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EXTENDED CHIRAL ALGEBRAS AND MODULAR INVARIANT PARTITION FUNCTIONS

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ABSTRACT

We show how the fusion rules can be used to associate with every rational conformal field theory a discrete group, the center. The center is generated by primary fields having unique fusion rules with any other field. The existence of a non-trivial center implies the existence of non-diagonal modular invariants, which are related to extended integer or fractional spin algebras. Applied to Kac-Moody algebras this method yields all known as well as many new infinite series of modular invariants. Some results on exceptional invariants are also presented, including an example of an exceptional integer spin invariant that does not correspond to a conformal embedding.

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1. Introduction

The study of two-dimensional conformal field theories is central to the investigation of both two-dimensional critical phenomena and string theories. A lot of efforts have been made in order to classify all (rational) conformal field theories. It was suggested that one way to understand conformal field theories is in terms of extended chiral algebras [1], which include the Virasoro algebra as a sub-algebra. A conformal field theory with $c > 1$ which has an infinite number of primary fields with respect to the Virasoro algebra may turn out to have only a finite number of primary fields with respect to the bigger extended algebra. Moreover, one can understand the appearance of some non-diagonal modular invariant theories as arising from a theory which is diagonal with respect to the extended chiral algebra generated by certain currents with integer conformal spin.

Starting with the important work of Verlinde [2], the central role of the fusion rules of the conformal field theory became clear [2], [3-6]. A great deal of the structure of the underlying conformal theory is encoded in the commutative and associative Verlinde algebra which these fusion rules define. There exists a deep relationship between the fusion algebra and the modular properties of the theory. The modular matrix S diagonalizes the fusion rules [2],[3]. The (integer) multiplicities appearing in the fusion rules can be computed in terms of elements of the modular S -matrix.

The partition function of a conformal field theory can be expressed in terms of the characters of the chiral algebra as

$$Z = \sum \chi_i^* M_{ij} \chi_j , \quad (1.1)$$

where the indices $\bar{i}, j$ go over the primary fields associated with the $\bar{A}$ and A chiral algebra respectively. Here M is a matrix with non-negative integer entries. Invariance under the modular transformation T implies that the matrix M pairs primary fields whose dimensions differ by an integer *i.e.* $h_{\bar{i}} - h_j \in \mathbb{Z}$ if $M_{\bar{i}j} \neq 0$. Invariance under the modular transformation S implies that $S^\dagger M S = M$, *i.e.* S , which is unitary, commutes with M . For every chiral algebra there always exists a diagonal modular invariant $M_{ij} = \delta_{ij}$. However, even when the maximal extended (integer spin) chiral algebra has been identified, and when all modular invariants are written in terms of its characters, some modular invariants may still be non-diagonal. This would be the case if the matrix M which appears in (1.1) defines

an automorphism of the fusion rules algebra under which $i \rightarrow j$ [4], [7]. Indeed known (generalized) diagonal and non-diagonal invariant theories could be explained in terms of either extended algebras or automorphism of the fusion rules.

In this paper we shall concentrate on the problem of identifying and constructing extended chiral algebras associated with any given conformal field theory. We will consider integer as well as fractional spin currents. Once we identify the extended current algebra, *i.e.* the currents and their operator product expansion, and the primary fields we can construct partition functions. Those are modular invariant bilinear combinations of the characters as in (1.1). The current algebras which we construct via our approach have the structure of a direct product of $\mathbf{Z}_N$ factors. The modular invariant combinations that we construct, besides the diagonal one, have an underlying generalized orbifold structure. By generalized orbifold structure we mean that the orbifold-like construction is done on the level of the abstract fusion rules rather than in the Hilbert space defined by a given Hamiltonian (this kind of generalization of the orbifold construction has also been discussed recently in [8]). Nevertheless, these two kinds of orbifold constructions are very similar from the point of view of conformal field theory. This allows us to borrow the well-known regular orbifold approach to construct new invariants. This amounts to performing GSO-like projections and summing over twisted sectors.

It has been suggested that all rational conformal field theories can be obtained by starting from an affine Kac-Moody algebra associated with a group G and performing the coset construction [9]. It is therefore natural to study the application of our results to Kac-Moody algebras. Furthermore this allows us to compare the scope of our method with already known ones. For the case $G = SU(2)$ all modular invariant combinations have been classified [10]. There are known infinite series of modular invariant theories associated with other Lie algebras [11-13]. Using our method we have been able to construct all those invariants and elucidate the extended current algebra associated with them. Moreover, we have been able to generate new infinite series of modular invariants for many Kac-Moody algebras. In addition some exceptional modular invariants (which exist only for a few special values of the level) are known [10], [14], [15],[7]. Most of these are related to some (integer spin) extended algebra, which in all cases known up to now could be associated with conformal sub-algebras [16] (we have however found one new integral spin exceptional invariant which does not correspond to a conformal embedding). These integral spin exceptional invariants cannot be obtained with the current algebras we derive from the

fusion rules, but they do clearly fall within the general class of modular invariants related to some sort of extended algebra. Actually among all known modular invariant theories the only ones which do not obviously belong to that class are the E_7 invariant of $SU(2)$ [10] and the recently found level 9 $SU(3)$ invariant [7]. Those two cases have been shown to be associated with new automorphisms of the fusion rules.

This paper is organized as follows; in section 2 we present our approach. We identify the currents and investigate the structure of the theory associated with the extended chiral algebra. This allows us to formulate a general procedure for constructing a class of modular invariant partition functions which are associated with the extended algebras. This is done by applying a generalized orbifold construction. A natural classification emerges in terms of the Z_N factors and their subgroups. In section 3 we apply our approach to generate modular invariant theories associated with Kac-Moody algebras. In this process we obtain besides the known results also some new classes of invariants. In section 4 we discuss the exceptional cases associated with embeddings and comment on the generality of our approach. Section 5 discusses applications to other conformal field theories, and to string construction in general. Most known constructions can be regarded as special cases of our method.

2. Modular Invariants of Rational Conformal Field Theories

In this section we discuss a general procedure for constructing a class of modular invariants which exists for many rational conformal field theories. These invariants are related to the existence of certain extended “algebras” generated by chiral currents of integer or fractional conformal spin. The word algebra should be understood here in a rather loose sense. We will define a “current algebra” through the fusion rules, by looking for a set of currents that closes under fusion. We will then demand that the operator products of all currents have simple monodromy properties (*i.e.* all powers of $(z - w)$ in the operator product of two currents must differ by integers), which is important to define (generalized) commutation relations. We will furthermore require that all currents have simple monodromy with all other fields in the theory, so that well-defined mode expansions exist. The simplest way to obtain simple monodromy is to require that the currents have unique fusion rules with all other primary fields. Many conformal field theories have such currents.

We will not pay much attention here to the algebras themselves. Apart from a few cases with small spins (such as Virasoro, super-Virasoro and Kac-Moody algebras, free fermions, parafermions [17] and Zamolodchikov's W-algebra [18]), we know very little about their algebraic structure. An attempt to extend chiral algebras to higher spin was made recently by Bouwknegt [19]. By requiring associativity he was able to find interesting constraints on the existence of such algebras, and he observed that the values of the central charge of some solutions coincide with those of the minimal series [20,21]. Unfortunately this kind of analysis, even with already rather severe restrictions on the fusion rules, cannot be pushed easily to very large spins or more than one current. Our approach is completely different, and does not require knowledge of the algebra (or, in other words, the operator products), but only of the fusion rules. This turns out to be sufficient to achieve our goal, which is to use these putative algebras in the construction of modular invariants. The existence of these modular invariants (in particular those that contain chiral integer spin currents) presumably implies that associative algebras can be defined, the associativity being a consequence of duality in the corresponding closed string theory.

2.1 Non-degenerate currents

Consider a conformal field theory $\mathcal{C}$, which we will assume to be unitary and have a finite number of primary fields Φ_i (these fields may be primary with respect to the Virasoro algebra alone, or with respect to some extension of it, such as Kac-Moody algebras, super-Virasoro algebras etc.). Suppose there exists a primary field J which has unique fusion rules with all other fields in the theory, *i.e.*

$$J \cdot \Phi = \Phi' , \tag{2.1}$$

with just one term appearing on the right-hand side for any primary field Φ in the theory. We will refer to such a primary field J as a *simple current*. The conjugate J^c of J is defined by the fusion rule $J \cdot J^c = 1$. Consider now the fusion rule of J with itself. The right-hand side is unique and is either the identity (in which case $J = J^c$) or a new primary field, which we will denote as J_2 (the fusion rule $J \cdot J = J$ would obviously imply that $J = 1$, as one can easily check by acting on both sides with J^c).

Using associativity of the fusion rules it is easy to see that J_2 is itself a simple current. One can now act with J on J_2 , and show by similar arguments that the result is either the

identity, or a new simple current J_3 . By repeating this process one defines an orbit of simple currents, which inevitably (since the total number of primary fields is finite) returns to the identity after a finite number of steps. Let the *order* N of J be the smallest positive integer so that $J_N = 1$. Obviously the fusion rules for the set of fields $J_n, n = 0, \dots, N-1$ (where $J_0 = 1$, and $J_1 = J$) generate a group isomorphic to $\mathbf{Z}_N$. Furthermore we have $J_n^c = J_{N-n}$.

The orbit generated by J may not exhaust the set of simple currents in the theory. Obviously, the complete set of simple currents in a conformal field theory forms an abelian group, with the fusion rule as group operation, and charge conjugation as the inverse. This group is isomorphic to $\mathbf{Z}_{N_1} \times \dots \times \mathbf{Z}_{N_k}$, and we will call it the *center* of the conformal field theory. The reason for this terminology is that for Kac-Moody algebras this group will turn out to be isomorphic to the center of the corresponding Lie algebra.

Consider now any simple current J of order N . Acting on the other primary fields, J and its powers J_n produces orbits which close back into themselves after N steps. The fields Φ_n produced by acting with $J_n, n = 1, \dots, N$ on a primary field $\Phi \equiv \Phi_0$ need not all be different, but clearly the total number of different fields that appears must be a divisor of N . Hence the current J combines all fields in the theory into orbits, whose length is N or a divisor (which may be 1) of N .

Uniqueness of the fusion rules (2.1) implies that every field Φ has definite monodromy with the current J (the converse is of course not true), most conveniently defined by means of the operator product. If $J(z)\Phi(w) \propto (z-w)^{-\alpha}\Phi'(w)$, then the monodromy of Φ with respect to J is $\alpha \bmod 1$. Because $J_N = 1$, Φ must have trivial monodromy with respect to N currents J , so that $\alpha = \frac{t}{N}$, $t \in \mathbf{Z}$. In particular the current-current monodromy must satisfy this condition. Hence

$$J(z)J(w) \propto (z-w)^{-r/N} J_2(w) , \quad (2.2)$$

for some integer r . A second important parameter of the “current algebra” is the L_0 eigenvalue h_J of J . The L_0 eigenvalue of J_2 can be expressed (modulo integers) in terms of r and h_J by means of (2.2), and by recursion one obtains then the following result for the L_0 eigenvalues h_n of J_n

$$h_n = -\frac{rn(n-1)}{2N} + nh_J \pmod{1} . \quad (2.3)$$

By imposing the condition $h_N = 0 \bmod 1$ we get the following restrictions on h_J

$$\begin{aligned} h_J &= \frac{p}{N} \bmod 1 && \text{for } N \text{ odd and for } N \text{ even, } r \text{ even} \\ h_J &= \frac{p}{2N} \bmod 1 \text{ (} p \text{ odd)} && \text{for } N \text{ even, } r \text{ odd.} \end{aligned}$$

Note that (2.3) does not satisfy in general the charge symmetry condition $h_n = h_{N-n}$, so that it allows the current and its charge conjugate to have different L_0 eigenvalues. It seems reasonable to impose charge symmetry (modulo one) as an additional constraint. This gives a relation between r and p , which leads to the following expression for h_n

$$h_n = \frac{rn(N-n)}{2N} \pmod{1}, \quad (2.4)$$

where for odd N the parameter r is defined modulo N as before, whereas for even N its range must be doubled to cover all cases, so that it is defined modulo $2N$. Thus the complete set of inequivalent values of r is $0, 1, \dots, N-1$ for N odd and $0, 1, \dots, 2N-1$ for N even. We will call r the *monodromy parameter* of the algebra. Due to the $\mathbf{Z}_N$ structure of our “current algebra”, (2.4) resembles the result obtained for the $\mathbf{Z}_N$ parafermionic theory [17] (where in the latter case $r = 1$).

The L_0 spectrum on an orbit generated by acting on some field Φ is completely determined by its monodromy $\frac{t}{N}$ with J and its L_0 eigenvalue $h(\Phi)$. For currents satisfying the charge conjugation condition given above, the result is

$$h(\Phi_n) = h(\Phi) + \frac{rn(N-n)}{2N} - \frac{tn}{N} \bmod 1. \quad (2.5)$$

We will call $\frac{t}{N} \pmod{1}$ the charge $Q(\Phi)$ of the field Φ . We will show in the next subsection that this charge is conserved. Using (2.5) one can compute the charges of all other fields on the orbit. The result is

$$Q(\Phi_n) = \frac{t}{N} + \frac{rn}{N}. \quad (2.6)$$

One can define such a charge for any current. The charge Q_n belonging to the current J_n is simply equal to $nQ \pmod{1}$. In particular for the conjugate J^c of J one finds $Q^c = -Q$. Using (2.4) one can derive the useful relation

$$Q_n(J_m) = \frac{nmr}{N}. \quad (2.7)$$

Note that if r and N are relative prime, the charge on the orbit can have all possible values $\frac{k}{N}$, $k = 0, \dots, N-1$. In particular there is then always a unique value of n (modulo

N) so that $Q(\Phi_n) = 0$. It is then natural to use this field Φ_n instead of Φ as the starting point of the orbit. The spectrum on such a “centered” orbit is of course given by (2.5) with $t = 0$, and is symmetric under the reflection $n \rightarrow -n \bmod N$.

The other extreme is $r = 0 \bmod N$. In that case Q is constant on each orbit. The possible values of the charge are $\frac{t}{N} \bmod 1$. As we will see, orbits with $Q = 0$ are of most interest. They include in particular the identity orbit. If N is odd, the current J has integral conformal spin; if N is even, the current has integral conformal spin if $r = 0 \bmod 4$, and half-integer conformal spin if $r = 2 \bmod 4$.

Between the two extremes one has the possibility that r and N have a common divisor. Let d be the largest common divisor of r and N . Then the charge $Q_{N/d}$ is constant on each orbit, and can take values $\frac{t}{d}, t = 0, \dots, d-1$ on different orbits. The charge Q takes d different values on each orbit, each occurring $\frac{N}{d}$ times.

In general, we consider two choices for $t \bmod N$ as *equivalent* if the charges and L_0 -eigenvalues on the corresponding orbits are related to each other by simply choosing a different field as the starting point of the orbit. For example, if N is prime, all possible choices for $t \bmod N$ are equivalent for $r \neq 0 \bmod N$, and they are all inequivalent for $r = 0 \bmod N$. One can always find a charge which is constant on all orbits, and whose eigenvalues distinguish the inequivalent orbits.

2.2 Modular Invariants

The characters of the conformal field theory $\mathcal{C}$ are defined in the usual way, *i.e.*

$$\chi(\Phi_i) = \text{Tr} e^{2\pi i \tau (L_0 - \frac{c}{24})},$$

where the trace is over all states obtained by acting on the ground state $\Phi_i(0)|0\rangle$ with all the negative modes of the generators of the algebra belonging to $\mathcal{C}$. Our goal is to find matrices M_{ij} with non-negative integer coefficients such that $\sum_{ij} \chi(\Phi_i) M_{ij} \chi^*(\Phi_j)$ is invariant under the modular transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\frac{1}{\tau}$ which generate the modular group of the torus.

We will now demonstrate that the presence of an orbit of simple currents in a conformal field theory implies the existence of certain modular invariants. More precisely, we will show that if a CFT has a center that is isomorphic to $\mathbf{Z}_N$, then there is a different modular

invariant for every divisor of N if N is odd or if N and the monodromy parameter r are both even, and for every divisor of $\frac{N}{2}$ if N is even and r is odd. The diagonal invariant $M_{ij} = \delta_{ij}$ is associated with the divisor 1.

Modular invariants corresponding to a divisor d of N are always obtained by considering a subgroup $\mathbf{Z}_d \subset \mathbf{Z}_N$, and treating all currents that do not belong to $\mathbf{Z}_d$ as ordinary primary fields. The $\mathbf{Z}_d$ subgroup has a monodromy parameter $r_d = r \frac{N}{d}$ and for its generating current one can take $J_{N/d}$.

Our goal is to construct an invariant that is really associated with $\mathbf{Z}_d$, rather than a subgroup of $\mathbf{Z}_d$. Thus we will require that the generating current J of $\mathbf{Z}_d$ plays a non-trivial rôle, as will be made more precise below. This ensures that modular invariants obtained by using different cyclic subgroups are really different.

Consider thus a cyclic group $\mathbf{Z}_N$ of simple currents J and its “fusion powers” J_n (where $\mathbf{Z}_N$ may be the full center or a subgroup of the center). Via the monodromy of J with any field one can define a charge $Q(\Phi_i)$ for every field Φ_i , as discussed above. Observe first of all that this charge is conserved. To see why, consider a typical operator product in the theory

$$\Phi_i(z)\Phi_j(w) = \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j} \Phi_k(w) + \text{descendants} \quad (2.8)$$

By considering the monodromy of J with respect to both sides of (2.8), and using the fact that J has unique monodromy with all fields, we can conclude that $Q(\Phi_i) + Q(\Phi_j) = Q(\Phi_k)$ for all fields Φ_k that appear on the right-hand side of (2.8). Since the charge is conserved in operator products, it is certainly conserved in all correlation functions, and hence corresponds to a symmetry of the theory.

It is this symmetry that we will use in an orbifold-like procedure to construct modular invariants. The simplest way to do this is to mimic a symmetric orbifold procedure. Since we do not have a target space interpretation of the symmetries we are using, we have to describe the procedure entirely in terms of conformal field theory, using the formalism developed in [22]. To do so we introduce a twist field $\mathcal{T}(z, \bar{z}) = J(z)J^c(\bar{z})$. The monodromy of any other field in the theory with respect to this twist field is well-defined. A primary field $\Phi_{ij}(z, \bar{z})$, which is a highest weight state with label “ i ” for the left-moving (holomorphic) algebra and with label “ j ” for the right-moving (anti-holomorphic) one, picks up a phase $e^{2\pi i(Q(\Phi_i)-Q^c(\Phi_j))}$ when carried around the twist field. In this way we identify the generator

of the symmetry which defines the orbifold twist as $Q = Q_L - Q_R^c = Q_L + Q_R$, which is indeed a conserved charge. Note that in more familiar target-space orbifold procedures one would start with a symmetry and then define the twist operator corresponding to it, whereas here we proceed in exactly the opposite order.

Having identified all the ingredients in the orbifold procedure, we can now immediately write down the corresponding modular invariant partition function. One starts with any modular invariant partition function (the diagonal invariant, for example), constructs the untwisted sector by removing all states that are not invariant under Q (*i.e.* those that do not have integer charge), and adds all twisted sectors, that is all states obtained from the untwisted sector by acting with T .

This procedure yields a modular invariant by standard orbifold arguments, and the only questions we still have to address are whether the invariant is new and non-trivial, whether there are additional modular invariants that are in some way associated with the center, but are missed in this way, and what happens if such twists are applied repeatedly. The last question is answered in the Appendix. One can write down an explicit formula for the twisted invariant, and show that repeated application of twists of order $d_1, \dots, d_m$ yields an answer that depends only on the smallest common multiple of $d_1, \dots, d_m$, and not on the order of the twists or the choice of the divisors d_i .

If the center consists of more than one $\mathbf{Z}^N$ factor, one can superimpose the invariants of every subgroup of every factor of the center, by applying the corresponding twists successively. In general, twists from *different* $\mathbf{Z}_N$ factors do *not* commute, however. We will not describe the possible results in detail here, but an example is discussed in the next section.

To answer the other questions we will discuss the various kinds of modular invariants that can be obtained, starting with the simplest cases. There are a few additional technical problems in the discussion for even N , and we will therefore restrict ourselves to odd N for a while.

Invariants for odd N

Consider first N prime, with $r \neq 0 \bmod N$. If one starts with the diagonal invariant, the untwisted sector consists of all left-right combinations of the orbit generators Φ with $Q = 0$, since this is the only way to get $Q_L + Q_R = 0$ on fields with the same left and right components. Now we add the twisted sectors, generated by acting with T and its powers

on the untwisted sector. The result is that while $J(z)$ generates the orbit, $J^c(\bar{z})$ generates it in reverse order. Instead of a diagonal pairing δ_{ij} one gets a pairing Π_{ij} , where Π is the transformation that interchanges simultaneously, in one chiral sector, all fields Φ_n and Φ_{-n} in each orbit. For example if $N = 5$ each orbit has fields $\Phi_i, i = 0, \dots, 4$ with characters χ_i , and the invariant has the form

$$\sum_{\text{all orbits}} \chi_0 \chi_0^* + \chi_1 \chi_4^* + \chi_2 \chi_3^* + \chi_3 \chi_2^* + \chi_4 \chi_1^* .$$

Modular invariants of this type always contain all primary fields of $\mathcal{C}$, but with a non-trivial left-right pairing Π_{ij} . This pairing defines an automorphism of the fusion rules [4]. We will refer to modular invariants of this type as *fractional spin invariants*, since the current J has fractional spin.

Consider now integer spin algebras, with $r = 0 \bmod N$ (where N is again an odd prime). Starting with the diagonal invariant, one keeps in the untwisted sector only those states which have *total* charge $Q = 0$. This corresponds to diagonal combinations within all the $Q = 0$ orbits of the *chiral* theory. It is easy to see that by adding the twisted sector one generates precisely all the off-diagonal terms in each $Q = 0$ orbit, so that the result can be written as a square. Using again $N = 5$ as an example, we get now

$$\sum_{Q=0 \text{ orbits}} \left| \sum_{i=0}^4 \chi_i \right|^2 .$$

This invariant has the form of a sum of squares, and looks like the diagonal invariant of some new conformal field theory, in which the one we started with is embedded. Although in general the currents J can have arbitrarily large integer spin, so that in general very little is known about the algebras they might generate, it is quite clear that at least as far as the modular invariant is concerned we can think of an extension of the original algebra by the set of currents J_n . The primary fields of the original algebra are grouped into representations of the extended algebra. These representations are precisely the $Q = 0$ orbits. In invariants of this type, which we will call *integer spin invariants*, not all primary fields of the original conformal field theory $\mathcal{C}$ are present.

We have indicated before that in general it can happen that some fields generate orbits whose length l is not N , but a divisor of N (if N is prime, the only new possibility is

of course $l = 1$). In that case the orbifold procedure will generate such orbits (provided they have $Q = 0$) with multiplicity $\frac{N}{l}$. In terms of the extended algebra these $\frac{N}{l}$ fields can presumably be distinguished, but they all have the same characters. A simple example of this situation occurs for Kac-Moody algebras of groups with complex representations. Complex conjugate ground state representations yield the same Virasoro-character, but can be distinguished by their transformation properties with respect to the Lie-algebra. In a diagonal invariant of such a Kac-Moody algebra (e.g. $SU(N)$, $N \geq 3$) complex representations appear with multiplicity two if one does not distinguish complex conjugate representations. The appearance of non-trivial multiplicities in integer spin invariants is also familiar from the D_{2l} -series of $SU(2)$ modular invariants (see e.g. [4]).

Having dealt completely with values of N equal to odd primes, let us now consider the case $N = p^i$, where p is an odd prime, and $i \in \mathbb{Z}$. There is nothing new to be said if r and N are relative prime, or if $r = 0 \bmod N$. One simply obtains exactly the same kinds of modular invariants as for prime N . If, however, r contains a factor p^j , $j \neq 0$ and $j < i$, the Q -charges on the orbits appear with degeneracy p^j , and there are p^j inequivalent choices (in the sense in section (2.1)) for the parameter t in (2.6). Furthermore the charge $Q_{p^{i-j}}$ is constant on every orbit, and takes values $\frac{t}{p^j}$, $t = 0, \dots, p^j - 1$. Orbits with $Q_{p^{i-j}} \neq 0$ are projected out completely, whereas in orbits with $t = 0$ p^j diagonal terms with $Q_L = Q_R = 0$ are kept in the untwisted sector.

To build the $\mathbf{Z}_N$ invariant it is simplest to consider first the $\mathbf{Z}_{p^j}$ invariant generated by $J_{p^{i-j}}$. It is easy to see that all $\mathbf{Z}_{p^j}$ currents have integer spin, so that one obtains an integer spin invariant as discussed earlier. To this invariant we apply then the twist $\mathcal{T} = J_L J_R^c$. In general two things may happen if one twists an integer spin invariant: either J appears in the integer spin invariant, or it does not. If it does not, applying $\mathcal{T}$ unravels the integer spin invariant partly or completely, and one recovers in the twisted sector some fields that were previously projected out. If J does appear, applying $\mathcal{T}$ leads to a “fractional spin invariant on top of an integer spin invariant”, which means that the primary fields appearing in the integer spin invariant are paired non-diagonally. Although, as shown in the appendix, it does not matter how one builds the $\mathbf{Z}_N$ invariant, it is certainly simplest to construct first the maximal integer spin invariant that does not project out the current J . It is easy to show that if r contains j factors of p , this maximal integer spin invariant is the one belonging to the subgroup $\mathbf{Z}_{p^j}$.

As an example, consider $N = 9$. If one chooses $r = 1$ one has the following three invariants

$$\begin{aligned} \mathbf{Z}_1 : & \sum_{\text{all orbits}} \sum_{i=0}^8 |\mathcal{X}_i|^2 \\ \mathbf{Z}_3 : & \sum_{\text{all orbits}} |\mathcal{X}_0 + \mathcal{X}_3 + \mathcal{X}_6|^2 \\ \mathbf{Z}_9 : & \sum_{\text{all orbits}} \sum_{i=0}^8 \mathcal{X}_i \mathcal{X}_{9-i}^* . \end{aligned}$$

The first of these is just the diagonal invariant. The $\mathbf{Z}_3$ invariant is generated by the current J_3 , and the $\mathbf{Z}_9$ invariant by J . If one chooses $r = 3$ the first invariant does not change, but for the other two we find

$$\begin{aligned} \mathbf{Z}_3 : & \sum_{Q_3=0 \text{ orbits}} |\mathcal{X}_0 + \mathcal{X}_3 + \mathcal{X}_6|^2 + |\mathcal{X}_1 + \mathcal{X}_4 + \mathcal{X}_7|^2 + |\mathcal{X}_2 + \mathcal{X}_5 + \mathcal{X}_8|^2 \\ \mathbf{Z}_9 : & \sum_{Q_3=0 \text{ orbits}} |\mathcal{X}_0 + \mathcal{X}_3 + \mathcal{X}_6|^2 + [(\mathcal{X}_1 + \mathcal{X}_4 + \mathcal{X}_7)(\mathcal{X}_2 + \mathcal{X}_5 + \mathcal{X}_8)^* + \text{c.c.}] . \end{aligned}$$

The last one is an example of a fractional spin invariant on top of an integer spin invariant. The charge Q_3 belongs to J_3 , and is equal to $3Q$ (all identities involving charges are modulo one, as before). Finally, if one chooses $r = 9$ the $\mathbf{Z}_1$ and $\mathbf{Z}_3$ invariants stay the same, and for the $\mathbf{Z}_9$ invariant one gets

$$\sum_{Q=0 \text{ orbits}} \left| \sum_{i=0}^8 \mathcal{X}_i \right|^2 .$$

To complete the discussion for odd N we have to consider combinations of different primes. This is simply a matter of combining the invariants of the various prime factors, which may be done in any order. For example, if $N = 15$ there is a $\mathbf{Z}_1$, a $\mathbf{Z}_3$, a $\mathbf{Z}_5$ and a $\mathbf{Z}_{15}$ invariant. If r and 15 are relative prime all (except $\mathbf{Z}_1$) are fractional spin invariants; if r contains a factor of 3, the $\mathbf{Z}_3$ invariant becomes an integral spin invariant, which may be twisted by $\mathbf{Z}_5$. If r is proportional to 15, all invariants are of integral spin type. There are no subtle new features appearing in these combinations.

All invariants discussed so far have a simple feature that distinguishes them uniquely, namely the presence of the combination of currents $(J_n)_L (J_{N-n})_R$. A $\mathbf{Z}_d$ invariant always contains this combination for $n = \frac{N}{d}$, but not for any smaller value of n . For example the

$\mathbf{Z}_3$ invariants given above all contain $\mathcal{X}_3\mathcal{X}_6^*$ but not $\mathcal{X}_1\mathcal{X}_8^*$; the latter combination appears only in the $\mathbf{Z}_9$ invariant. Obviously it follows that for each divisor of N one finds a *different* invariant.

Invariants for even N

In dealing with even N we face a problem that is most easily illustrated by discussing first $N = 2$. The twist operator $\mathcal{T} = J(z)J(\bar{z})$ obviously acts trivially on the diagonal invariant. Hence one does not get a new invariant by using $\mathcal{T}$. Nevertheless, in some cases a new invariant does exist.

If $N = 2$, the current can have conformal spin $0, \frac{1}{4}, \frac{1}{2}$ or $\frac{3}{4}$ (modulo 1) (see (2.4)). The half-integer conformal spin algebras are generalizations of the familiar Neveu-Schwarz-Ramond (spin $\frac{1}{2}$) and super-Virasoro (spin $\frac{3}{2}$) algebras. The primary fields of the theory fall into two classes, Neveu-Schwarz fields (which are local with respect to the current) and Ramond fields (which have a square-root branch cut). The modes of the current increase h by half-integer or integer values respectively when acting on these fields. If the conformal spin of the current is $\frac{1}{4}$ or $\frac{3}{4}$ the primary fields of the original conformal field theory are also paired by the current. The multiplets one obtains in this case can have conformal spins $(h, h + \frac{1}{4} \bmod 1)$ or $(h, h + \frac{3}{4} \bmod 1)$. In fact these two kinds of multiplets are not really distinct: if one interchanges the role of Φ and its “descendant” $J\Phi$ in one of them, one gets the structure of the other.

If the current has integer spin, we can perform an orbifold construction with the current itself (i.e. $J_L 1_R$ rather than $J_L J_R^c$) as a chiral twist field. This yields an integral spin invariant with a structure completely analogous to the ones encountered before.

If the current has half-integer spin, we can apply the familiar method of GSO projections to construct a new modular invariant. One uses the fact that the “PP-sector” (which in path-integral language corresponds to integrands that are periodic along the two non-contractible loops on the torus) is separately modular invariant. The PP-characters are obtained by computing the trace of $e^{2\pi i L_0 - \frac{c}{24}} (-1)^F$ in the Ramond sector, where F counts the number of times the current has acted on the ground state (for the ground state one may choose either one of the two primary fields that form a Ramond multiplet). This modular invariant is not positive, but one can obtain a positive modular invariant by subtracting it from the diagonal invariant. The result is diagonal in the Neveu-Schwarz sector, but off-diagonal in the Ramond sector. The off-diagonal terms are always left-right combinations

of pairs of fields that form a Ramond multiplet. This is precisely the structure of the D_n series of $SU(2)$, level k modular invariants for *odd* values of n .

The last possibility for $N = 2$ is that the current has conformal spin $\frac{1}{4} \bmod \frac{1}{2}$. This is the only case where no new modular invariant can be defined that is associated with the current J , because there are no states that preserve their T -eigenvalue under the action of J (here $T = e^{2\pi i L_0}$). All invariants we have encountered so far contain some off-diagonal combinations of characters from the same orbit. But if there are no degenerate T -eigenvalues within any orbit, there is clearly no way to construct an invariant (at least not one that has anything to do with the current J).

This can also be understood in terms of level-matching conditions [23] for orbifolds. In terms of twist fields, the level matching conditions for ordinary orbifolds imply that the conformal spin of the twist field must be $\frac{k}{N}$ if the order of the twist is N . This condition is automatically satisfied if we use the twist operator $J_L J_R^c$, which has integer conformal spin. However this twist operator gives a trivial answer if $N = 2$. Instead we can try to use an asymmetric orbifold procedure by using J itself as a chiral twist operator. This is the procedure we followed above for integer and half-integer spin currents. This will work only if the conformal spin of the chiral current satisfies level matching, which it does for N odd and for N even, r even, but not for N even, r odd. In the latter case we should therefore not expect to find a modular invariant.

Consider now $N = 4$. Now the symmetric orbifold procedure using $\mathcal{T} = (J_1)_L (J_3)_R$ does produce a non-diagonal invariant, characterized by the presence of $\mathcal{T}$ in its field content. However, it is easy to see that one obtains in this way simply the modular invariant associated with the $\mathbf{Z}_2$ subgroup of $\mathbf{Z}_4$, discussed above. Note that the monodromy parameter of $\mathbf{Z}_2$ is twice that of $\mathbf{Z}_4$, and hence it is always even. Therefore the $\mathbf{Z}_2$ modular invariant always exists if $\mathbf{Z}_2$ is embedded in a $\mathbf{Z}_4$ subgroup of the center.

This raises the question whether there is yet another invariant associated with the full $\mathbf{Z}_4$ rather than a subgroup. The answer is again suggested by level matching: we can expect a new invariant if r is even. This invariant can probably be constructed by means of some generalization of the asymmetric orbifold procedure, but we have not succeeded yet in formulating it. However, we have checked in some non-trivial cases that this conjectured invariant does indeed exist.

All of this can easily be generalized to other values of N . If $N = 2 \bmod 4$, one does

not obtain a new invariant by twisting with $J_L J_R^c$. This can be seen most easily by using the result of the appendix, which shows that such an invariant can also be obtained by performing first a Z_2 and then a $Z_{N/2}$ twist. But the former (when performed symmetrically) has no effect, so that one merely obtains the $Z_{N/2}$ invariant. To get the genuine Z_N invariant (which exists only for r even) one has to perform the Z_2 twist in the way explained above.

If N contains more than one factor of 2, we expect a genuine Z_N invariant whenever level matching is satisfied, *i.e.* if r is even. A convincing argument for the existence of such Z_N modular invariants can be given if Z_N appears as a subgroup of a Z_{2N} group of simple currents. One can then construct an invariant using the twist field $\mathcal{T} = J_L J_R^c$, where J is the current that generates Z_{2N} .

Whether one associates this invariant with Z_N or Z_{2N} is a matter of taste, but the Z_4 example suggests that it is most naturally associated with Z_N . In any case the following is true. Suppose $N = 2^p N_{\text{odd}}$, where N_{odd} is odd. Suppose J is a current generating a Z_d subgroup of Z_N . Then the symmetric twist procedure using $J_L J_R^c$ generates a *different* invariant for every value of d that is odd or divisible by four (as for odd N , different invariants are uniquely characterized by the presence of $J_L J_R^c$). If d is divisible by four one obtains invariants that are most naturally associated with $Z_{\frac{d}{2}}$. Hence this procedure does not produce an invariant for values of d that are divisible by 2^p , the maximal number of factors of 2. Because of the level matching conditions for the Z_d current there is probably no such invariant if r is odd. But we do expect such an invariant if r is even.

While we do not know an orbifold construction for invariants with r even and d having the maximal number of factors of two, we do know their exact form. A useful mnemonic way of constructing it is to double the field content of the conformal field theory by doubling N and dividing r by two. To do so one must double all orbits in length, so that the original fields correspond to (2.5) with n even. New “phantom” fields are added for the odd values of n . Now one generates the Z_N invariant by using the phantom Z_{2N} current in a symmetric orbifold procedure. It is easy to see that all phantom fields are paired with phantom fields. If one removes all terms with phantom fields one obtains the (conjectured) Z_N modular invariant. We emphasize once more that this procedure is only a mnemonic device, and does not have a physical meaning (as far as we know).

To illustrate this perhaps somewhat complicated series of arguments, we have worked out all the modular invariants one can obtain for $N = 8$, with $r = 1, 2, 4, 8$ and 16. The

results are as follows (we omit the diagonal invariant, which is associated with $\mathbf{Z}_1$)

$$\begin{aligned}
r = 1 \quad \mathbf{Z}_2 & \sum_{\text{all orbits}} [|\mathcal{X}_0 + \mathcal{X}_4|^2 + |\mathcal{X}_2 + \mathcal{X}_6|^2] \\
\mathbf{Z}_4 & \sum_{\text{all orbits}} \sum_{i=0}^7 \mathcal{X}_i \mathcal{X}_{-i}^* \\
\mathbf{Z}_8 & \text{Level matching condition not satisfied} \\
r = 2 \quad \mathbf{Z}_2 & \sum_{Q_4=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_5|^2 + |\mathcal{X}_3 + \mathcal{X}_7|^2 + |\mathcal{X}_2 + \mathcal{X}_6|^2] \\
\mathbf{Z}_4 & \sum_{Q_4=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_4|^2 + \{(\mathcal{X}_1 + \mathcal{X}_5)(\mathcal{X}_3 + \mathcal{X}_7)^* + \text{c.c}\} + |\mathcal{X}_2 + \mathcal{X}_6|^2] \\
\mathbf{Z}_8 & \sum_{\text{all orbits}} \sum_{i=0}^7 \mathcal{X}_i \mathcal{X}_{-t-i}^* \\
r = 4 \quad \mathbf{Z}_2 & \sum_{Q_4=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_5|^2 + |\mathcal{X}_3 + \mathcal{X}_7|^2 + |\mathcal{X}_2 + \mathcal{X}_6|^2] \\
\mathbf{Z}_4 & \sum_{Q_2=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_2 + \mathcal{X}_6 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_3 + \mathcal{X}_5 + \mathcal{X}_7|^2] \\
\mathbf{Z}_8 & \sum_{Q_4=0 \text{ orbits}} \sum_{i=0}^3 (\mathcal{X}_i + \mathcal{X}_{i+4})(\mathcal{X}_{-t/2-i} + \mathcal{X}_{-t/2-i+4})^* \\
r = 8 \quad \mathbf{Z}_2 & \sum_{Q_4=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_5|^2 + |\mathcal{X}_3 + \mathcal{X}_7|^2 + |\mathcal{X}_2 + \mathcal{X}_6|^2] \\
\mathbf{Z}_4 & \sum_{Q_2=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_2 + \mathcal{X}_6 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_3 + \mathcal{X}_5 + \mathcal{X}_7|^2] \\
\mathbf{Z}_8 & \sum_{Q_2=0 \text{ orbits}} \sum_{i=0}^1 (\mathcal{X}_i + \mathcal{X}_{i+2} + \mathcal{X}_{i+4} + \mathcal{X}_{i+6}) \\
& \quad \times (\mathcal{X}_{-t/4-i} + \mathcal{X}_{-t/4-i+2} + \mathcal{X}_{-t/4-i+4} + \mathcal{X}_{-t/4-i+6})^* \\
r = 16 \quad \mathbf{Z}_2 & \sum_{Q_4=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_5|^2 + |\mathcal{X}_3 + \mathcal{X}_7|^2 + |\mathcal{X}_2 + \mathcal{X}_6|^2] \\
\mathbf{Z}_4 & \sum_{Q_2=0 \text{ orbits}} [|\mathcal{X}_0 + \mathcal{X}_2 + \mathcal{X}_6 + \mathcal{X}_4|^2 + |\mathcal{X}_1 + \mathcal{X}_3 + \mathcal{X}_5 + \mathcal{X}_7|^2] \\
\mathbf{Z}_8 & \sum_{Q=0 \text{ orbits}} \left| \sum_{i=0}^7 \mathcal{X}_i \right|^2
\end{aligned}$$

Note that in general the contribution of an orbit depends on the value of the parameter t . In those cases where all choices of t are equivalent (in the sense explained earlier) it is

simplest to choose $t = 0$. Any other choice would merely correspond to a relabelling of the characters in the orbit. However for the $\mathbf{Z}_8$ invariants not all orbits that contribute are equivalent, and then there is no canonical choice for t ; hence the appearance of the parameter t for the $\mathbf{Z}_8$ invariants. The reason that we have not encountered this problem before is that for *odd* N all orbits appearing in a modular invariant are always equivalent. The simplest example of non-equivalent orbits occurs for $N = 2$ and half-integer spin: the Ramond and Neveu-Schwarz orbits are clearly not equivalent.

The modular invariants for other r -values have the same structure as the one given above, if the number of factors of two matches. However the precise t -dependence depends on r in a more complicated way. For 5 or more factors of 2 one gets the same answer as for $r = 16$. In this example, the invariants that cannot be obtained by the symmetric twist procedure or by GSO projections are the $\mathbf{Z}_8$ invariants for $r = 2$ and $r = 4$.

3. Modular Invariants of Kac-Moody algebras

In this section we apply the formalism of the previous section to Kac-Moody algebras. To prevent confusion we emphasize that our interest is not in the Kac-Moody currents themselves, but in additional currents that define an extension of the Kac-Moody algebra (which itself is an extended Virasoro algebra). Thus the Kac-Moody algebra serves here simply as an example of a rational conformal field theory.

We begin by describing the most interesting case, the level k $SU(N)$ Kac-Moody algebras. To identify the simple currents we need first of all some information about the fusion rules.

The primary fields of $SU(N)$ level k are most conveniently characterized by means of Young tableaux. The representations are built on ground states that belong to $SU(N)$ representations $[m_1, \dots, m_{N-1}]$ with $m_1 \leq k$, where m_i is the length of the i^{th} row. One can in principle obtain all the fusion rules from the matrix S representing the modular transformation $\tau \rightarrow -\frac{1}{\tau}$, for which a general formula exists [24,25]. To obtain the fusion rules one simply substitutes S into the formula obtained in [2]

$$\begin{aligned}\Phi_i \cdot \Phi_j &= \sum_k N_{ijl} \Phi_l \\ N_{ijl} &= \sum_n \frac{S_{in} S_{jn} S_{nl}^*}{S_{1n}}\end{aligned}$$

where the indices label all primary fields, with “1” representing the identity. Since the

formula for S involves a sum over the entire Weyl-group this approach is not very practical except for computer calculations. In addition using S explicitly goes against the spirit of this paper, which is to derive modular invariants directly from the fusion rules.

It should be possible to obtain the fusion rules directly using Young tableaux, with some algorithm analogous to that for computing tensor products of $SU(N)$ Lie-algebra representations. It is already known how to do that for $SU(2)$ [26]. Although the fusion rules for $SU(N)$ are considerably more complicated in general, we need them only in a few special cases, where they are still fairly easy to obtain. The fields that can appear in the fusion of two primary fields Φ_i and Φ_j characterized by two Young tableaux Y_i and Y_j are a subset of the ones appearing in tensor product of Y_i and Y_j , computed in the usual way. Thus if

$$Y_i \otimes Y_j = \sum_l M_{ijl} Y_l ,$$

then $N_{ijl} \leq M_{ijl}$. One might have hoped that it is sufficient to determine whether Y_l represents a Kac-Moody primary field, but unfortunately this is not always unambiguous. To appreciate this, observe that Y_l has the general form $[m_1, \dots, m_{N-1}, m_N]$, with m_N not necessarily equal to zero (as it was by assumption for Y_i and Y_j). In computing $SU(N)$ tensor products one may always cancel the first m_N columns of Y_l , but it is not *a priori* clear whether one should impose the Kac-Moody condition $m_1 \leq k$ before or after doing that. In other words, one cannot determine in this way whether or not a Young-tableau with $m_1 > k$ but $m_1 - m_N \leq k$ corresponds to a term in the fusion rule of Φ_i and Φ_j . Both possibilities can occur in practice.

Rather than trying to formulate general rules to deal with these ambiguous cases, we concentrate here on the unambiguous ones. The unambiguous part of the fusion rules can be obtained as follows. One should start with two Young tableaux Y_i and Y_j with a first column of length smaller than N . Then one computes the tensor product of the Young tableaux in the usual way, without cancelling columns of length N in the result. Any Young tableau with a first row of length not larger than k must represent a term in the fusion rules, since it implies that there is coupling between three primary fields. Any Young tableau with $m_1 - m_N > k$ does not correspond to a primary field even after cancelling columns, and hence does not correspond to a term in the fusion rules. The remaining terms in the product are ambiguous, and one has to use more sophisticated techniques to determine whether or not they should appear.

Most primary fields can be eliminated as candidate simple currents by considering their fusion rules with the primary field represented by $Y_F = [1, 0, \dots, 0]$. Every primary field yields at least two others in its fusion rule with Y_F , unless it is represented by a Young tableau Y_n , with n rows of length k . Consider first the primary field $Y_1 = [k, 0, \dots, 0]$. The fusion rule of this current with any other field is very simple to obtain: one simply increases the length of the first k columns by one, a procedure which clearly yields a valid new Young-tableau. It is manifestly the *only* valid one if none of the columns of the original tableau has length $N - 1$, since in that case there is no column cancelling ambiguity. One can deal with the remaining cases, *i.e.* $m_{N-1} \neq 0$, by induction in m_{N-1} . First write

$$[m_1, \dots, m_{N-1} + 1] = [m_1, \dots, m_{N-1}] \otimes [1, 0, \dots, 0] - \text{extra terms} ,$$

where one subtracts all terms in the tensor product that are not equal to the left-hand side. All these terms have an $(N - 1)^{\text{th}}$ column of length m_{N-1} or less. The induction assumption is that the fusion rule of Y_1 with all primary fields with an $(N - 1)^{\text{th}}$ row not longer than m_{N-1} is obtained by adding 1 to the first k columns. Now compute the fusion rule of the right-hand side with $Y_1 = [k, 0, \dots, 0]$. Using commutativity and associativity of the fusion rules, one gets for the first term $([m_1, \dots, m_{N-1}] \otimes Y_1) \otimes [1, 0, \dots, 0]$, which has an unambiguous answer by the induction assumption (one should of course cancel columns within the brackets before multiplying with $[1, 0, \dots, 0]$). After subtracting the product of the extra terms with Y_1 , one obtains the expected result.

Thus we have proved that Y_1 represents a simple current. This is then also true for its orbit $Y_n, n = 0, \dots, N - 1$, obtained by repeated application of the fusion rules. Hence we have identified the N primary fields corresponding to the Young-tableaux Y_n as the only simple currents. The abelian group generated by these currents is isomorphic to the center Z_N of $SU(N)$.

It is straightforward to compute the eigenvalues of L_0 on these currents. The result is

$$h_n = \frac{k}{2N} n(N - n) .$$

This agrees with the general formula (2.4) if we identify the monodromy parameter r with the level k . Note that (2.4) was derived without any reference to Kac-Moody algebras.

To find examples of Kac-Moody modular invariants we can now simply apply the rules formulated in section 2. We find a different modular invariant for any divisor of N if N is odd or if k is even, and a different invariant for any divisor of $\frac{N}{2}$ if N is even and k is odd.

In comparison with the modular invariants presented in all other work we are aware of we conclude the following. For $SU(2)$, our results give the A and D series of modular invariants found in [10]. The D_n -invariants for even n are identified as integer spin invariants, while for n odd they are half-integer spin invariants. In the next section we will comment on the exceptional invariants. Modular invariants for $SU(N)$ at level 1 have been presented in [13], and were found to be labelled by the divisors of N , so that they are undoubtedly identical to ours.* In [11] several infinite series of modular invariants were found for all simple Lie algebras with a non-trivial center. From our point of view, these invariants correspond all to integer and half-integer spin algebras, and for $SU(N)$ the results of [11] for this sub-class agree with ours. Finally, in [12] some invariants that we would associate with fractional spin currents have been identified. For $SU(N)$ with N prime we do not find anything new, but for other values of N there are many new solutions, which are obtained either as “fractional spin invariants on top of integer spin invariants” or as invariants belonging to subgroups of $\mathbf{Z}_N$. Indeed, in [12] modular invariants are found only if N and the level k are relative prime, a condition which implies the absence of simple integer spin currents.

It is straightforward to construct new modular invariants explicitly. For example, one may use the formulas given in section 2 for $N = 5$, $N = 8$ and $N = 9$. To apply these formulas, one simply has to combine the $SU(N)$ primary fields into orbits generated by the appropriate currents. Since the $SU(N)$ fusion rules for simple currents have a very simple realization in terms of Young tableaux, this is an easy task.

In $SU(N)$ (and in any Lie-algebra with complex representations) there is always an obvious way to obtain new modular invariants, namely by multiplying a matrix M representing a modular invariant by the charge conjugation matrix $C = S^2$. In particular $M = C$ defines a modular invariant. If one is only interested in the τ -dependence of a modular invariant,

* All our modular invariants for simply laced algebras at level 1 can be represented in terms of self-dual lattices. This is true because all extended currents of simply laced level 1 algebras (remember that we are *not* interested in the Kac-Moody currents themselves) can be represented by exponentials of free bosons. Hence our construction reduces to the shift-vector method for constructing self-dual lattices. Furthermore it is easy to see that we get all possible self-dual lattices for a given simply Lie-algebra, since all those lattice can be mapped into each other by using combinations of weights as shift vectors. Hence the classification of modular invariants of simply laced algebras at level 1 is complete, unless there exist invariants that do not correspond to self-dual lattices (which seems very unlikely).

these solutions are not really different, since a representation and its conjugate have the same character. For some small values of k our invariants are trivial in this sense, but in general they are not. Indeed, for $k \geq 2$ the representation $[1, 0, \dots, 0]$ and its conjugate are not even in the same orbit, so that they can never be paired. On the other hand this implies that there are some invariants (albeit trivial ones), that exist for arbitrary k , and cannot be obtained with our method.

To conclude this section we describe the analogous results for the other simple Lie-algebras. First of all, the center of the Kac-Moody conformal field theory (as defined in the previous section) is always isomorphic to the center of the Lie-algebra, so that there are no new modular invariants of this type for G_2 , F_4 and E_8 . For the remaining algebras, the simple currents correspond to representations with highest weights kw_i , where w_i is the highest weight of the vector representation or the two inequivalent fundamental spinor representations of the Lie-algebra D_n , the (27) or ($\overline{27}$) of E_6 , the (56) of E_7 , the vector representation of B_n and the representation with Dynkin labels $(0, 0, \dots, 0, 1)$ for C_n . These highest weights for these currents can be inferred from the results of [11] and [12], by interpreting the results of these papers in terms of current algebras. Note that for simply laced groups these are the weights of smallest length in each conjugacy class, but this rule does not apply for the non-simply laced algebras. The L_0 eigenvalues of these currents are always given by $\frac{k}{2}(w_i, w_i)$, in a normalization so that the longest root has norm 2 (so that for C_n one finds then $h = \frac{kn}{4}$). We have checked by explicit calculation (using the Kac-formula for the matrix S to get the fusion rules) that they are indeed simple currents for the lowest values of k and the rank of the Lie-algebra. With the help of a more efficient algorithm for the fusion rules of these algebras one should be able to derive these simple currents algebraically for arbitrary rank and level, just as for $SU(N)$.

Since the order of the center is 2 or 3 for B_n , C_n , E_6 and E_7 , we only recover the already known invariants of [11] and [12] for these Kac-Moody algebras. For D_n the problem is a bit more complicated. If n is odd the center is Z_4 , and the conformal spins of the currents with highest weights kw_s, kw_v, kw_c are $\frac{kn}{8}, \frac{k}{2}, \frac{kn}{8}$. This matches the general formula (2.4) for $kn = 3r \bmod 8$, which always has a solution for odd n . One always has an integer or half-integer spin invariant associated with the Z_2 subgroup of Z_4 , and one gets one for Z_4 if $k = 0 \bmod 4$. All these invariants are already included in [11]. If k is odd, level matching is not satisfied for Z_4 . Finally, if $k = 2 \bmod 4$ we get a new invariant associated with Z_4 (here we need the even- N conjecture of section 2). This spin- $\frac{1}{4}$ invariant has (probably)

been found already in [12].

For D_n with n even the center is $\mathbf{Z}_2 \times \mathbf{Z}_2$, and we get in some cases new invariants coming from combinations of the two factors, namely whenever the conformal spins of both the vector and the spinor currents ($\frac{k}{2}$ and $\frac{kn}{8}$ respectively) are (half)-integral.

A perhaps somewhat too trivial example is D_4 level 1. The primary fields correspond just to the conjugacy classes $(0), (v), (s)$ and (c) , which are all simple currents. The modular invariants generated by these currents are respectively

$$\begin{aligned} & (0,0) + (v,v) + (s,s) + (c,c) \\ & (0,0) + (v,v) + (s,c) + (c,s) \\ & (0,0) + (v,c) + (s,s) + (c,v) \\ & (0,0) + (v,s) + (s,v) + (c,c) , \end{aligned}$$

in a notation which should be self-explanatory. The first one is the diagonal invariant, while the last three are generated by the spin- $\frac{1}{2}$ currents $(v), (s)$ and (c) . These invariants are all contained in [11]. One can make new invariants by applying these twists successively. For example by applying first (v) and then (s) we get

$$(0,0) + (v,c) + (s,v) + (c,s) .$$

Note that this invariant is not left-right symmetric, and that the two twist do not commute (this does not disagree with the results of the appendix, since the two twists generate two separate $\mathbf{Z}_2$ factors). By applying all possible combinations of twists, we get all invariants of the form

$$(0,0) + \sum_{i=v,s,c} (i, \Pi(i)) ,$$

where Π is an element of the permutation group S_3 acting on the three conjugacy classes $(v), (s)$ and (c) .

This example has the disadvantage that the characters $(v), (s)$ and (c) are identical because of triality, so that the six modular invariants are not really different. However, the same $\mathbf{Z}_2 \times \mathbf{Z}_2$ center (with half-integer spin currents) appears always for $D_n, n = 0 \bmod 4, k = 1 \bmod 2$, and in general the invariants are not all the same. Other cases

not included in [11] occur when the spinor currents have half-integer spin, and the vector current integer spin ($k = 0 \bmod 2, n = 2 \bmod 4$) or vice versa ($k = 1 \bmod 2, n = 0 \bmod 8$). In these cases one can twist the integer spin invariant by means of the half-integral spin current. Finally it may happen that all three currents have integer spin (namely if $k = 0 \bmod 2, kn = 0 \bmod 8$), and a new invariant can then be obtained by using all three of them in the identity block of the new extended algebra. For example, D_4 level 2 has an invariant^{*}

$$|\mathcal{X}_{0000} + \mathcal{X}_{2000} + \mathcal{X}_{0020} + \mathcal{X}_{0002}|^2 + 4|\mathcal{X}_{0100}|^2,$$

where the subscripts denote the Dynkin-labels of the ground states. In addition there are the invariants generated by each of the three currents separately, which have already been described in [11].

4. Exceptional Invariants: Conjectures and Counterexamples

An interesting and important question is to which extent the discussion in the foregoing section can be extended to the complete set of modular invariants of a rational conformal field theory. It turns out that one has to be very careful in drawing general conclusions. We will illustrate that here for Kac-Moody algebras by listing some tempting, but incorrect generalizations and the corresponding counterexamples.

It would obviously be a mistake to conjecture that the results of section 2 could yield all modular invariants of a given theory. The E_6 , E_7 and E_8 invariants of $SU(2)$ do not come out of our previous analysis, and hence are counterexamples to that conjecture. A second rather general method for constructing modular invariants has been discussed in [27]. These authors point out that one can obtain modular invariants for a conformal sub-algebra $\mathcal{H}$ of an algebra $\mathcal{G}$ by simply writing a $\mathcal{G}$ modular invariant in terms of primary fields of $\mathcal{H}$. All such invariants are easy to obtain since all conformal sub-algebras have been classified in [16]. However, this does not complete the classification of modular invariants either: a counterexample is the E_7 -invariant of $SU(2)$.

The conformal embedding invariants are (with a few exceptions) not described by the arguments of section 2, but they are a rather natural extension, since they are clearly obtained by extending the Kac-Moody algebra by additional integral spin currents. Such

^{*} All modular invariants in this section and the next one have been verified by explicit calculation.

integral spin invariants can be recognized easily by the presence of off-diagonal terms involving the identity and some other primary field of integral conformal spin. In all cases we know one can complete squares, so that the modular invariant has the form

$$|\chi_0 + \sum_i m_i \chi_i|^2 + \dots, \quad (4.1)$$

where χ_0 is the character of the identity, and the χ_i 's are characters of integral spin fields, appearing with multiplicity m_i . The remaining terms do not contain χ_0 . In the simplest cases these other terms are also a sum of squares of linear combinations of primary fields, and then the invariant can be regarded as a diagonal invariant of some extended algebra generated by the integral spin currents. However, in some cases one can choose a non-trivial pairing of the characters of the extended algebra. Examples are the E_7 -invariant of $SU(2)$, as discussed in [4], [7], the $SU(3)$ -invariant at level 9 found in [7], and furthermore such examples appear in abundance among the modular invariants discussed in section 2. A popular and still unchallenged conjecture is that any positive modular invariant in which the identity appears off-diagonally can be written as a diagonal or “twisted” invariant of an integral spin algebra (see *e.g.* [4]).

Integral spin invariants obtained via conformal embeddings can always be recognized by the presence of a spin-1 current in the set (4.1). These spin-1 currents are primary fields of $\mathcal{H}$, which combine with the Kac-Moody generators of $\mathcal{H}$ to form those of $\mathcal{G}$. This structure is quite obvious in the E_6 invariant of $SU(2)$. In general there are however additional integral spin currents in the set (4.1), as is clearly demonstrated by the E_8 -invariant of $SU(2)$.

The modular invariants discussed so far in the literature still allow the conjecture that all integral spin modular invariants (in the sense defined above) are either part of the regular series described in section 2, or they are conformal embeddings. We have looked rather systematically for integral spin invariants for Kac-Moody algebras at sufficiently small levels (*e.g.* $k \leq 30$ for A_2 and G_2 , $k \leq 25$ for B_2 , $k \leq 10$ for A_3, B_3, D_3 and F_4 , $k \leq 6$ for other rank 4 algebras, etc.), and we have found just one counterexample to this conjecture, namely the following invariant of F_4 at level 6

$$|\chi_{0000} + \chi_{0004} + \chi_{0030} + \chi_{1100}|^2 + |\chi_{0003} + \chi_{0006} + \chi_{0021} + \chi_{2010}|^2 \\ + |\chi_{0101} + \chi_{1012}|^2 + |\chi_{0102} + \chi_{2000}|^2,$$

where the subscripts are the Dynkin-labels of the ground states. Since F_4 has a trivial center, this invariant cannot be described by the method of section 2. Furthermore, since

the three currents have conformal spin 2, 3 and 2 respectively, it cannot be a conformal embedding.

Another tempting conjecture is that perhaps *all* modular invariants are associated with currents of integral or fractional spin, by some generalization of the class discussed in section 2. While it is clear how one would generalize the integral spin invariants (as discussed above), it is not at all clear what would be a meaningful generalization of the fractional spin ones (with the exception of half-integer spin). In any case there are two known modular invariants that pose a serious challenge to this conjecture, namely the E_7 invariant of $SU(2)$, level 16, and the one found in [7] for $SU(3)$, level 9. We have not succeeded in finding a convincing fractional spin interpretation of these invariants (although it is not hard to find unconvincing ones).

The regular series, as well as the exceptional invariants discussed so far may suggest incorrect conclusions regarding the general form of integral spin invariants. So far we have only encountered cases where the coefficients m_i introduced in (4.1) are 1 or 0. However, among the conformal embeddings one can find counterexamples to that suggestion. The simplest is $SO(9)$ level 2, which has a modular invariant

$$|\chi_{0000} + 2\chi_{0010} + \chi_{2000}|^2.$$

This example is quite easy to understand since it can be obtained from the embedding $SO(9) \subset SO(16) \subset E_8$. There are in fact two such embeddings, related by a change of chirality of the $SO(16)$ spinor. The branching rules of the (248) of E_8 are for these two embeddings

$$(I) \quad (248) \rightarrow (120) + (128) \rightarrow ((36) + (84)) + (128)$$

and

$$(II) \quad (248) \rightarrow (120) + (128)' \rightarrow ((36) + (84)) + ((44) + (84)).$$

The second embedding yields two copies of the representation (84) with Dynkin labels (0010), and corresponds to the example; the first embedding yields a second modular invariant (in general the determination of the form of the modular invariant from the conformal embedding is not so straightforward).

A second, more complicated example of higher multiplicities of an integer spin current occurs for $SO(5)$ at level 12, and corresponds to the conformal embedding $SO(5) \subset E_8$. The invariant is

$$|\chi_{0,0} + \chi_{12,0} + \chi_{3,2} + \chi_{7,2} + 2\chi_{4,4} + \chi_{0,6} + \chi_{6,6} + \chi_{1,8} + \chi_{3,8}|^2.$$

This invariant has also been obtained in [15], but without the factor of 2 for the character $\chi_{4,4}$.

A final example of an exceptional invariant that looks somewhat unusual is the following one for $SU(4)$ level 6

$$\begin{aligned} & |\chi_{000} + \chi_{060} + \chi_{202} + \chi_{222}|^2 + |\chi_{006} + \chi_{600} + \chi_{022} + \chi_{220}|^2 \\ & + |\chi_{002} + \chi_{240} + \chi_{212}|^2 + |\chi_{200} + \chi_{042} + \chi_{212}|^2 \\ & + |\chi_{004} + \chi_{420} + \chi_{121}|^2 + |\chi_{400} + \chi_{024} + \chi_{121}|^2 \\ & + |\chi_{012} + \chi_{230} + \chi_{303}|^2 + |\chi_{032} + \chi_{210} + \chi_{303}|^2 \\ & + |\chi_{103} + \chi_{321} + \chi_{030}|^2 + |\chi_{301} + \chi_{123} + \chi_{030}|^2 \end{aligned}$$

This example corresponds to the conformal embedding $SU(4) \subset SU(20)$, and shows that it may happen that one character (for example χ_{212}) appears in more than one square in the sum. Furthermore it does not seem to satisfy the rule that the lengths of all “orbits” are divisors of the length of the identity orbit. This kind of invariant never occurs among the regular ones discussed in section 2. A nice way of understanding these strange features is to construct this invariant by using the regular $\mathbf{Z}_2$ integral spin invariant as an intermediate extended algebra. It turns out that $\mathbf{Z}_2$ -invariant contains some fields with multiplicity two (namely precisely those that occur twice in the example above). Hence characters that appear twice in the example belong presumably to two different fields of the intermediate extended algebra. Furthermore both the length four and the length three orbits turn out to be length two orbits of the intermediate algebra: the latter has among its characters for example $\chi_{000} + \chi_{060}$, $\chi_{202} + \chi_{222}$, $\chi_{002} + \chi_{240}$ and two characters χ_{212} . Hence this invariant is quite normal after all.

We have constructed all conformal embeddings that occur for sufficiently small values of k and the rank of the Lie algebra, but we will not list them all here. These invariants were

obtained by means of a systematic procedure to look for integer spin invariants, and not by using the embeddings directly. For $SU(3)$, $SO(5)$ and G_2 all such invariants were listed in [15]. However, apart from the level 12 $SO(5)$ invariant already mentioned above, we also disagree with the results of this paper for the conformal embedding $E_7/SU(3)$, which gives a level 21 invariant for $SU(3)$. For this invariant we find

$$|\chi_{0,0} + \chi_{4,4} + \chi_{6,6} + \chi_{10,10} + [\chi_{21,0} + \chi_{13,4} + \chi_{10,1} + \chi_{9,6} + \text{c.c.}]|^2 \\ + |[\chi_{15,6} + \chi_{15,0} + \chi_{10,7} + \chi_{10,4} + \chi_{7,4} + \chi_{6,0} + \text{c.c.}]|^2 ,$$

where “c.c.” refers to the characters of the complex conjugate representations. The second term is missing in [15]. We agree with the other invariants listed in this paper.

5. Other Applications

Although we have discussed Kac-Moody algebras as an example, the fact that our results are not limited to Kac-Moody algebras follows already from the fact that many other RCFT’s can be obtained using the coset construction [9]. The existence of simple currents in a Kac-Moody algebra always implies the existence of such currents in a coset theory constructed from it (see *e.g.* [28] for several examples and further references). For example, the members of the minimal series of Virasoro algebras, with

$$c = 1 - \frac{6}{m(m+1)} \\ h(r,s) = \frac{[rm - s(m+1)]^2 - 1}{4m(m+1)} ,$$

for $m \geq 3$, have a simple current for $(r,s) = (1, m-1)$ or equivalently $(m,1)$ with spin $\frac{1}{4}(m-1)(m-2)$. These currents can be used to construct the regular modular invariants for these theories (the relation between the D_{even} invariants and integer spin fields has already been observed in [7], [29]). Some of the simple currents have an already known interpretation in terms of extended algebras. For $m = 3$ one finds a simple current of spin $\frac{1}{2}$, implying the free fermion interpretation of the Ising model (more about this below). For $m = 4$ the simple current has spin $\frac{3}{2}$, and its presence hints at the possible existence of a super-Virasoro algebra. Indeed, this model (the tricritical Ising model) is known to be equivalent to the $c = \frac{7}{10}$ member of the $N = 1$ discrete series. A CFT may have many spin- $\frac{3}{2}$ primary

fields, but this does not always imply the existence of a super-Virasoro algebra. The spin $\frac{3}{2}$ current in the $m = 4$ model has the crucial property that it has unique fusion rules. The spin 3 current occurring for $m = 5$ generates a W -algebra, and the corresponding modular invariant has been explained in [30]. For $m = 6$ the simple current corresponds to the spin 5 current found in [19] by requiring associativity. On the other hand, the half-integer spin examples found in [19] do not correspond to simple currents. These currents (of spin $\frac{7}{2}$ and $\frac{9}{2}$) belong to the $m = 11$ and $m = 12$ members of the minimal series, and it is natural to suspect that they are in some way related to the (E_6, A) modular invariant which occurs precisely for these values of m (and in which the half-integer spin fields do indeed appear). (There are no other simple currents in the minimal series that fall within the range of spins considered in [19].)

The systematic construction of modular invariants described in section 2 should be quite useful in constructions of string theories, and has in fact already been used extensively in various special cases. One may regard our results as a generalization of the shift vector method in lattice constructions to arbitrary conformal field theories. The primary fields of free bosonic theories on lattices all have unique fusion rules with each other, because the fusion rules are nothing but lattice momentum conservation. Hence any vector $\vec{v}$ defines a simple current $e^{i\vec{v}\cdot\vec{X}}$, and applying the procedure of section 2 simply amounts to using $\vec{v}$ as a shift vector (for a more detailed explanation of the shift vector method see *e.g.* [31]). The shift vector method relies on two important ingredients: the existence of an inner product (used to “project out” some of the lattice vectors), and vector addition, used to define the shifted lattice.

The main obstacle to applying this procedure to arbitrary conformal field theories is that fusion rules are not unique, so that for an arbitrary primary field the analog of inner products and vector addition for lattice vectors cannot be defined. Our conclusion is that there exists a compromise: it is not necessary that *all* primary fields have unique fusion rules, but all that is needed is that some currents, the analog of the operators $e^{i\vec{v}\cdot\vec{X}}$, have that property.

Nearly all string constructions that have appeared in the literature (see [31] for references) can be understood as special cases of the method of section 2. This is obviously true for orbifold constructions (in the original sense of tori twisted by automorphisms), except that there one applies our construction backwards: a torus partition function is an integral

spin invariant (the currents include at least the field ∂X), in which some primary fields (the twist fields) do not appear at all; this integral spin modular invariant is unraveled by the twist fields.

Tensor product constructions are an especially fruitful application of our results. If the elements of the tensor product contain simple currents, then one can construct huge numbers of such currents in the tensor product by combining them in all possible ways. Consider for example the Ising model. Of the three primary fields $1, \psi$ and σ with $h = 0, \frac{1}{2}, \frac{1}{16}$ respectively, only ψ is a non-trivial simple current. This spin- $\frac{1}{2}$ does not generate anything interesting when applied to the Ising model itself, since the Ramond sector is rather trivial. However, in the tensor product of two Ising models one can use the spin 1 current $\psi_1\psi_2$ to twist the diagonal invariant. The result is, by straightforward application of the procedure of section 2

$$|1_1 1_2 + \psi_1 \psi_2|^2 + |1_1 \psi_2 + \psi_1 1_2|^2 + 2|\sigma_1 \sigma_2|^2 ,$$

where we use the fields to denote their characters. Note that the last term appears with a factor of 2 because the $\sigma_1 \sigma_2$ orbit has length 1. This implies that in the new theory there are two different fields, $(\sigma_1 \sigma_2)^+$ and $(\sigma_1 \sigma_2)^-$ with the same character. One can construct a new S -matrix and new fusion rules for these four fields and in particular for the spin fields one finds

$$\begin{aligned} (\sigma_1 \sigma_2)^+ (\sigma_1 \sigma_2)^- &= (1_1 1_2 + \psi_1 \psi_2) \\ (\sigma_1 \sigma_2)^+ (\sigma_1 \sigma_2)^+ &= (1_1 \psi_2 + \psi_1 1_2) . \end{aligned} \tag{5.1}$$

Here we observe a new phenomenon: it may happen that the set of simple currents of an extended algebra is enlarged. From the Ising model, one would only expect to have as a simple current $(1_1 \psi_2 + \psi_1 1_2)$, but it follows from (5.1) (as well as the other fusion rules we did not write down) that the spin fields combine into simple currents as well.

The new extended theory is nothing but the D_1 lattice, with conjugacy classes $(0), (v), (s)$ and (c) . One option one has in constructing new modular invariants for tensor products of Ising models is to pair them first into D_1 lattices, in order to exploit the enlarged set of simple currents. The construction proceeds then by using combinations of D_1 conjugacy classes as shift vectors. This corresponds to string construction with complex fermions with periodic and anti-periodic boundary conditions, a problem that is well understood both from a fermionic [32,33] and bosonic [34] point of view. The more general possibility of real

fermion boundary conditions corresponds to starting with Ising models that are not paired first into D_1 lattices. Note that in general we cannot guarantee that using simple currents one will find *all* solutions (there may be “exceptional invariants”), but it is clear that at least for lattices one does.

Although we do not get anything new by tensoring Ising models, our method can just as easily be used for all other members of the minimal series. For bosonic strings tensoring of minimal models has already been considered in [35]. However, only combinations of separately modular invariant building blocks were considered in this paper. It is now possible to construct systematically a large class of modular invariants which do not factorize in terms of the building blocks. To appreciate the difference, note that there is just one modular invariant if one tensors N separately modular invariant Ising models, whereas we know from fermionic constructions that the total number of solutions grows very rapidly with N .

To apply the foregoing to fermionic strings in fewer than 10 dimensions one would first have to construct a world sheet supercurrent out of the primary fields of the minimal models, a problem that has been solved for the Ising model [36]. This is a very interesting problem, which, however, is well beyond the scope of this paper. A much simpler approach is to start with $N = 1$ or $N = 2$ building blocks. As is well known, a proper realization of world sheet supersymmetry requires that all Neveu-Schwarz states of the building blocks appear in combination with space-time bosons, and all Ramond states with space-time fermions.

To achieve this one can always use the following procedure. First one builds a modular invariant out of the $N = 1$ or $N = 2$ building blocks. This may be just the product of all diagonal invariants,* or of other invariants of the separate building blocks, or it may be anything one constructs from such invariants by using combinations of simple currents. To obtain a proper realization of world sheet supersymmetry, one tensors the result with the space-time NSR-sector of the theory, and builds a new modular invariant by using all simple currents of the form $\psi^i(T_F)_l$, where ψ^i is an NSR-fermion, and $(T_F)_l$ is the supercurrent in the l^{th} building block. Obviously these spin-2 currents remove all states that are not pure Ramond or Neveu-Schwarz states. In type-II string theories one has to perform this projection in both sectors, in heterotic strings only in one.

* We are implicitly assuming here that a left-right symmetric construction is being used. This is not strictly necessary for the rest of the discussion. However, in constructions that use different CFT's in the left- and right-moving sectors one first has to find at least one modular invariant.

If one wishes to obtain a space-time supersymmetric theory one further projection is needed. One would like to add to the set of operators of the theory a spin-1 operator which act like a space-time supercharge. Such an operator can only be obtained by multiplying the NSR spin operator with the Ramond ground states of all the factors in the tensor product (not all $N = 1$ minimal models have such a ground state). One can use our procedure if this spin-1 operator is a simple current. The NSR spin operator is a simple current in even space time dimensions. The crucial point is now that the operators creating the Ramond ground states in $N = 1$ and $N = 2$ models are in general not simple currents. For example, in [37] one finds for the Ramond ground state $[\frac{1}{24}]_R$ in the $N = 1$, $c = 1$ model the following fusion rule

$$[\frac{1}{24}]_R \times [\frac{1}{24}]_R = [0]_{NS} + [\frac{1}{6}]_{NS} + [1]_{NS} .$$

Here we use the notation of [37], but there is no need to explain it: it suffices to observe that there is more than one term on the right-hand side. This is in general a problem in $N = 1$ models, and for this reason it is not easy (though not necessarily impossible) to build space-time supersymmetric strings out of $N = 1$ building blocks (note that although it is true that space-time supersymmetry implies $N = 2$ world sheet supersymmetry, this does not mean that every building block must have $N = 2$ world sheet supersymmetry).

On the other hand, building such models with $N = 2$ models has been shown to be quite easy [38]. Why is it so much easier? From our point of view the reason is that among the Ramond ground states of the $N = 2$ theories, there is always at least one that has unique fusion rules, because it carries the label of an $l = 0$ representation of $SU(2)$. Hence one can now construct a simple current that behaves like a space-time supercharge, and apply the techniques of section 2. We believe that upon closer inspection this procedure will turn out to be equivalent to that of [38]. However it has the advantage that one never needs to worry about the modular transformation properties of the $N = 2$ characters; all one needs are the fusion rules (and in fact only a small subset of them). Modular invariance is automatically guaranteed. It may be possible to construct new modular invariants for these theories by exploiting systematically all possible combinations of simple currents of the building blocks. Whether this extends the already large set of solutions even further remains to be seen.

6. Conclusions

Projections of various kinds have been a favorite tool in string construction. They give a great deal of control over the result of a construction, since one can influence which states are in the spectrum, and which are not. Up to now, this kind of procedure has been restricted to free two-dimensional theories. Finding modular invariants for other theories was usually a rather laborious process, although in some cases (for example those of [38]) that process could be distilled down to a simple recipe. The main result of this paper is to generalize the notion of projections in conformal field theory, by showing that any simple current can be used for this purpose. Using simple currents one can construct off-diagonal invariants directly from the fusion rules, of which only partial knowledge is required. Indeed, once one has identified the simple currents, all one needs to know are their fusion rules with all other primary fields or, equivalently, the orbits they generate.

We have tested this method by applying it to Kac-Moody algebras, and reproduced all known infinite series of modular invariants. Furthermore we found many new infinite series, which provide a natural extension (and perhaps completion) of the previously known set. Although speculations tend to be incorrect in this area, it is nevertheless natural to speculate that our method produces all modular invariants of a rational conformal field theory, apart from a few “exceptional” invariants.

APPENDIX

In this appendix we will show that different ways of constructing invariants related to subgroups of $\mathbf{Z}_N$ yield the same result. In the following J is the generator of $\mathbf{Z}_N$, J_n its n^{th} power under fusion and $Q_n(\Phi_i)$ is the charge of the primary field Φ_i with respect to the current J_n (i.e. Φ_i and J_n have a relative monodromy $e^{2\pi i Q_n(\Phi_i)}$). The character of the primary field Φ_i is denoted by $\chi(\Phi_i)$.

If one uses the orbifold procedure described in section 2 repeatedly, it is not *a priori* obvious that one will always obtain the same result, irrespective of the order of the twists. For example, to obtain a modular invariant related to $\mathbf{Z}_{15}$, one could apply first a $\mathbf{Z}_3$ twist and then a $\mathbf{Z}_5$ twist, or one could interchange the order, or one could directly perform a $\mathbf{Z}_{15}$ twist. We will show that the result is always the same.

We start with a modular invariant $\sum_{ij} M_{ij} \chi(\Phi_i) \chi^*(\Phi_j)$. If J_n has order N_1 in $\mathbf{Z}_N$, the modular invariant one obtains by applying the corresponding twist is

$$\frac{1}{N_1} \sum_{l=0}^{N_1-1} e^{2\pi i l [Q_n(\Phi_i) + Q_n(\Phi_j)]} \sum_{k=0}^{N_1-1} M_{ij} \chi(J_{nk} \Phi_i) \chi^*(J_{nk}^c \Phi_j) , \quad (\text{A1})$$

where $nN_1 = N$. Here the sum on l yields a projection on charge singlet states, whereas the sum on k yields the twisted sectors.

If one applies two successive twists, generated by currents J_{n_1} and J_{n_2} of order N_1 and N_2 respectively, one obtains

$$\begin{aligned} \frac{1}{N_1 N_2} \sum_{l_2=0}^{N_2-1} \sum_{l_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \sum_{k_1=0}^{N_1-1} e^{2\pi i l_2 [Q_{n_2}(J_{n_1 k_1} \Phi_i) + Q_{n_2}(J_{n_1 k_1}^c \Phi_j)]} e^{2\pi i l_1 [Q_{n_1}(\Phi_i) + Q_{n_1}(\Phi_j)]} \\ M_{ij} \chi(J_{n_2 k_2 + n_1 k_1} \Phi_i) \chi^*(J_{n_2 k_2 + n_1 k_1}^c \Phi_j) . \end{aligned} \quad (\text{A2})$$

The charges satisfy the addition rule $Q(\Phi_i \Phi_j) = Q(\Phi_i) + Q(\Phi_j)$, so that we get

$$Q_{n_2}(J_{n_1 k_1} \Phi_i) = Q_{n_2}(J_{n_1 k_1}) + Q_{n_2}(\Phi_i) .$$

Using formula (2.7) we find

$$Q_{n_2}(J_{n_1 k_1}) = \frac{n_1 n_2 k_1 r}{N} .$$

On the other hand one has

$$Q_{n_2}(J_{n_1 k_1}^c) = Q_{n_2}(J_{-n_1 k_1}) = -\frac{n_1 n_2 k_1 r}{N} .$$

Hence the contribution of the currents to the exponentials in (A2) cancels, and one may replace $Q_{n_2}(J_{n_1 k_1} \Phi_i) + Q_{n_2}(J_{n_1 k_1}^c \Phi_j)$ by $Q_{n_2}(\Phi_i) + Q_{n_2}(\Phi_j)$. This makes the expression manifestly symmetric in the two twists.

One might expect to obtain the same result by using a single twist of order M , where M is the smallest common multiple of N_1 and N_2 . The sum over the twists (i.e. over k_1 and k_2) clearly covers all $\mathbf{Z}_M$ twists. If N_1 and N_2 are not relative prime the set of twisted states is covered more than once, but this is only a matter of normalization (to prevent such overcounting we should have used for the order of the second twist instead of N_2 the order of J_{n_2} in the theory twisted by J_{n_1}). What remains to be shown is thus that the projections that result from the sums on l_2 and l_1 are the same ones one would obtain in the $\mathbf{Z}_M$ -twisted theory.

Defining $m = \frac{N}{M}$ we find for the partition function of the $\mathbf{Z}_M$ -twisted theory

$$\frac{1}{M} \sum_{l=0}^{M-1} e^{2\pi i l [Q_m(\Phi_i) + Q_m(\Phi_j)]} \sum_{k=0}^{M-1} \chi(J_{km} \Phi_i) \chi^*(J_{km}^c \Phi_j) . \quad (\text{A3})$$

The projection used in (A2) selects all terms with $Q_{n_2} = Q_{n_1} = 0$, whereas the one used in (A3) selects those with $Q_m = 0$.

To see why this is the same note that since $n_1 = m \frac{M}{N_1}$ the current J_{n_1} is the $\frac{M}{N_1}$ th power of J_m under fusion. If $Q_m = 0$ on some field, that field is local with respect to J_m and hence also with respect to J_{n_1} (and analogously J_{n_2}). Hence $Q_{n_1} = Q_{n_2} = 0$ on such a field.

To check the converse, consider a field with vanishing Q_{n_1} and Q_{n_2} , and assume that $Q_m = \frac{i}{M}$, $i \in \mathbf{Z}$. Then we can use the relation $J_{n_1} = (J_m)^{M/N_1}$ derived above to find Q_{n_1} and Q_{n_2} :

$$Q_{n_1} = \frac{i}{N_1} \quad ; \quad Q_{n_2} = \frac{i}{N_2} .$$

Since both charges are zero (modulo 1), it follows that i must be a multiple of N_1 as well as N_2 , and hence it is a multiple of M . Consequently $Q_m = 0 \bmod 1$, and the projections used in (A2) and (A3) are in fact identical.

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