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PHASE TRANSITION OF A SCALAR FIELD IN A DE SITTER BACKGROUND:
THE NUCLEATION RATE AT THE WKB APPROXIMATION

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ABSTRACT

We rederive the Coleman-de Luccia and Hawking-Moss results on the probability of decay of a scalar field from a false to a true vacuum state by a straightforward generalization to the de Sitter background of the standard calculation in Minkowski space-time.

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During the early phase of the history of the Universe, as described by the standard hot Big-Bang scenario (see, e.g., Ref. [1] for a critical review), it may have happened that some scalar field got trapped in a metastable "false vacuum" state, thus giving rise to an effective cosmological constant that produced a phase of exponential growth of the scale factor (for reviews of this "inflationary" scenario, see, e.g., Refs. [2-4]). Various physical processes susceptible to end this period of inflation and force the Universe to resume its standard evolution have been envisaged. They could be the fluctuations of the scalar field, as in the so-called "new" and "chaotic" inflation scenarios [2-5]. Another possibility, proposed by Guth [6] and based on a work by Coleman [7], is that bubbles of the stable phase (corresponding to an effective cosmological constant close to zero) nucleate in the metastable phase via quantum tunnelling of the scalar field through the potential barrier separating the two phases. Although this mechanism had to undergo a number of major improvements (see, e.g., [8-9]) it remains, within the framework of the "new inflationary" scenario [2-4], a favoured means of ending the period of inflation [10-15].

The process of bubble nucleation in Minkowski space-time is well understood [7,11,16-18]: the nucleation rate can be computed from the solutions to the scalar field equation written in an auxiliary Euclidean (++++) flat space, and the configurations of the bubbles at the moment of their formation and their subsequent evolution can be obtained by analytically continuing these Euclidean solutions back to Minkowski space-time (--++). In the presence of gravity, the process was similarly analysed by Coleman and de Luccia [19], who deduced the nucleation rate and the initial configurations of the bubbles from an analysis of the coupled Einstein and Klein-Gordon equations in an auxiliary Riemannian (++++) geometry. Now, as emphasized in [18], the "Euclideanization" of the Klein-Gordon equation in the absence of gravity is a conceptual short-cut, wholly justified by the underlying quantum field theory in Minkowski space-time, but the Coleman-de Luccia "Riemannization" of the Einstein-Klein-Gordon equations, which seems to involve a quantization of both the scalar and gravitational fields, appears as a mere prescription which, as such, is poorly justified, however plausible and computationally fruitful it may be.

The purpose of this note is to show that if one assumes that the background gravitational field remains classical, static and unperturbed during the nucleation process, this prescription can in fact be justified by a straightforward extension of the quantum field theory of a scalar field in Minkowski space-time. The conditions imposed on the gravitational field, especially that of staticity and the neglect of the back reaction during the nucleation process, may seem very restrictive, and indeed they are, but they are sufficient to recover, as we shall see, the Coleman-de Luccia [19] and Hawking-Moss [10] results (see also [9,11,12,18]). They moreover may be necessary if one is reluctant to leave the firm ground of

quantum field theory in curved space-time where staticity ensures that the concept of a vacuum state remains well defined and where the neglect of the back reaction eliminates the difficult problem of predicting the reaction of the geometry to an essentially a-causal quantum jump.

We consider the classical action for a scalar field $\phi(x^i)$ coupled to a gravitational field $g_{ij}(x^\ell)$:

$$S^T = S_\phi + \kappa^{-2} S_E + \xi \kappa^{-2} S_I \quad (1)$$

with

$$\begin{aligned} S_\phi &= - \int d^4x \sqrt{|g|} \left(\frac{1}{2} \partial_i \phi \partial^i \phi + V(\phi) \right), \\ S_E &= \int d^4x \sqrt{|g|} \frac{1}{2} R, \\ S_I &= - \int d^4x \sqrt{|g|} \frac{1}{2} \phi^2 R. \end{aligned} \quad (2)$$

Our conventions are: signature $(-+++)$; $R_{ij} = +\partial_k \Gamma_{ij}^k$; $K = 8\pi/M_p^2$, where M_p is the Planck mass; $\xi = 0$ (or $1/6$) in the case of minimal (or conformal) coupling. As for the potential $V(\phi)$, we shall assume that it possesses a local minimum at $\phi = \phi_i = 0$, $V(0) > 0$ (the "false vacuum"), separated by a potential barrier at $\phi = \phi_{top}$ from a global minimum at $\phi = \phi_s$, such that $V(\phi_s) = 0$ (the "true vacuum"). An example of such a potential is the (nearly) Coleman-Weinberg potential [12,20].

The Euler variation of (1)-(2) with respect to the metric g_{ij} and the field ϕ yields the standard field equations:

$$R_{ij} = \frac{\kappa}{1 - \xi \phi^2} \left[\partial_i \phi \partial_j \phi + g_{ij} V(\phi) - \xi \left(\nabla_i \nabla_j \phi^2 + \frac{1}{2} g_{ij} \square \phi^2 \right) \right] \quad (3)$$

$$-\square \phi + \frac{dV}{d\phi} + \kappa^{-2} \xi \phi R = 0. \quad (4)$$

We assume now that in some domain of the manifold the field ϕ is in a ground state $\phi = \phi_0 = \text{constant}$. Then from (3) the geometry of that domain is that of an Einstein space:

$$R_{ij} = \Lambda g_{ij} ; \quad \Lambda = \frac{\kappa V(\phi_0)}{1 - \xi \phi_0^2} \quad (5)$$

and ϕ_0 is determined by (4):

$$\left. \frac{dV}{d\phi} \right|_{\phi_0} + \frac{4\xi}{1-\xi\phi_0^2} \phi_0 V(\phi_0) = 0. \quad (6)$$

Hence, when $\xi \neq 0$, the ground states do not necessarily correspond to the extrema of the potential. Thus $\phi = \phi_{\text{top}}$ is not a solution of the system (3)-(4); the false and true vacua ϕ_i and ϕ_s , on the other hand, are, since $\phi_i = 0$ and $V(\phi_s) = 0$.

The fields being initially in a false vacuum state, they may evolve towards the true vacuum configuration either because of fluctuations of the field ϕ , which solve the linearized Eq. (4) in the background (5)-(6), and act back on the gravitational field [2-5,21-22]; or else one may invoke a non-linear process of quantum tunnelling through the barrier separating the two vacua [7,9-15,18,19]. In the latter approach, the probability of nucleation as well as the initial configurations of the bubbles of the true vacuum after formation were obtained [9,10,12-15,19] by a "Riemannization" prescription [19], whose validity ought to be made explicit [18]. We shall show here that these results can in fact be obtained by a straightforward procedure of canonical quantization of the scalar field in the fixed geometry of the false vacuum state when this geometry is assumed (and that will be crucial) to be static.

We then consider a static false vacuum state obeying Eq. (5) with $\phi_0 = \phi_i = 0$ and a foliation orthogonal to the orbits of its timelike Killing vector. Moreover, we choose a co-ordinate system with zero shift, but leave the lapse arbitrary for the moment (this will simplify the notations and meet our needs). The metric and the curvature scalar can then be written as:

$$ds^2 = -N^2 dt^2 + \gamma_{\alpha\beta} dx^\alpha dx^\beta \quad (7)$$

$$R = P - \frac{2}{N\sqrt{\gamma}} \partial_\alpha (\sqrt{\gamma} \gamma^{\alpha\beta} \partial_\beta N) = 4\pi V(0) \quad (8)$$

where the lapse N and the metric $\gamma_{\alpha\beta}$ are given functions of the spatial co-ordinates x^μ only, and P is the curvature scalar formed with $\gamma_{\alpha\beta}$. Since the gravitational field is now considered as an external field, the action (1,2) becomes a functional of $\phi(t, x^\alpha)$ only and reads:

$$S = \int dt d^3x (L_\phi + \kappa^{-1} \xi L_I) \quad (9a)$$

with

$$L_\phi = N\sqrt{g} \left[\frac{1}{2N^2} \dot{\phi}^2 - \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right] \quad (9b)$$

$$L_I = -2\kappa N\sqrt{g} \phi^2 V(0), \quad (9c)$$

where $\dot{\phi} \equiv \partial/\partial t$.

The canonical formulation of the theory of a scalar field in a given curved space-time is well understood (see, e.g., [23-24]). In the present context, it yields the following Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \int d^3x H\left(\phi, \frac{\delta S}{\delta \phi}\right) = 0 \quad (10)$$

where the Hamiltonian density is defined as

$$H(\phi, \pi_\phi) = N\sqrt{g} \left[\frac{1}{2g} \pi_\phi^2 + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) + 2\xi V(0) \phi^2 \right] \quad (11a)$$

π_ϕ being the momentum density conjugate to ϕ :

$$\pi_\phi \equiv \frac{\delta L_\phi}{\delta \dot{\phi}} = \frac{\sqrt{g}}{N} \dot{\phi}. \quad (11b)$$

Since, and this is the crucial point, we restricted ourselves to static backgrounds described by time-independent metrics of the type (7), the variables t and ϕ can be separated, just as in flat space-time:

$$S = -Et + S_0(\phi) \quad (12)$$

where E is a constant that we choose so that a false vacuum configuration $\phi = \phi_1 = 0$, $\delta S_0/\delta \phi = 0$ indeed solves Eqs. (10) and (11):

$$E = \int d^3x N\sqrt{g} V(0). \quad (13)$$

The Hamilton-Jacobi equations (10) and (11) then reduce to

$$\int d^3x N\sqrt{g} \left[\frac{1}{2g} \left(\frac{\delta S_0}{\delta \phi} \right)^2 + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) - V(0) + 2\xi \phi^2 V(0) \right] = 0. \quad (14)$$

In the vicinity of the ground state configuration ($\phi \approx 0$), the integrand of (14)

is positive definite since $V''(0) > 0$ and $V(0) > 0$ and since the classical action S_0 must remain real. Hence, classically, the field ϕ is trapped in the false vacuum configuration, as it should be.

We therefore quantize it (without touching the background geometry) by introducing a correspondence principle which turns the classical Hamilton-Jacobi equation (10) into an operator acting on a state vector $\psi(t, \phi)$, yielding a Schrödinger-type equation. In the WKB approximation one sets $\psi = A \exp iS$, and at leading order the Schrödinger equation for the eikonal S reduces to nothing but the Hamilton-Jacobi equations (10)-(11). The eikonal S , however, is no longer restricted to be real. We therefore look for a complex solution of (10) and (11):

$$S = -Et - i \tilde{S}_0(\phi) \quad (15)$$

where E is given by (13) and $\tilde{S}_0$ satisfies

$$\int d^3x \sqrt{\gamma} \left\{ -\frac{1}{2\gamma} \left(\frac{\delta \tilde{S}_0}{\delta \phi} \right)^2 + \frac{1}{2} \gamma^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) - V(0) + 2\xi \phi^2 V(0) \right\} = 0. \quad (16)$$

The space of the state vectors ψ having been given (heuristically at least [24]) a Hilbert space structure, then

$$P = \sum_{\phi_f} \frac{|\Psi(\phi_f)|^2}{|\Psi(\phi_i)|^2} \propto \sum_{\phi_f} \exp \left\{ +2 [\tilde{S}_0(\phi_f) - \tilde{S}_0(\phi_i)] \right\} \quad (17)$$

is interpreted as the probability for the field ϕ to tunnel out of the false vacuum $\phi_i = 0$; the configurations ϕ_f are the "points" where the solutions of (16) smoothly join those of (14) and ϕ resumes a classical behaviour, and therefore satisfy

$$\left. \frac{\delta \tilde{S}_0}{\delta \phi} \right|_{\phi_f} = 0 \quad \text{or} \quad \pi_{\phi_f} = 0, \quad (18)$$

[see Eqs. (10) and (11)]. The probability P is dominated by the configuration(s) ϕ_f , obeying (18), that yields for $\tilde{S}_0$ the maximum maximum value.

In ordinary quantum mechanics where ϕ is a spatial co-ordinate, one can proceed by actually solving (16) by quadrature. In field theory where (16) is not trivially integrable (or in quantum mechanics as an alternative procedure), one uses the trick of considering another, closely-related auxiliary dynamical system. Following Ref. [7] (see also [18, 25-26]) we hence consider the auxiliary Riemannian metric:

$$d\tilde{s}^2 = + N^2 d\tau^2 + \gamma_{\alpha\beta} dx^\alpha dx^\beta \quad (19)$$

that is deduced from (7) by a Wick rotation $t = -i\tau$, and whose associated curvature scalar is still given by (8) since the metrics (7) and (19) do not depend on time. In that background the action (1,2) for the scalar field reduces to

$$\tilde{S} = \tilde{S}_\phi + \xi \kappa^{-1} S_\tau$$

with

$$\tilde{S}_\phi = \int d\tau d^3x N\sqrt{\gamma} \left[-\frac{1}{2N^2} \phi'^2 - \frac{1}{2} \gamma^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right] \quad (20)$$

and S_τ given by (9b) (here $' \equiv \partial/\partial\tau$). After separation of the variables

$$\tilde{S} = -E\tau + \tilde{S}_0(\phi) \quad (21)$$

with E given by (13), the canonical formulation of the theory of a scalar field in the background (19) yields, instead of the Hamilton-Jacobi equation (14), the equation (16) which in this context is therefore to be regarded as a classical Hamilton-Jacobi equation for a field whose action is (20). Hence the action S ([15]) of the real system is related to the action $\tilde{S}$ ([21]) of the auxiliary system by

$$S = -i\tilde{S}. \quad (22)$$

Now the configurations ϕ that maximize (21) and (20) are the solutions of the Klein-Gordon equation (4) written in the background (19),(8):

$$-\frac{1}{N^2} \phi'' - \frac{1}{N\sqrt{\gamma}} \partial_\beta (N\sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha \phi) + \frac{dV}{d\phi} + 4\xi V(0) \phi = 0 \quad (23)$$

The configuration ϕ_f that we look for is therefore the solution of (23) which, when inserted in (20), gives the smallest value.

The difference between the "Riemannization" prescription of Ref. [19] and the procedure which is presented here (which follows step by step Ref. [18] in its treatment of the same problem in Minkowski space-time) is that our background is fixed: it is a static false vacuum state defined by (5),(6) with $\phi_0 = \phi_1 = 0$. We therefore do not "Wick-rotate" the Einstein field equations (3), which only serve to define the background geometry.

In order to find the searched-for solution of (23), we must be more specific about the geometry of the false vacuum. Indeed, Eq. (5) (with $\phi_0 = 0$) does not fully determine it, even when staticity is required (see [15]). We shall hence make the additional assumption that it is locally maximally symmetric. The metric (7)-(8) is then the de Sitter metric written in a static co-ordinate system and (19) its Wick rotation:

$$N^2 = 1 - \frac{r^2}{A^2} ; \quad \gamma_{\alpha\beta} dx^\alpha dx^\beta = \frac{dr^2}{1 - r^2/A^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (24)$$

where $A^2 = 3/\Lambda = 3/(kV(0))$. Equation (23) is then the Klein-Gordon equation (4) written in the Riemannian metric (19,24). It can be solved in any co-ordinate system. Thus in the transformation

$$T = \int \frac{dr}{\sqrt{1 - r^2/A^2}} , \quad R = t, \quad (25)$$

the metric (19,23) becomes

$$ds^2 = dT^2 + \cos^2 T/A dR^2 + A^2 \sin^2 T/A (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (26)$$

which is the metric on a four-dimensional sphere embedded in a five-dimensional flat Euclidean space. Another change of co-ordinates:

$$\cos \zeta = \cos \frac{R}{A} \cos \frac{T}{A} ; \quad \sin \eta = \sin \frac{T}{A} \sqrt{1 - \cos^2 \frac{R}{A} \cos^2 \frac{T}{A}} \quad (27a)$$

then casts it into the standard form:

$$ds^2 = d\zeta^2 + A^2 \sin^2 \zeta/A [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)] . \quad (27b)$$

In this new co-ordinate system the Klein-Gordon equation (23) reads

$$-\tilde{\square} \phi + \frac{dV}{d\phi} + 4\zeta V(0) \phi = 0 \quad (28)$$

where $\tilde{\square}$ is the Laplacian associated with the Riemannian metric (27).

Following Coleman and de Luccia [19], we now assume that the solution of (28) that will give the smallest value for the action (20) has (as in flat space-time [17]) an $O(4)$ symmetry and hence is a function of ζ only. Equation (28) then reduces to

$$\frac{d^2\phi}{d\zeta^2} + \left(\frac{3}{A} \cotg \frac{\zeta}{A} \right) \frac{d\phi}{d\zeta} = \frac{dV}{d\phi} + 4\xi V(0)\phi. \quad (29)$$

This equation has been thoroughly studied [10,12,13,18]. If ϕ is interpreted as a spatial position and ζ as the (absolute) time, it describes the Newtonian motion of a particle in the potential $-[V(\phi) + 2\xi V(0)\phi^2]$, subjected to a time-dependent frictional force. The bounded solutions are such that $d\phi/d\zeta = 0$ at $\zeta = n\pi A$. One of them is the Hawking-Moss ^{almost} solution [10,12,13,18], $\phi = \bar{\phi} = \text{constant} \neq 0$ such that

$$\left. \frac{dV}{d\phi} \right|_{\phi=\bar{\phi}} + 4\xi V(0) \bar{\phi} = 0. \quad (30)$$

As shown in [12] this Hawking-Moss solution, when $V(\phi)$ is sufficiently flat, is indeed the "channel" ϕ_f which yields for the action (20) the smallest value. The action (20) is written in the static co-ordinate system (24) but can be computed in the system (26) and reads, when $\phi = \bar{\phi}$:

$$\tilde{S} = -\frac{4}{3}\pi^2 A^4 [V(\bar{\phi}) + 2\xi \bar{\phi}^2 V(0)] \quad (31)$$

[with $A^2 = 3/kV(0)$].

The auxiliary Riemannian metric (19) having fulfilled its purpose, we return to the real system by means of Eq. (22) and obtain the probability of tunnelling via Eqs. (15) and (17):

$$P \propto \exp \left\{ \frac{-24\pi^2}{\kappa^2 V^2(0)} [V(\bar{\phi}) - V(0) + 2\xi \bar{\phi}^2 V(0)] \right\} \quad (32)$$

where $\bar{\phi}$ satisfies (30). If the potential $V(\phi)$ is approximated near $\phi = 0$ by

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \quad (33)$$

then the probability (32) is

$$P \propto \exp \left[-\frac{2\pi^2}{3\lambda} \frac{(m^2 + 4\xi V_0)^2}{H^4} \right] \quad (34)$$

with $H^2 = 1/A^2 = kV_0/3$ - a result which coincides with Eq. (76) of Ref. [18] and shows that the dependence of P on the coupling ξ to gravity amounts to a renormalization of the mass of the scalar field: $m^2 \rightarrow m^2 + 4\xi V_0$.

When the solutions of (29) (the "bounces") cannot be approximated by the Hawking-Moss solution (30), one must resort to numerical integration (cf., e.g.,

[9,12,13,15,18]). The "bounces" then not only allow us to estimate the probability of tunnelling (17), but also give the configurations of the bubbles at the moment of nucleation [7]. Indeed, the configuration of the bounce $\phi(\zeta)$ at the "time" $\tau = 0$ such that $\partial\phi/\partial\tau|_{\tau=0} \equiv \phi' = \dot{\phi} = 0$ is the "turning point" where (18) is satisfied [taking account of (11b)], and where the field ϕ , having penetrated the barrier, resumes its classical behaviour. This behaviour is then governed by the classical coupled Einstein Klein-Gordon equations (3)-(4). Indeed, the approximation of a static fixed background is only required to compute the probability of nucleation and the initial data for the subsequent evolution of the bubbles.

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