}{2}m_\rho^2 \vec{\rho}_\mu \cdot \vec{\rho}^\mu + \mathcal{L}_{\pi,\Delta} , \quad (.1)$$

where ψ denotes the 8-component nucleon field, the neutral and charged vector meson field strengths are given by

$$w^{\mu\nu} = \partial^\mu v^\nu - \partial^\nu v^\mu ,$$

$$\vec{b}^{\mu\nu} = \partial^\mu \vec{\rho}^\nu - \partial^\nu \vec{\rho}^\mu$$

respectively, and the SU(2)-covariant derivative is

$$D^\mu(x) = \partial^\mu + ig_v v^\mu(x) + ig_\rho \frac{1}{2} \vec{\tau} \cdot \vec{\rho}^\mu(x) = \partial^\mu + iu^\mu(x) .$$

The field $u^\mu(x)$ is the sum of the isoscalar and isovector four-vector fields. The space-time dependent effective nucleon mass is given by

$$m(x) = m_N - g_s \phi(x) .$$

Neglecting the π and Δ Lagrangian, the equations for the field operators are

$$i\gamma_\mu D^\mu(x) - m(x) \psi(x) = 0 , \quad (.2)$$

$$\partial_\mu \partial^\mu + \frac{\delta}{\delta \phi} U(\phi) = g_s \bar{\psi} \psi , \quad (.3)$$

$$\partial_\mu w^{\mu\nu} + m_v^2 v^\nu = g_v \bar{\psi} \gamma^\nu \psi , \quad (.4)$$

$$\partial_\mu \vec{b}^{\mu\nu} + m_\rho^2 \vec{\rho}^\nu = g_\rho \bar{\psi} \gamma^\nu \frac{1}{2} \vec{\tau} \psi . \quad (.5)$$

In the mean-field approximation the source of the mean scalar meson field is the scalar baryon density, $\rho_s = \langle \bar{\psi} \psi \rangle$. The vector meson field couples to the four-current, $j^\mu = \langle \bar{\psi} \gamma^\mu \psi \rangle$, of the nucleons. The ρ -meson is generated by the nucleon isospin current, $j^\mu = \langle \bar{\psi} \gamma^\mu \frac{1}{2} \vec{\tau} \psi \rangle$.

In symmetric nuclear matter the Dirac-equation yields the dispersion relation for the nucleons

$$(E - g_v v_0)^2 = (\vec{p} - g_v \vec{v})^2 + (m - g_s \phi)^2 .$$

Observe that the effective energy, mass, and momentum depend strongly on the local scalar and vector densities of the system. Even if the spatial part of the vector field, $\vec{v}$, is neglected one obtains an effective momentum which is not on mass-shell.

The inclusion of delta and pion degrees of freedom would require the specification of $\mathcal{L}_{\pi,\Delta}$. However, a satisfactory *relativistic* treatment of pions and Δ 's consistent with both low energy pion nucleus scattering data and nuclear saturation properties is not yet known and remains one of the outstanding problems in nuclear theory as emphasized in ref.[88] and, thus, also for a QHD transport theory.

On the other hand, the role of momentum dependent forces and quantum effects in nuclear transport theory can be studied already in the simplified QHD model without pions and deltas [85]-[87]. For that purpose one derives the equation of motion for the nucleon Wigner operator [13] in the Heisenberg picture, cf. Chap. 3,

$$\hat{W}_{\alpha\beta}(x, p) = \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \bar{\psi}_\beta(x + \frac{1}{2}y) \psi_\alpha(x - \frac{1}{2}y) . \quad (.6)$$

Here the indices, α and β , are double indices for spin and isospin, and the Wigner operator is an (8×8) -matrix. The ensemble averaged currents can be expressed in terms of $\hat{W}$ by

$$j_\Gamma(x) = \int d^4 p \text{Tr} \Gamma \langle \hat{W}(x, p) \rangle , \quad (.7)$$

where the brackets indicate the ensemble average. The trace has to be taken over spin and isospin indices, Γ denotes products of spin- and isospin matrices.

Next, we consider the Wigner operator in the mean-field approximation, i.e. we neglect quantum fluctuations of the meson fields and treat them as classical c-number fields generated by mean sources of the form (.7). Whereas the important collision terms can be derived [13,87] from terms arising from meson field fluctuations, there are no collision terms presently nor are there any loop corrections like vacuum polarization and self-energy corrections. Hence, one truncates a hierarchy of equations for the nucleon Wigner function at the one-body level.

The quantum equation of motion for the nucleon Wigner function follows, similarly as in Chap. 3, from the Dirac equation (.2) by noting that

$$[i\gamma_\mu D^\mu(x_-) - m(x_-)] \hat{\rho}(x_-, x_+) = 0 ,$$

where $\hat{\rho}_{\alpha\beta}(x_-, x_+) = \bar{\psi}_\beta(x_+) \psi_\alpha(x_-)$ is the Wigner kernel, and $x_\pm = x \pm \frac{1}{2}y$. Integrating with $\int d^4 y \exp(-iy \cdot p)$ and replacing $i\partial_y$ by $-p$ and y by $i\partial_p$, the operator in the square brackets can be removed from the integrand to obtain finally [85]

$$[\gamma \cdot K - M(x)] W(x, p) = 0 , \quad (.8)$$

where

$$K^\mu = \Pi^\mu + \frac{1}{2}i\hbar\nabla^\mu = [p^\mu - \cos(\hbar\Delta)u^\mu] + \frac{1}{2}i\hbar[\partial_x^\mu + \frac{2}{\hbar}\sin(\hbar\Delta)u^\mu(x)] , \quad (.9)$$

$$M = m_R - i\hbar m_I = \cos(\hbar\Delta)m - i\sin(\hbar\Delta)m , \quad (.10)$$

and where $\Delta = \frac{1}{2}\partial_x \cdot \partial_p$ is the “triangle operator” with ∂_x acting only on the scalar and vector fields. In eqs.(.8-.10) we have explicitly inserted $\hbar$ to facilitate the study of the semiclassical limit, and $W \equiv \langle \hat{W} \rangle$ is the ensemble averaged Wigner function. Since Δ has dimensions of inverse action, the semiclassical limit is obtained, as before, by expanding the operators K^μ and M in powers of the triangle operator.

In order to extract the Vlasov equation from eq.(8) without cumbersome spin-dependent effects, only locally spin and isospin saturated systems were studied in ref.[85]. In this case the ρ -field is zero, W is diagonal in isospin, the pseudo-scalar and the pseudo-vector densities vanish. The resulting spin decomposition is [41]

$$\langle W \rangle = F + \gamma_\mu V^\mu + \frac{1}{2} \sigma_{\mu\nu} S^{\mu\nu} , \quad (.11)$$

where $F = \frac{1}{4} \text{tr} \langle \hat{W} \rangle$, $V^\mu = \frac{1}{4} \text{tr} \langle \gamma^\mu \hat{W} \rangle$, $S^{\mu\nu} = \frac{1}{4} \text{tr} \sigma^{\mu\nu} \langle \hat{W} \rangle$, and tr denotes the trace with respect to the spinor indices only. Taking appropriate traces $\Gamma \hat{W}$, where Γ runs over the basis of the Clifford algebra, one obtains a decomposition of eq.(8). It finally amounts to 22 coupled real equations [85].

In the classical limit, formally $\hbar \rightarrow 0$, this system of equations reduces to

$$S^{\mu\nu} = 0 , \quad (.12)$$

$$V^\mu = \frac{p^\mu - g_v v^\mu}{m} F , \quad (.13)$$

$$[(p - g_v v)^2 - m^2] F = 0 , \quad (.14)$$

$$(p_\mu - g_v v_\mu)(\partial_x^\mu + g_v \Delta v^\mu) \frac{F}{m} + \Delta m F = 0 \quad (.15)$$

Obviously, in the classical limit the only independent density is $F(x, p)$ for spin saturated systems. Eq.(.14) shows that it is non-zero only if the expression in brackets vanishes, i.e. only if the energy is constrained locally to be

$$p_0(x) = g_v v_0(x) \pm \{ [\vec{p} - g_v \vec{v}(x)]^2 + m^2(x) \}^{\frac{1}{2}} , \quad (.16)$$

as expected. Note that while $p^2 \neq m^2$, the *kinetic* momentum, $k^\mu = p^\mu - g_v v^\mu$, is locally on shell, $k^2 = m^2(x)$. It is therefore convenient to define the phase-space density involving the kinetic momentum via

$$\tilde{F}(x, k) = \frac{m_N}{m(x)} F(x, k + g_v v(x)) = \frac{m_N}{m(x)} F(x, p) . \quad (.17)$$

In consequence of

$$\partial_{x; p=\text{const}}^\mu = \partial_{x; k=\text{const}}^\mu - g_v (\partial_x^\mu v_\nu) \partial_k^\nu ,$$

where ∂_k is the partial derivative with respect to the kinetic momentum, k , and $\partial_{x; p, k=\text{const}}$ is evaluated with respectively p or k being kept constant, this Lorentz scalar density satisfies a relativistic Vlasov equation

$$k_\mu (\partial_x^\mu - g_v w^{\mu\nu} \partial_\nu^k) + m (\partial_x^\nu m) \partial_\nu^k \tilde{F}(x, k) = 0 . \quad (.18)$$

In the *nonrelativistic limit*, where $k_0 \approx m_N$, eq.(.18) reduces to the Vlasov part of the classical Vlasov-Uehling-Uhlenbeck equation

$$(\partial_t + \frac{\vec{k}}{m_N} \cdot \vec{\nabla}_x - \vec{\nabla}_x U \cdot \vec{\nabla}_k) f(\vec{x}, \vec{k}, t) = C(f)$$

with the identification of $U(x) = g_v v_0(x) + m(x) - m_N$. However, in the *classical limit* generally there is an important difference: the effective potential, U , acquires a momentum dependence due to relativistic effects as expected. We note again that the absence of a collision term $C(f)$ in eq.(.18) is due to the mean-field approximation used here.

The above transport equations can be applied to describe nuclear collisions and are most often evaluated numerically using so-called molecular dynamics codes as discussed in refs.[84,85]. (Note, by comparison, how much more complicated the quark-gluon plasma transport equations are. In fact, they have been studied numerically only for very idealized situations so far [31].)

Quantum corrections to the average single-particle behavior as described by the semiclassical QHD transport equations, i.e. neglecting correlations, can be calculated using the zeroth order solution as input and by expanding the full set of quantum equations in powers of the Δ -operator. In a practical calculation one would monitor the size of the first order corrections to see their order of magnitude. The corrections will depend on the location in phase-space and, in this way, one could map out the “safe” regions where hopefully quantum effects can be neglected and the classical results trusted (cf. [49]).

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