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**$N = 2$ SUPERCONFORMAL MODELS, LANDAU-GINSBURG
HAMILTONIANS AND THE ϵ -EXPANSION**

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Abstract

The anomalous dimensions of a class of operators, operator product expansions and the central charge, c , are calculated in a family of $N = 2$ supersymmetric two-dimensional Landau-Ginsburg models. The results allow the identification of these theories with $N = 2$ minimal superconformal models. A key role is played by the $N = 2$ non-renormalisation theorem.

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Landau-Ginsburg Hamiltonians have played an important role in the discussion of critical phenomena associated with second-order phase transitions in systems for which two phases coexist at the critical point[1]. The corresponding Landau-Ginsburg (LG) Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(\partial_\mu \phi)^2 + g \frac{\phi^4}{4!} \quad (1)$$

This phenomenological Hamiltonian can be thought of as arising from applying the renormalisation group to the underlying microscopic Hamiltonian. Generalisations of this Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2}(\partial_\mu \phi)^2 + g \frac{\phi^{2n}}{2n!} \quad , \quad n \geq 3 \quad (2)$$

have been used in the study of two-dimensional critical phenomena with particular emphasis on tricritical phenomena ($n = 3$)[2,3]. For general n the potential of (2) describes a system where n phases may coexist. Systems at a critical point are characterised by long range fluctuations which are a manifestation of the conformal symmetry which is present precisely at this point. Since the LG Hamiltonians are not classically conformally invariant, it follows that there must be essentially quantum-mechanical fixed points if LG models are to describe critical behaviour. Computations with the Hamiltonians (2) in perturbation theory about $g = 0$ are plagued with off-shell infra-red divergences. At zeroth order in this perturbation theory the dimension of ϕ is zero and hence all powers of ϕ are equally relevant as operators. While $g = 0$ perturbation theory will provide corrections to this behaviour it will not radically alter the overall picture. In fact, the presumed infra-red fixed point of the theory defined by the Hamiltonian (2) must be outside of $g = 0$ perturbation theory. In order to use the Hamiltonian (2) to describe critical phenomena it is therefore necessary to provide a scheme of calculation which has the fixed point within its range.

Zamolodchikov [4] has suggested that there exists a fixed point of the LG Hamiltonian (2) at which the theory corresponds to the n th member of the minimal series of BPZ [5], i.e. the one with central charge $c = 1 - \frac{6}{(n+1)(n+2)}$. In addition, Zamolodchikov proposed that the composite operators ϕ^m in the LG model should be identified with certain of the operators $\Phi_{(r,s)}$ occurring in the minimal models,

$$\phi^m \approx \begin{cases} \Phi_{(m,m)} & \text{for } m = 0, 1, \dots, n-1, \\ \Phi_{(m-n+1, m-n+2)} & m = n, \dots, 2n-2 \end{cases} \quad (3)$$

Zamolodchikov advances the following suggestive argument in favour of the identification of the LG model with the minimal model. He assumes that there exists a calculational scheme for the LG model (within whose range the infra-red fixed point lies) and that the composite operators ϕ^m can be defined within this scheme to obey the operator product expansion

$$\begin{aligned} \phi^k(x)\phi(0) \approx & \text{terms involving } \phi^l, \quad l \leq k \quad + \\ & \frac{1}{|x|^{d_k+d_1-d_{k+1}}} \phi^{k+1}(0) \quad + \\ & \text{less singular terms.} \end{aligned} \quad (4)$$

With the identifications of (3), this operator product expansion coincides with that of the n th minimal model. Further, the operator product of $\Phi_{(2,2)}$ (corresponding to ϕ) with $\Phi_{(n-1,n)}$ in the n th minimal model contains a descendant of $\Phi_{(2,2)}$, i.e. $\partial_\mu \partial^\mu \Phi_{(2,2)}$. It is therefore natural to identify this descendant with ϕ^{2n-1} so obtaining the equation of motion for the LG model.

In refs.[4,6] the extension of these ideas to the case of LG models and conformal models with $N = 1$ supersymmetry is discussed. A similar identification of LG models and minimal models has been proposed [6] in the case of $N = 2$ supersymmetry. $N = 2$ supersymmetric Hamiltonians of LG type have been much studied recently [6,7,8,9] particularly in the context of trying to provide a more concrete realisation of Gepner's conjecture [10] that $N = 2$ sigma models on Calabi-Yau manifolds can be obtained from tensor products of minimal $N = 2$ superconformal models. These minimal models are assumed to be described by the $N = 2$ generalisation of the Hamiltonian (2) (equation (5) below), and tensor products are described by Hamiltonians with superpotentials which are particular polynomials of several chiral superfields. It has been observed that these polynomials are the ones whose vanishing specifies Calabi-Yau manifolds as complex submanifolds of $\mathbf{CP}^m$. Further interesting work has connected these ideas with catastrophe theory [7,8].

One method of defining a perturbation theory which contains the infra-red fixed point is the ϵ -expansion [1]. In a previous paper [11] this technique was applied to the Hamiltonian (2), and, although some features such as the number of relevant operators come out as expected, the results are not in agreement with the minimal model. The reader is referred to the paper [11] for more details and a discussion of these results.

In this paper the ϵ -expansion is applied to $N = 2$ supersymmetric LG Hamiltonians. In contrast to the $N = 0$ case it is found that the correspondence with minimal conformal models can be established. In particular, it will be shown that various conformal weights, the operator product expansion and the value of the central charge c coincide with those of certain minimal $N = 2$ supersymmetric models. In the derivation of these results essential use is made of the $N = 2$ non-renormalisation theorem (which is basically the same as the $N = 1, d = 4$ non-renormalisation theorem[12]). In general this theorem has been established only for perturbation theory around zero coupling and can be violated by non-perturbative effects. However, the ϵ -expansion can be set up in such a way that the non-renormalisation theorem continues to hold.

The $N = 2$ supersymmetric generalisation of (2) is

$$\mathcal{H} = - \int d^2x d^2\theta d^2\bar{\theta} \bar{\phi}\phi - (g \int d^2x d^2\theta \phi^n + c.c.) \quad (5)$$

where the minus sign is an artefact of our conventions. These conventions are as follows. Euclidean $N = 2$ superspace has real coordinates $(x^\mu, \theta^{\alpha i})$ where $\alpha = 1, 2$ is a spinor index and $i = 1, 2$ is an internal symmetry $SO(2)$ index. The supercharges obey the algebra

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2i\delta_{ij}(\gamma^a)_{\alpha\beta}P_a \quad (6)$$

where P_a is the momentum operator. The corresponding covariant derivatives $D_{\alpha i}$ obey

$$\{D_{\alpha i}, D_{\beta j}\} = 2(\gamma^a)_{\alpha\beta}\partial_a \quad (7)$$

It is convenient to work with the complex spinors $\theta^\alpha = \theta^{\alpha 1} + i\theta^{\alpha 2}$, $\bar{\theta}^\alpha = \theta^{\alpha 1} - i\theta^{\alpha 2}$ for which the corresponding covariant derivatives are

$$D_\alpha = \frac{1}{2}(D_{\alpha 1} - iD_{\alpha 2}); \quad \bar{D}_\alpha = \frac{1}{2}(D_{\alpha 1} + iD_{\alpha 2}) \quad (8)$$

they obey the algebra

$$\begin{aligned} \{D_\alpha, \bar{D}_\beta\} &= (\gamma^a)_{\alpha\beta} \partial_a \\ D_\alpha D_\beta &= \epsilon_{\alpha\beta} D^2, \quad \bar{D}_\alpha \bar{D}_\beta = \epsilon_{\alpha\beta} \bar{D}^2 \end{aligned} \quad (9)$$

which defines the operators D^2 and $\bar{D}^2$. The chiral fields appearing in the Hamiltonian (5) obey the constraint

$$\bar{D}_\alpha \phi = 0 \quad (10)$$

and the integration measures are defined by

$$\begin{aligned} \int d^4\theta V &\equiv \int d^2\theta d^2\bar{\theta} V = (D^2 \bar{D}^2 V)| \\ \int d^2\theta \phi &= (D^2 \phi)| \end{aligned} \quad (11)$$

where V is an arbitrary superfield, ϕ is a chiral superfield and the horizontal bar denotes a superfield evaluated at $\theta = 0$.

It will also be useful to consider using complex bases for the spinor and vector indices. As usual $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$ while for the spinors we define $\theta^+ = \theta^1 + i\theta^2$, $\theta^- = \theta^1 - i\theta^2$, $\bar{\theta}^+ = \bar{\theta}^1 + i\bar{\theta}^2$ and $\bar{\theta}^- = \bar{\theta}^1 - i\bar{\theta}^2$. The complex conjugate of θ^+ is $\bar{\theta}^-$. For the covariant derivatives we have $D_+ = \frac{1}{2}(D_1 - iD_2)$, $D_- = \frac{1}{2}(D_1 + iD_2)$, etc. so that the D-algebra in this basis is

$$\{D_+, \bar{D}_+\} = \partial_z, \quad \{D_-, \bar{D}_-\} = \partial_{\bar{z}} \quad (12)$$

with all other anti-commutators vanishing. Explicitly, the D 's are given by

$$D_+ = \frac{\partial}{\partial\theta^+} + \frac{1}{2}\bar{\theta}^+\partial_z, \quad \bar{D}_+ = \frac{\partial}{\partial\bar{\theta}^+} + \frac{1}{2}\theta^+\partial_z \quad (13)$$

and similarly for the D_- 's.

To set up the ϵ -expansion for the Hamiltonian (4) we take the dimension of space to be $d > 2$ but we retain the spinors as they are in two dimensions. This is therefore akin to the procedure of dimensional reduction used to regulate supersymmetric theories [13]. The starting Hamiltonian becomes

$$\mathcal{H} = - \int d^d x d^2\theta d^2\bar{\theta} \bar{\phi}\phi - g \left(\int d^d x d^2\theta \frac{1}{n!} \phi^n + c.c. \right) \quad (14)$$

The critical dimension d_c is defined to be the dimension where the infra-red divergences become logarithmic (as opposed to power behaved in $d < d_c$). This is the dimension in

which the dimension of the coupling g is zero. Since the (naive) dimension of ϕ is $\frac{1}{2}(d-2)$ the dimension of g is $d-1-\frac{n}{2}(d-2)$. Thus the critical dimension is

$$d_c = 2 + \frac{2}{n-2} \quad (15)$$

To carry out the ϵ -expansion one therefore starts in $d = d_c - \epsilon$ dimensions and sets $\epsilon = \frac{2}{n-2}$ at the end of the calculation.

The superspace Feynman rules for the Hamiltonian (14) are computed in the standard way [14]. They are

$$\text{propagators: } \Delta_{12} = \frac{1}{p^2} \delta_{12}$$

chiral (anti-chiral) vertices: a factor of $\bar{D}^2$ (D^2)

on each line but one leaving the vertex, but

omit a factor $\bar{D}^2$ (D^2) for each external line

$$\text{integrations: } \int \prod_{\text{external lines}} \frac{d^d p}{(2\pi)^d} \delta(\sum p) \prod_{\text{loops}} \frac{d^d k}{(2\pi)^d} \prod_{\text{vertices}} d^2 \theta d^2 \bar{\theta} \quad (16)$$

where $\delta_{12} = \delta^2(\theta_1 - \theta_2) \delta^2(\bar{\theta}_1 - \bar{\theta}_2)$ is the δ -function in the odd part of superspace.

It follows from these Feynman rules that the non-renormalisation theorem carries over straightforwardly to the perturbation expansion in g and ϵ . For any contribution to the effective action (free energy) from a Feynman graph the D-algebra can be carried out reducing the expression to an integral over all the odd coordinates at one of the vertices. It follows that one never encounters divergences which are integrals over the chiral subspace, so that the potential term in the Hamiltonian is not renormalised.

The theory is renormalised by making the following standard changes in the Hamiltonian,

$$\begin{aligned} \phi &\rightarrow \phi_o = Z^{\frac{1}{2}} \phi \\ g &\rightarrow g_o = \mu^{\frac{(n-2)\epsilon}{2}} Z_g g \end{aligned} \quad (17)$$

The non-renormalisation theorem then implies that

$$Z_g Z^{\frac{n}{2}} = 1 \quad (18)$$

Defining the β function and the anomalous dimension of ϕ , γ_ϕ , in the usual way by

$$\beta = \mu \frac{\partial}{\partial \mu} g \Big|_{g_o, \Lambda}, \quad \gamma_\phi = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z \Big|_{g_o, \Lambda} \quad (19)$$

where Λ is the cut-off, we obtain

$$\beta(g) = -\frac{(n-2)}{2} \epsilon g + n g \gamma_\phi(g) \quad (20)$$

using (17). At the fixed point, g^* , $\beta(g^*) = 0$, by definition, so we get

$$\gamma_\phi(g^*) = \frac{(n-2)\epsilon}{2n} = \frac{1}{n} \quad (21)$$

To compute the anomalous dimension of the operator ϕ^p we add to the Hamiltonian a chiral source term

$$\int d^2x d^2\theta K_p \frac{\phi^p}{p!} \quad (22)$$

which is again not renormalised. Therefore we have

$$Z_p = Z^{-\frac{p}{2}} \quad (23)$$

where $K_p \rightarrow K_{po} = Z_p K_p$ under renormalisation. The anomalous dimension of ϕ^p is

$$\gamma_{\phi^p} = -\mu \frac{\partial}{\partial \mu} \ln Z_p = p \gamma_\phi \quad (24)$$

At the fixed point this is

$$\gamma_{\phi^p}(g^*) = \frac{p}{n} \quad (25)$$

Clearly these results are exact to all orders in perturbation theory, i.e. all orders in g and ϵ .

It is instructive to find the fixed point and confirm the above results directly from Feynman graphs. The lowest order contribution to the wave-function renormalisation is given by the graph of figure 1. In momentum space this graph yields the contribution

$$g^2 k_1 \int \frac{d^d p}{(2\pi)^d} d^4 \theta_1 d^4 \theta_2 \{ \phi(-p, \theta_1) \bar{\phi}(p, \theta_2) \times \\ \int \prod_{i=1}^{i=(n-2)} \frac{d^d k_i}{(2\pi)^d} \prod_{i=1}^{i=(n-2)} \left\{ \frac{1}{(k_i)^2} \frac{1}{(p + \sum_{j=1}^{j=n-2} k_j)^2} \right\} \delta_{12} [D_1^2 \bar{D}_2^2 \delta_{12}]^{N-2} \} \quad (26)$$

to the effective action. The combinatoric factor is $k_1 = \frac{1}{(n-1)!}$. The D -algebra is easily evaluated using the relation $\delta_{12} D_1^2 \bar{D}_2^2 \delta_{12} = \delta_{12}$. The contribution (26) can be rewritten in the x -space form

$$k_1 g^2 \int d^d x_1 d^d x_2 d^4 \theta \phi(x_1, \theta) \bar{\phi}(x_2, \theta) \frac{1}{|x_1 - x_2|^{(d-2)(n-1)}} a^{n-1} \quad (27)$$

The x -space propagator is $a|x_1 - x_2|^{-(d-2)} \delta_{12}$ where $a \equiv (d-2)^{-1} \Omega_d^{-1}$ and Ω_d is the volume of the $(d-1)$ -sphere, $\Omega_d = \frac{2(\pi)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. This gives

$$k_1 g^2 \frac{\Omega_c}{\epsilon(n-2)} a^{n-1} + O(\epsilon) \quad (28)$$

with $\Omega_c \equiv \Omega_{d_c}$. The divergence can be extracted by using ϵ as a regulator parameter as well as an expansion parameter, as has been done in the non-supersymmetric case [1]. The wave-function renormalisation constant is therefore given at this order by

$$Z = 1 - k_1 g^2 \frac{\Omega_c}{\epsilon(n-2)} a^{n-1} \quad (29)$$

Using (18) and (19) we find that the β and γ functions are given at this order by

$$\begin{aligned} \beta(g) &= -\frac{(n-2)}{2} \epsilon g + \frac{n}{2} g^3 k_1 \Omega_c a^{n-1} \\ \gamma_\phi(g) &= \frac{k_1}{2} g^2 \Omega_c a^{n-1} \end{aligned} \quad (30)$$

The infra-red stable fixed point is therefore at

$$(g^*)^2 = \frac{(n-2)\epsilon}{k_1 n \Omega_c a^{n-1}} \quad (31)$$

Substituting the fixed point of (31) into the formula (30) for the anomalous dimension of ϕ we of course recover the result of (20).

We now compare these results for the LG model with the $N = 2$ minimal model which has central charge (in suitable units) $c = (1 - \frac{2}{n})$. The left-moving primary fields $\Phi_{j,m}$ of the $N = 2$ minimal models [15] are labelled by their conformal weights and $U(1)$ charges, the allowed values of these quantum numbers being

$$\begin{aligned} h_{j,m} &= \frac{j(j+1) - m^2}{n} \\ q &= \frac{m}{n} \end{aligned} \quad (32)$$

Similar statements hold for the right-moving fields. It has been proposed that the field ϕ^p of the LG model should be identified with $\Phi_{(p,p)}$ of the conformal model. The relation between the conformal weight and the anomalous dimension of a composite operator in the LG model can be determined by using the renormalisation group. The two-point function $\langle \phi \bar{\phi} \rangle$ behaves as $|x|^{-(d-2+2\gamma_\phi)} = |x|^{-2\gamma_\phi}$ while the two-point function of ϕ^p and $\bar{\phi}^p$ behaves as $|x|^{-(d-2+2\gamma_p)} = |x|^{-2\gamma_p}$. From conformal field theory we know that the two-point function for a primary field of weight $(h, \bar{h})$ is $z^{-2h} \bar{z}^{-2\bar{h}}$. For a spinless field $h = \bar{h}$ which gives a propagator function $|x|^{-4h}$. We conclude that $h = \frac{\gamma_\phi}{2}$ for ϕ and $h = \frac{\gamma_p}{2}$ for ϕ^p . From equation (25) we therefore compute h for the operator ϕ^p to be $\frac{p}{2n}$, $p = 1, 2, \dots, n-1$, which agrees with the weight of the primary field $\Phi_{(p,p)}$ calculated from (3). An $N = 2$ chiral conformal field has weights $(h, \bar{h})$ which are half of its left and right $U(1)$ charges. The $U(1)$ charges of the operators ϕ^p (given below) and their conformal weights derived above are indeed related in this manner.

To show that the operator product expansion in the LG model coincides with that of the minimal conformal model we consider, as an example, the product of $\phi^p(x, \theta, \bar{\theta})$ with

$\phi^q(0)$. Since these two operators are both chiral, and since $\bar{D}^2 \bar{\phi} \propto \phi^n$ by the equation of motion, it follows that only chiral operators can appear in their OPE. In addition ϕ has $U_V(1)$ charge $\frac{2}{n}$, so that the only operator that can occur in this product is ϕ^{p+q} . Thus we must have

$$\phi^p(x, \theta, \bar{\theta}) \phi^q(0) \approx C(x, \theta, \bar{\theta}, g, \mu) \phi^{p+q} + \dots \quad (33)$$

Using standard renormalisation group arguments we can show that C satisfies

$$(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}) C(x, \theta, \bar{\theta}, g, \mu) = 0 \quad (34)$$

because the anomalous dimension of ϕ^{p+q} is equal to the sum of the anomalous dimensions of ϕ^p and ϕ^q . At the fixed point C is super-scale invariant and hence constant. The OPE of the chiral operators ϕ^p, ϕ^q at the fixed point is thus in agreement with the OPE of $\Phi_{(p,p)}$ and $\Phi_{(q,q)}$ in the minimal model.

We now turn to the evaluation of the central charge, c , in the LG model. The supercurrent is a vector superfield, J_a , [16] which obeys

$$(\gamma^a)_\alpha{}^\beta D_\beta J_a = 0 \quad (35)$$

at the fixed point, i.e. it is a superconformal current. It splits into $(J_z, J_{\bar{z}})$ with

$$D_- J_z = \bar{D}_- J_{\bar{z}} = 0 \Rightarrow \partial_{\bar{z}} J_z = 0 \quad (36)$$

and similarly for $J_{\bar{z}}$. From (35) J_z depends on only $(z, \theta^+, \bar{\theta}^+)$ and hence defines a $(2, 0)$ superfield, denoted by $\mathcal{J}_z$, which satisfies the conservation condition [17]

$$\partial_{\bar{z}} \mathcal{J}_z = 0 \quad (37)$$

The components of $\mathcal{J}_z$ are $(j_z, G_{+z}, \bar{G}_{+z} \text{ and } T_{zz})$, these being the $U_L(1)$ current, the left-handed supersymmetry currents and the holomorphic part of the energy-momentum tensor, respectively. Similarly, $J_{\bar{z}}$ defines a $(2, 0)$ multiplet of right-handed currents. The central charge can be defined through the two-point function for $\mathcal{J}_z$ [15],

$$\langle \mathcal{J}_z(Z), \mathcal{J}_z(W) \rangle = \frac{c}{(Z - W)^2} \quad (38)$$

where $Z \equiv (z, \theta^+, \bar{\theta}^+)$, $W \equiv (w, \zeta^+, \bar{\zeta}^+)$ and $(Z - W) = z - w - \frac{1}{2}(\theta^+ \bar{\zeta}^+ + \bar{\theta}^+ \zeta^+)$. This shows the relation of c to the superconformal algebra. The central charge c can also be interpreted as the coefficient of an anomaly. To do this the model is first coupled to $(2, 0)$ supergravity [17] which, at the linearised level, is described by an unconstrained superfield $V_{\bar{z}\bar{z}}$ whose components are the right-handed graviton, $h_{\bar{z}\bar{z}}$, the gravitino, $\psi_{\bar{z}-}, \bar{\psi}_{\bar{z}-}$, and the $U_L(1)$ gauge field $b_{\bar{z}}$. The linearised interaction is simply

$$\int d^2x d\theta^+ d\bar{\theta}^+ \mathcal{J}_z V_{\bar{z}\bar{z}} = \int d^2x (j_z b_{\bar{z}} + G_{z+} \psi_{\bar{z}-} + \bar{G}_{z+} \bar{\psi}_{\bar{z}-} + T_{zz} h_{\bar{z}\bar{z}}) \quad (39)$$

The two-point function of equation (40) is the second functional derivative with respect to $V_{\bar{z}\bar{z}}$ of an effective action, $\Gamma[V]_{\bar{z}\bar{z}}$ computed at $V_{\bar{z}\bar{z}} = 0$. On the other hand, by computing $\partial_{\bar{z}}\delta\Gamma/\delta V_{\bar{z}\bar{z}}$ without setting $V_{\bar{z}\bar{z}} = 0$, one finds that there is a supergravitational anomaly of the form [17],

$$\partial_{\bar{z}}\mathcal{J}_z = c\mathcal{A} \quad (40)$$

where $\mathcal{A}$ is the anomaly multiplet which is a function of the supergravity fields. An explicit calculation of this type was done for a free field in [17]. The component anomalies are the $U_L(1)$ anomaly, the local supersymmetry anomaly and the gravitational anomaly. Thus c can be evaluated simply by computing the $U_L(1)$ anomaly which is proportional to the sum of the $U_L(1)$ charges of the fermion fields. $U_L(1)$ acts on $(2, 2)$ superspace by

$$\delta z = 0 \quad ; \delta\theta^+ = a\theta^+ \quad ; \delta\bar{\theta}^+ = -a\bar{\theta}^+ \quad (41)$$

the right-handed coordinates, $(z, \theta_-, \bar{\theta}_-)$, being inert. The potential term $\int d^d x d\theta^+ d\theta^- \frac{1}{n!} \phi^n$ will therefore be invariant if ϕ has $U_L(1)$ charge $q_L = \frac{1}{n}$. Hence the fermion fields λ_+ and λ_- of the chiral multiplet, ϕ , have $U_L(1)$ charges $q_L - 1$ and q_L , respectively. The $U_L(1)$ anomaly is therefore proportional to $1 - 2q_L = 1 - \frac{2}{n}$, and hence

$$c \propto 1 - \frac{2}{n} \quad (42)$$

For a free theory $c = 1$ and so the constant of proportionality in (43) is 1. Thus $c = 1 - \frac{2}{n}$ for the LG model at the fixed point in agreement with the value of c for the corresponding minimal model. For a theory with several chiral fields ϕ_i the central charge is given by

$$c = \sum_i (1 - 2q_{Li}) \quad (43)$$

This type of argument for the central charge is similar in some respects to the anomalies argument for the finiteness of certain rigid supersymmetric theories [18]. These results agree with the phenomenological formula of ref.[8] and the result of ref.[9], which was derived using some aspects of one-loop $g = 0$ perturbation theory. An alternative derivation using representation theory and spectral flow arguments was given in ref.[19].

Identical arguments to those given above hold in the right-handed sector so that $\bar{c} = c$ is also given by formula (43). Since the model is left-right symmetric, it is possible to add finite local counterterms to the effective action (in the presence of $(2, 2)$ background supergravity) in such a way as to remove the local supersymmetry and gravitational anomalies. This procedure induces conformal and S-supersymmetry anomalies. The anomaly multiplet is a chiral $(2, 2)$ superfield $\bar{C}$ and equation (37) is modified to

$$(\gamma^a)_\alpha{}^\beta D_\beta J_a = \bar{D}_\alpha \bar{C} \quad (44)$$

For this form of the anomaly the vector current (the first component of J_a), the γ -trace of the supersymmetry current and the trace of the energy-momentum current are not conserved. There is an alternative version of the supercurrent with a twisted chiral anomaly

[16] (equivalent to a linear multiplet anomaly in two dimensions, and the existence of these two multiplets is paralleled by two minimal forms of supergravity [20]).

The calculations in this paper can easily be generalised to models with several chiral fields with potentials of the form $\sum_i (\phi_i)^n$. More complicated potentials can also be treated. As an example consider an LG model with two chiral fields ϕ_1, ϕ_2 with a potential term

$$\int d^2x d^2\theta (g_1 (\phi_1)^{\frac{n}{2}} + g_2 \phi_1 (\phi_2)^2) \quad (45)$$

Let us assume that the bare couplings g_{1o} and g_{2o} have the same dimension which is taken to be that for which the kinetic term for ϕ_1 has the standard form. (This implies that the kinetic term for ϕ_2 is non-standard unless $n = 6$). We will then have

$$g_{io} = \mu^{\frac{(n-2)\epsilon}{2}} \sum_j Z_{ij} g_j \quad (46)$$

and

$$\phi_{io} = Z_i^{\frac{1}{2}} \phi_i \quad (47)$$

The non-renormalisation theorem gives

$$\begin{aligned} g_1 &= \sum_j Z_{1j} g_j Z_1^{\frac{n}{4}} ; \\ g_2 &= \sum_j Z_{2j} g_j Z_1^{\frac{1}{2}} Z_2 \end{aligned} \quad (48)$$

At the fixed point (47) and (48) imply, using the same arguments as before, that the anomalous dimensions, γ_i , are given by

$$\gamma_1 = \frac{2}{n} ; \gamma_2 = \frac{n-2}{2n} \quad (49)$$

The conformal weight of ϕ_i is $\frac{1}{2}\gamma_i$, and these weights are in agreement with those of ref.[6] where it was proposed that the LG model with the above potential should be identified with the n th $N = 2$ superconformal model of the $D_{\frac{n}{2}+2}$ series. By equation (44) this model has central charge $c = 1 - 2/n$ as the $U_L(1)$ charges of ϕ_1, ϕ_2 are $\frac{2}{n}$ and $\frac{n-2}{2n}$ respectively.

It is clear that the results given for chiral operators in this paper depend crucially on the non-renormalisation theorem. Indeed, suppose we have a scheme in which this theorem holds and which includes the infra-red fixed point g^* in its range. We will have

$$g_o = \mu^{d_g} Z_g g \quad (50)$$

where g is taken to be dimensionless and d_g is the dimension of g_o . The non-renormalisation theorem implies that $Z_g Z^{\frac{n}{2}} = 1$ and hence, using the same arguments as above, that

$$\gamma_\phi(g) = \frac{d_g}{n} \quad (51)$$

In two dimensions, $d_g = 1$, giving the same result for γ_ϕ as we had before. It may therefore be possible to find a scheme that works directly in two dimensions and obtain the same results as one gets with the ϵ -expansion. In passing we note that such arguments can be applied in other supersymmetric theories, such as the massless Wess-Zumino model in four dimensions which has a cubic chiral potential. This model is classically conformally invariant, and g_o is dimensionless. This would imply that the anomalous dimension of ϕ is zero, and is a signal that this model has no non-trivial fixed points [21].

To summarise, in this paper we have used the ϵ -expansion to calculate the fixed point of the LG Hamiltonian (5), the conformal weights of the chiral operators ϕ^p , $p = 1, \dots, (n-2)$, their OPE's and the central charge. Essential use was made of the $N = 2$ non-renormalisation theorem which remains valid in the double expansion in g and ϵ . The results are in complete agreement with the corresponding $N = 2$ minimal model and in effect establish the identification of the LG model with the minimal model in the $N = 2$ case.

We note that the successes of the ϵ -expansion and the LG model in calculations of critical exponents in the three-dimensional Ising model have not been repeated in two-dimensional bosonic models. In a previous paper [11] we showed that the agreement between the LG model, calculated in the ϵ -expansion, and minimal models is not good for the case of two dimensions. The results are much better for the $N = 2$ case as we have seen, and indeed, those exhibited in this paper (except the value given for the fixed point) are exact. This is due in large part to the non-renormalisation theorem. In supersymmetric theories the UV divergences are softened due to cancellations between the bosons and the fermions. In a superspace formulation it is as if the odd coordinates act in some way as negative dimensions. On the other hand, at the critical point there are long-range, infrared, fluctuations, which become more difficult to control as the dimension is decreased. It therefore seems that supersymmetry has the opposite effect in this domain, as the infrared divergences are also better behaved in the supersymmetric theory as opposed to the bosonic one.

One point which should be mentioned is the status of the dimensional reduction scheme. It is known that this scheme suffers from ambiguities although there are fewer problems with Wess-Zumino type models than there are with gauge models. If the WZ model is formulated in components a version of the ϵ -expansion which is closer to conventional dimensional regularisation can be used. In such a scheme the use of the Fierz rearrangement formula is suspended until the end of the calculation when two dimensions is returned to [22]. The supersymmetry Ward Identity is violated in the regulated theory but graphs which contribute to this breaking of supersymmetry can be shown to give no contribution in the two-dimensional limit. Indeed, this is an alternative way of deriving the non-renormalisation theorem [23]. This suggests that the use of the naive dimensional reduction scheme in superspace is indeed justified for the LG hamiltonians we have been discussing.

To conclude we point out some further directions which suggest themselves. Firstly there are a number of other calculations that can be done in the simplest LG model. In a future paper we shall report on the anomalous dimensions of other operators, such as $\phi^p \phi^q$, and compute their OPE's. This should hopefully lead to a complete dictionary

between the LG model and the corresponding minimal model. Secondly, it would be interesting to consider other supersymmetries. For example, in the $(2, 0)$ case there is also a non-renormalisation theorem which is however weaker than the $N = 2$ one as a potential term requires that the model have chiral bosonic and right-handed fermionic superfields. Thus the non-renormalisation theorem in this case gives a relation between the renormalisation constants for the coupling and the two field operators. For $N = 1$ there is no non-renormalisation theorem and it may be that the results in this case are as bad as in the bosonic case. On the other hand, $N = (4, p)$ models are UV finite in two dimensions for $p = 0, \dots, 4$ [24] and it would be interesting to see what can be said about them from the point of view of this paper.

Another interesting case is the three-dimensional supersymmetric Ising model for which the critical dimension is $d_c = 4$ and ϵ must be set equal to 1 at the end of the calculation. The anomalous dimensions of ϕ and ϕ^2 are easily calculated as before and turn out to be $\frac{1}{6}$ and $\frac{1}{3}$ respectively.

Finally, we note that the LG model with the general potential $\sum_n g_n 1/n! \phi^n$ has an infinite number of fixed points and that there exists a perturbation expansion around each one. It would be of interest to understand more about these perturbation expansions. For example, whether they are asymptotic series or convergent series, and, if the latter, what the radius of convergence is. It is tempting to think that it would be the distance to the next fixed point in coupling constant space. It may be possible to relate the different perturbation theories to each other to obtain a patching of coupling constant space. This is in some way related to the notion of renormalisation group flow. Further one could speculate that there is an expression which contains the perturbation expansions about all the fixed points.

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Fig. 1: Graph contributing to wave function renormalization