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INTEGRABILITY FOR MULTIDIMENSIONAL LATTICE MODELS

Jean Michel MAILLET [†]

CERN, CH-1211 Geneva 23, Switzerland

and

Frank NIJHOFF

Department of Mathematics and Computer Sciences

CLARKSON University, Potsdam, NY 13676, USA

Abstract

The generating principle for the algebraic construction of the hierarchy of the d -simplex equations generalizing the Yang-Baxter equation in any dimensions d is given. Following this principle, we construct the generalization of the Lax equations for multidimensional Integrable Systems. We show that this construction leads, as in $d = 2$, to the existence of an infinite series of conserved charges for such models. In the $d = 3$ case a reinterpretation of these results is given in term of functionals of loops, leading to the notion of extended gauge symmetry and gauge fields associated to quantum groups.

[†]Permanent Address: LPTHE, Université Pierre et Marie Curie, Paris, France
bitnet: MAILLET@CERNVM

1. The development of generic algebraic structures of complete integrability for multidimensional models of Field Theory and Statistical Mechanics is nowadays one of the major outstanding problems in the theory of Completely Integrable Systems. The purpose of this note is to give a first account of our proposal for such structures of integrability in any dimension d . One of the main ideas from our point of view is that the notion of classical integrability is intimately linked to the possibility of constructing an overdetermined but solvable system of equations in a given dimension d , the compatibility condition of which gives rise to the non-linear field equation one is interested in, and which does not reduce trivially to a lower dimensional one. In the $d = 2$ situation, this idea has appeared under various guises and it is well-known that the overdetermined systems in this case are generically given by linear spectral problems [1]. In addition, and following the original idea of Lax [1] for the two-dimensional case, the existence of such a system (Lax pair in $d = 2$) has to lead, in particular, to the construction of an infinite series of constants of motion, this property being at the basis of the exact solvability of the associated non-linear model.

In the following, we present our generalization of the notion of Lax-pair and Lax equations (or Zakharov-Shabat equations) in any dimension d that preserves these basic features.

In order to achieve this program, the question we will have to address in the first place is the identification of the objects in terms of which such a generalization of the notion of Lax-pair and Lax equations may be written in a given dimension d . We will gain some insight into this problem by examining another aspect of integrability, i.e, the question of its associated canonical structure, given in dimension two by the Yang-Baxter equation.

It is well-known that the Yang-Baxter (triangle) equations [2] and their associated R matrices are the fundamental building blocks in the theory of $2-d$ Completely Integrable Systems. Therefore it is natural if one is interested in extending the algebraic structure of Completely Integrable Systems into dimensions higher than two, to consider the possibility of generalizing the Yang-Baxter equations. Such a generalization exists. It was proposed by Zamolodchikov in 1980 for the case $d = 3$ [3] and extended to arbitrary d in [4]. From their very nice geometrical representation, these equations are called the d -simplex equations [4,5,6].

The idea we wish to push forward in this note is that it is possible to identify the concept that underlies the logical construction of this hierarchy of d -simplex equations (there is one equation for any given dimension d) step by step, starting from the 1-simplex equation, and moreover that it is possible to use this generating principle to extend the

notion of Lax equations generically to any dimension d . Moreover our constructions lead us to define the notion of extended gauge symmetry and gauge fields associated to quantum groups.

2. Let us first describe very briefly the d -simplex equations in a form that is suitable for our discussion (for more details and alternative presentation see [5,6]). In a given dimension $d \geq 1$, the d -simplex equation may be written in terms of $d+1$ objects $S^{(k)}$, $k = 1, 2, \dots, d+1$ which are linear operators acting on a vector space V_d that is the tensor product of $\frac{1}{2}d(d+1)$ vector spaces $V^{(kj)}$, $k, j = 1, 2, \dots, d+1$, $k \neq j$ (with the convention that $V^{(kj)} = V^{(jk)}$), namely $V_d = \otimes_{k < j} V^{(kj)}$, such that each object $S^{(k)}$ acts non-trivially only on the d vector spaces $V^{(kj)}$, $j \neq k$, its action on the other vector spaces in the tensor product V_d being equal to the identity operator. Then the d -simplex equation may be written as [5,6] :

$$S^{(1)} * S^{(2)} * \dots * S^{(d+1)} = S^{(d+1)} * \dots * S^{(2)} * S^{(1)} \quad (1)$$

where $*$ stands for the operator product. Let us note that in eq.(1) any two operators $S^{(k)}$, $S^{(j)}$, $j \neq k$, are connected via operator products by their action on one and only one common vector space $V^{(kj)}$ in V_d . Conversely, there are only two operators in eq.(1) acting non-trivially on a given vector space $V^{(kj)}$, namely $S^{(k)}$ and $S^{(j)}$. So the r.h.s. of eq.(1) is obtained from the l.h.s. by reversing at the same time all these products between all the possible couple of operators $S^{(k)}$, $S^{(j)}$. Hence, eq.(1) may be considered as a generalization of matrix commutativity for multiple indices objects (tensors) $S^{(k)}$. In addition, each object $S^{(k)}$ depends on $\frac{1}{2}d(d-1)$ continuum parameters (spectral parameters), $\frac{1}{2}d(d+1)-1$ only being independent in eq.(1) (see [5,6] for more details). Let us give now some explicit examples of eq.(1) for $d = 1, 2, 3$, the objects $S^{(k)}$ being defined in tensor notation (in a particular basis of V_d) by $S^{(1)} = A$, $S^{(2)} = B$, etc. We have :

$$\text{d=1: } A_i^k B_k^j = B_i^k A_k^j \\ \text{(Matrix Commutativity Equation)}$$

$$\text{d=2: } A_{i_1 i_2}^{k_1 k_2} B_{k_1 i_3}^{j_1 k_3} C_{k_2 k_3}^{j_2 j_3} = C_{i_2 i_3}^{k_2 k_3} B_{i_1 k_3}^{k_1 j_3} A_{k_1 k_2}^{j_1 j_2} \\ \text{(Yang-Baxter Equation)}$$

$$\text{d=3: } A_{i_1 i_2 i_3}^{k_1 k_2 k_3} B_{k_1 i_4 i_5}^{j_1 k_4 k_5} C_{k_2 k_4 i_6}^{j_2 j_4 k_6} D_{k_3 k_5 k_6}^{j_3 j_5 j_6} = D_{i_3 i_5 i_6}^{k_3 k_5 k_6} C_{i_2 i_4 k_6}^{k_2 k_4 j_6} B_{i_1 k_4 k_5}^{k_1 j_4 j_5} A_{k_1 k_2 k_3}^{j_1 j_2 j_3} \\ \text{(Tetrahedron Equation)}$$

where the sum over repeated indices is understood. In order to avoid a plethora of indices we introduce at this point a convenient abbreviating notation. Instead of writing $S_{i_1 i_2 i_3 \dots i_p}^{j_1 j_2 j_3 \dots j_p}$ we will write simply $S_{1,2,3,\dots,p}$ representing any pair of indices (i_p, j_p) by their corresponding label p . Whenever such a label is repeated it is understood that a matrix product in terms of the indices corresponding to the repeated labels is performed. Adding new labels in a product will effectively mean that we take a (matricial) tensor product. In this fashion we can rewrite the d -simplex equations for $d = 1, 2, 3$ as follows :

$$d=1: A_1 \cdot B_1 = B_1 \cdot A_1$$

$$d=2: A_{12} \cdot B_{13} \cdot C_{23} = C_{23} \cdot B_{13} \cdot A_{12}$$

$$d=3: A_{123} \cdot B_{145} \cdot C_{246} \cdot D_{356} = D_{356} \cdot C_{246} \cdot B_{145} \cdot A_{123}$$

Our main purpose is now to describe the principle that enables us to generate the complete hierarchy of the d -simplex equation starting from the 1-simplex (commutativity) equation. In fact we will show that it is possible to generate the $d + 1$ -simplex equation from the d -simplex equation for any $d \geq 1$. For that purpose, let us consider $d + 1$ collections of objects ${}_{\alpha_k} S^{(k)}$ defined in a similar way as the objects $S^{(k)}$ we just used in eq.(1) but now depending on an additional (vector) index α_k . However, we will not suppose that these operators do verify the d -simplex equation (1) for any values of the indices α_k , but rather that there exists a tensor $R_{\alpha_1 \dots \alpha_{d+1}}^{\beta_1 \dots \beta_{d+1}}$ (depending on these additional vector indices α_k) that describes an obstruction to eq.(1) for the family of operators ${}_{\alpha_k} S^{(k)}$ in the following way :

$$R_{\alpha_1 \dots \alpha_{d+1}}^{\beta_1 \dots \beta_{d+1}} \cdot {}_{\beta_1} S^{(1)} * {}_{\beta_2} S^{(2)} * \dots * {}_{\beta_{d+1}} S^{(d+1)} = {}_{\alpha_{d+1}} S^{(d+1)} * \dots * {}_{\alpha_2} S^{(2)} * {}_{\alpha_1} S^{(1)} \quad (2)$$

where the sum over repeated indices β_k is understood and the $*$ product between operators ${}_{\alpha_k} S^{(k)}$ is defined as in eq.(1). Note that if $R_{\alpha_1 \dots \alpha_{d+1}}^{\beta_1 \dots \beta_{d+1}} = \delta_{\alpha_1}^{\beta_1} \dots \delta_{\alpha_{d+1}}^{\beta_{d+1}}$, eq.(2) would reduce to eq.(1) for any choice of the indices α_k . However, if R is not a trivial tensor, it describes an obstruction to eq.(1). In addition, this tensor will depend on $\frac{1}{2}d(d + 1)$ continuum parameters which are functions of those present in eq.(1) [5,6].

Then, requiring the associativity (in a generalized sense cf. [6]) of the operator product $*$ in eq.(2), it is possible to show that the tensor R has to satisfy, as a consistency condition, the $d + 1$ -simplex equation [6]. In other words, an obstruction to the d -simplex equation as given in equation (2) has to satisfy the $d + 1$ -simplex equation. The

most simple example of this mechanism is given by considering the case of an obstruction to the 1-simplex (commutativity) equation. Eq.(2) leads in that case to :

$$R_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \beta_1 A \cdot \beta_2 B = \alpha_2 B \cdot \alpha_1 A \quad (3)$$

where the dot product between the matrices A and B stands for the standard matrix product. This relation implies as a consistency condition (given by requiring the associativity of the products in eq.(3)) the following constraint on the R -matrix :

$$R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12} \quad (4)$$

which is just the Yang-Baxter equation (2-simplex equation) for R .

In the same way, the object that realizes an obstruction to the Yang-Baxter equation in the sense of eq.(2) has to verify, as a consistency condition, the tetrahedron (3-simplex) equation of Zamolodchikov [6].

3. This concept of obstruction we have just developed for the d -simplex equation hierarchy is in fact very suggestive and it is possible along this line of thinking to generalize the notion of Lax system and of Lax equations (Zakharov-Shabat equations) in any dimension.

In order to make the parallel with the concepts introduced in the previous section as close as possible, we will first explain our results in the case of lattice models and for simplicity restrict ourselves to the case $d=3$, even though it will be made clear that our construction and results extend recursively to any d .

But before explaining these new structures, let us first recall very briefly some important features of the well-known $2-d$ classical integrable lattice systems [7]. In that case, the Lax system reads :

$$\Phi^l(m, n) = L_l(m, n; \lambda_l) \cdot \Phi(m, n) \quad (5)$$

where $l = 1, -1, 2, -2$ denotes the four possible directions on the $2-d$ lattice starting from a given point, L_l are matrices acting on some vector space V , depending on a set of fields φ_i of our lattice model, and on spectral parameters λ_l ¹; $\Phi(m, n)$ is a vector function of the point (m, n) on the lattice, with $m, n \in \mathbf{Z}$. A superscript l will denote

¹Note that the spectral parameters λ_l and $\lambda_{l'}$ will be subject to a constraint, which is generically of the type $\lambda_l + \lambda_{l'} = c_{ll'}$, where $c_{ll'}$ are constants characteristic of the lattice model we are looking at. Hence for given $c_{ll'}$, λ_l and $\lambda_{l'}$ will be determined as a function of one spectral parameter λ .

a shift in direction l , namely we have that $\Phi^{-1}(m, n) \equiv \Phi(m+1, n)$, and similarly, $\Phi^2(m, n) \equiv \Phi(m, n+1)$, $\Phi^{-1}(m, n) \equiv \Phi(m-1, n)$, $\Phi^{-2}(m, n) \equiv \Phi(m, n-1)$. In fact the matrices L_l have to be considered as line objects living on the links of our 2-d lattice. In the following the site index (m, n) will be omitted. Note that the matrices L_l are invertible on the solution Φ of eq.(5) and verify for any point on the lattice and any values of l and λ the following relation : $L_{-l}^l \cdot L_l = 1$. If now we require eq.(5) to have non-trivial solutions Φ which are functions defined on the 2-d lattice, i.e., such that the shifts operators commute on it, namely $\Phi'' \equiv \Phi'^l$, we obtain the following constraint for the L_l matrices :

$$L_l'^l \cdot L_{l'} = L_l^l \cdot L_l \quad (6)$$

In fact eq.(6) may be interpreted as a zero curvature condition associated to a lattice gauge potential given by L_l in the following way :

$$L_l = P \exp \left\{ \int_{\underline{x}}^{\underline{x}+\underline{l}} A_\mu dx^\mu \right\} \quad (7)$$

where P stands as usual for the path-ordered product prescription, $\underline{x}$ for the vector coordinate of a point on the 2-d lattice, and $\underline{l}$ for the unit vector in directions $l = 1, -1, 2, -2$. Moreover, as is expected from eq.(7), the following gauge invariance of eqs.(5-7) holds :

$$\Phi \mapsto \Phi' = g \cdot \Phi \text{ and } L_l \mapsto L'_l = g^l \cdot L_l \cdot g^{-1}$$

where g is a matrix function of a point of the 2-d lattice. The linear system (5) is said to be a Lax system associated to our lattice model iff the equations (6) for any values of the spectral parameter λ are equivalent to the non-linear equations of motion for the fields φ_i we started with.

The importance of the existence of such a Lax representation for a lattice model is that one can in particular extract from it an infinite series of functionals of the fields φ_i that are constants of motion. To see that, let us first identify the direction 1 on the lattice with time and the direction 2 with space, the size of the lattice being set equal to M in the space direction 2. We now consider the monodromy matrix $T_2(t; \lambda)$ at constant lattice time t associated to eq.(5), constructed as a path-ordered product of L_2 matrices from the point $(t, 1)$ to the point (t, M) on the lattice :

$$T_2(t; \lambda) = L_2(t, M; \lambda) \cdots L_2(t, 2; \lambda) \cdot L_2(t, 1; \lambda)$$

As a consequence of eq.(6) we obtain for its time evolution (shift in direction 1) :

$$T_2(t+1; \lambda) = L_1(t, M+1; \lambda) \cdot T_2(t; \lambda) \cdot L_1^{-1}(t, 1; \lambda) \quad (8)$$

If we then impose periodic boundary conditions in the space direction 2 for the matrices L_l , we have for the time evolution of the trace of the matrix $T_2(t; \lambda)$:

$$\text{tr} \{ T_2(t+1; \lambda) \} = \text{tr} \{ T_2(t; \lambda) \} \quad (9)$$

Therefore $\text{tr} \{ T_2(t; \lambda) \}$ is a conserved object under shifts in time direction 1 and by an expansion in power series in λ we just obtain an infinite series of functionals of the fields φ_i of our lattice model that are constants of motion. Moreover, the associated canonical structure being given by the Yang-Baxter (2-simplex) equation, it is possible to show that these conserved quantities are all commuting under canonical brackets.

4. Let us now describe our proposal for the generalization of this scheme on a 3-d lattice. Following the obstruction principle given in section 2 we consider new (line) objects ${}_\alpha L_l$, for $l = 1, -1, 2, -2, 3, -3$, that are matrices as the previous L_l but now having an additional vector index α . We will suppose that these new line objects no longer satisfy the zero curvature condition (6) but rather that there exist tensors $(K_{ll'})^\gamma_\alpha{}^\delta_\beta$, that describe an obstruction to it in the following way (see for comparison eq.(2,3))² :

$$(K_{ll'})^\gamma_\alpha{}^\delta_\beta \cdot {}_\gamma L_l^{l'} \cdot {}_\delta L_{l'} = {}_\beta L_{l'}^l \cdot {}_\alpha L_l \quad (10)$$

Here the dot product between the ${}_\alpha L_l$ matrices stands for the usual matrix product we already used in eq.(6) ; the tensors $K_{ll'}$, which may be viewed as surface objects living on plaquettes with directions l, l' on the 3-d lattice, are functions of the fields φ_i , and on a set of spectral parameters $\lambda_{ll'}$. Let us note that if $(K_{ll'})^\gamma_\alpha{}^\delta_\beta$ is the constant tensor $\delta_\alpha^\gamma \delta_\beta^\delta$ (identity tensor on the spaces 1 and 2) eq.(10) would reduce to the zero curvature condition (6) for any values of the indices α . However if $(K_{ll'})^\gamma_\alpha{}^\delta_\beta$ is not a trivial tensor, it describes an obstruction to eq.(6). Now we wish to consider eqs.(10) as equations for the matrices ${}_\alpha L_l$, the $K_{ll'}$ playing the role of given potentials, in the same way as eqs.(5) were to be considered as equations for the vector functions Φ , the potentials L_l being given.

Equation (10) is our proposal for a generalization of the 2-d Lax system on a 3-d

²Here again upper indices l, l', l'' denote shifts in the directions l, l', l'' on the 3-d lattice, taking the six possible values 1, -1, 2, -2, 3, -3.

lattice. Introducing again compact tensor product notations, we will rewrite eq.(10) as³ :

$$(K_{ll'})_{12} \cdot \{(L_l^{l'})_1 \cdot (L_{l'}^l)_2\} = \{(L_{l'}^l)_2 \cdot (L_l^{l'})_1\} \quad (10')$$

Note that from eq.(10) the tensors $K_{ll'}$ are invertible on the solutions ${}_\alpha L_l$, i.e., we have that $(K_{ll'})_{12} \cdot (K_{l'l})_{21} = \mathbf{1} \otimes \mathbf{1}$. In order for this system of equations to have non-trivial solutions we have to require some compatibility condition to be satisfied, namely the associativity of the products in eq.(10) and the commutations of the lattice shifts on the matrices ${}_\alpha L_l$. This leads to the following constraint on the tensors $K_{ll'}$:

$$(K_{ll'})_{12} \cdot (K_{l''l'}^{l'})_{13} \cdot (K_{l'l''})_{23} = (K_{l'l''}^l)_{23} \cdot (K_{ll'})_{13} \cdot (K_{ll''}^{l'})_{12} \quad (11)$$

where the dot product $\cdot$ has the same definition as in eq.(4). Then we will say that the system of equations (10) is associated to a particular lattice model iff the non-linear equations of motion for its fields φ_i are equivalent to the cubic condition (11).

This is our proposal for the generalization of the 2- d Lax (Zakharov-Shabat) equation (6) for a 3- d lattice model. Note that in the present case we have no longer a linear spectral problem as overdetermined system, but instead the system has become *bilinear*. However we will show in section 5 how to reformulate it as a *linear* system of equations on a space of functionals of loops drawn on our 3- d lattice.

It should be noted furthermore that, as in $d = 2$, eqs.(10,11) have the following (extended) gauge-type invariance⁴ [8] :

$$L_l \mapsto g_l \cdot L_l \text{ and } K_{ll'} \mapsto (g_l \otimes g_{l'}^l) K_{ll'} (g_l^{l'}{}^{-1} \otimes g_{l'}^{-1})$$

where g_l are matrices function of a point on the 3- d lattice and depending on a direction l (i.e., there is one matrix g_l for each direction l on the lattice), acting on the vector indices α of the ${}_\alpha L_l$ matrices. Hence this symmetry is a non trivial extension of the usual gauge symmetry we had in the $d = 2$ case.

We further note that the cubic condition (11) realizes a local version of the Yang-Baxter (2-simplex) equation (4) in the same way as the Lax equation (5) realizes a local

³Here the indices ll' in $(K_{ll'})_{12}$ are plaquette indices, i.e., denote two directions on the 3- d lattice, and the indices 1, 2 stand as in eq.(4) for the tensor indices of $(K_{ll'})_{12}$ which is acting on the vector indices α of the two line objects ${}_\alpha L_l$, i.e., 1, 2 denote the two vector spaces V_1 and V_2 on which $(K_{ll'})_{12}$ is acting as a linear operator.

⁴We adopt in the following a simplified notation by identifying in $(K_{ll'})_{12}$ the plaquette indices l, l' and vector space indices 1, 2. Hence we will just set $(K_{ll'})_{12} \equiv (K_{ll'})$.

version of the commutativity (1-simplex) equation. Hence, these considerations lead us [8] to the notion of (extended) gauge theories associated not to a group but to a quantum group [9], the (extended) gauge potential being now related not to a line element L_l as in eq.(7) but to a surface-like object $K_{ll'}$. We will return to this issue in a future publication [8]. We wish also to stress that any solution of the Yang-Baxter equation is also a solution (that may in fact be considered as a stationary solution) of eq.(11). This provides us with a *classical* interpretation of the solutions of the *quantum* Yang-Baxter equation [8].

Now a non-trivial check of the relevance of the concept of "obstruction" we have used in obtaining eqs.(10,11), in analogy with the d -simplex equation hierarchy, is to show that, as in the $d = 2$ case, it is possible to construct from eqs.(10,11) an infinite series of functionals of the fields φ_i that are constants of motion for such a lattice model. In order to see that, let us again, as in the $d = 2$ case, define some new objects constructed from the potentials $K_{ll'}$. The direction 1 on the lattice will be identified with time, and 2, 3 will be space directions, the lattice size in directions 2 and 3 being set equal to M and N respectively.

We first define new objects $J_{2\hat{3}}$ and $J_{\hat{2}3}$ as path-ordered products of the elementary plaquette potentials $K_{ll'}$ for $l = 1, 2$ and $l' = 3$ at constant lattice time coordinate t , space coordinate m in direction 2 and for a given set of spectral parameters $\lambda_{ll'}$:

$$J_{2\hat{3}}(t, m; \lambda_{23}) \equiv (J_{23})_{\alpha_0; \beta_1 \dots \beta_N}^{\alpha_N; \gamma_1 \dots \gamma_N}(t, m; \lambda_{23})$$

where we have :

$$\begin{aligned} & (J_{2\hat{3}})_{\alpha_0; \beta_1 \dots \beta_N}^{\alpha_N; \gamma_1 \dots \gamma_N}(t, m; \lambda_{23}) \equiv \\ & \equiv (K_{23})_{\alpha_0 \beta_1}^{\alpha_1 \gamma_1}(t, m, 1) \cdot (K_{23})_{\alpha_1 \beta_2}^{\alpha_2 \gamma_2}(t, m, 2) \cdot \dots \cdot (K_{23})_{\alpha_{N-1} \beta_N}^{\alpha_N \gamma_N}(t, m, N) \end{aligned} \quad (12)$$

Using again compact tensor product notations eq.(12), may be rewritten as :

$$J_{2\hat{3}} = K_{23^{(1)}} \cdot K_{23^{(2)}}^3 \cdot K_{23^{(3)}}^{2*3} \cdot \dots \cdot K_{23^{(N)}}^{(N-1)*3} \quad (12')$$

where an upper index $k * 3$ means k shifts in direction 3, the index $\hat{3}$ stands for the collection of vector space indices $(3^{(1)}, 3^{(2)}, \dots, 3^{(N)})$ and also indicates the total shift content of $J_{2\hat{3}}$, i.e., N shifts in direction 3. So in eqs.(12) there is a matrix product between the plaquette objects $K_{23^{(i)}}$ on their common vector space 2 and a tensor product on the spaces $3^{(i)}$, $i = 1, \dots, N$. Replacing the space 2 by the space 1, we also

define $J_{1\hat{3}}$. We similarly define $J_{\hat{2}3}$ (dual to $J_{2\hat{3}}$) as :

$$J_{\hat{2}3} = K_{2(M)3}^{(M-1)*2} \cdot K_{2(M-1)3}^{(M-2)*2} \cdots K_{2(1)3} \quad (13)$$

Now as a consequence of eq.(11), these new objects verify the following relation :

$$K_{12} \cdot J_{1\hat{3}}^2 \cdot J_{\hat{2}3} = J_{1\hat{3}}^1 \cdot J_{1\hat{3}} \cdot K_{12}^{\hat{3}} \quad (14)$$

where again the superscript $\hat{3}$ means N shifts in direction 3 and the product between the J 's is performed on the multiple common tensor index $\hat{3}$. This is in fact a new representation of eq.(11) with extended surface like objects J , but also a matrix representation (in the additional α indices) of equation (10) which is now homogeneous in the potential K_{uv} . We now proceed by defining one more quantity $T_{\hat{2}3}$, which will be a surface-like object obtained from $J_{\hat{2}3}$ as a path-ordered product on the index $\hat{3}$ that we write using the above tensor notations as :

$$T_{\hat{2}3} \equiv J_{2(M)\hat{3}}^{(M-1)*2} \cdot J_{2(M-1)\hat{3}}^{(M-2)*2} \cdots J_{2(1)\hat{3}} \quad (15)$$

It should be noted here that eq.(15) defines in fact $T_{\hat{2}3}$ as a *surface-ordered product* of the elementary plaquette potentials K_{23} . The definition of such a surface-ordered product has been made possible by the fact that K_{23} is a tensor acting as a linear operator in the tensor product of *two* spaces, or in other words, the dimension of the geometrical object (plaquette) K_{23} is living at, i.e. *two*, equals the number of its matrix spaces, also *two*, hence generalizing the 2- d situation where the potentials L_l are living on *one*-dimensional geometrical objects (links) of the lattice and have also a number of matrix spaces equal to *one*. Therefore our construction suggests an interesting generalization of the usual non-Abelian gauge field that defines, through path-ordered exponentials, line elements as L_l , i.e., the existence of non-Abelian (extended) gauge potentials, associated now not to a group but to a quantum group, that may be used to construct plaquette objects as K_{uv} and defining *surface-ordered products*. We will explain these ideas in full in a future publication [8]. Now as a consequence of eq.(11-15) we have :

$$J_{1\hat{2}} \cdot J_{1\hat{3}}^{\hat{2}} \cdot T_{\hat{2}3} = T_{\hat{2}3}^1 \cdot J_{1\hat{3}} \cdot J_{1\hat{2}}^{\hat{3}} \quad (16)$$

where the objects J are here only depending on the values of the plaquette potentials K_{23} at the boundaries of our 3- d lattice. In addition, as a consequence of the invertibility of K_{uv} , $J_{1\hat{2}}$ and $J_{1\hat{3}}$ are also invertible, so that eq.(16) defines effectively the time evolution (shifts in direction 1) of $T_{\hat{2}3}$. Imposing now as in $d = 2$ some definite boundary conditions

on the plaquette potentials K_{uv} and on the J 's, i.e. here, $J_{1\hat{2}} \cdot J_{1\hat{3}}^{\hat{2}} = J_{1\hat{3}} \cdot J_{1\hat{2}}^{\hat{3}}$, we obtain for the time evolution of the trace of $T_{\hat{2}\hat{3}}$ on the spaces $\hat{2}$ and $\hat{3}$:

$$tr \{ T_{\hat{2}\hat{3}} (t+1 ; \lambda_{23}) \} = tr \{ T_{\hat{2}\hat{3}} (t ; \lambda_{23}) \} \quad (17)$$

Hence the trace of $T_{\hat{2}\hat{3}} (t+1 ; \lambda_{23})$ is a conserved quantity under time evolution, and by an expansion in the set of spectral parameters λ_{23} we obtain, as in $d=2$, a set of infinite series of functionals of the fields φ_i of our 3- d lattice model that are constants of motion. Moreover, the associated canonical structure being now given by the tetrahedron (3-simplex) equation, it can be proved that these conserved quantities are all commuting under canonical brackets [6,8].

Therefore our scheme gives a 3- d generalization of the Lax [1] and Yang-Baxter [2] original structures and ideas for the 2- d Integrable Systems.

We wish to stress here that our considerations and results extend recursively to any dimension d , hence defining a generic notion of Complete Integrability in arbitrary dimensions. In the $d=4$ case, for example, we would introduce new tensors ${}_{\alpha}K_{uv}$, $l, l' = 1, 2, 3, 4$, living on the elementary plaquettes of the 4- d lattice that will no longer obey eq.(11), but rather will be such that there exist new potentials, say $(\Gamma_{uv})^{\lambda\mu\nu}_{\alpha\beta\gamma}$ living on elementary volume cells of the 4- d lattice that describes an obstruction to eq.(11). A compatibility condition for these potentials may then be written, that realizes a gauged version of the tetrahedron (3-simplex) equation. The conserved charges will now be given as the trace of a *volume-ordered product* of the potentials Γ_{234} (direction 1 is time), the associated canonical structure being now given by the 4-simplex equation.

5. To conclude, we wish to give a brief account of a reinterpretation of the generalized Lax equations we have obtained here from a more geometrical point of view that we will develop in [8].

Let us again take the three-dimensional lattice models as a generic example. We now consider the space of oriented closed and non-intersecting loops C_Q drawn on this 3- d lattice. We define such a loop by giving a point Q on the lattice and an ordered string of numbers l_i , each taking the six possible values 1, -1, 2, -2, 3, -3 that denote the six possible directions in which one can go on a 3- d lattice starting from a given point. For example, the sequence $(Q ; 1, 2, -1, -2)$ defines a closed loop around an elementary plaquette in the plane (1, 2) starting from a given point Q . To each loop C_Q of this type defined by an ordered string of numbers l_i one can associate the following quantity $\Phi[C_Q]$, constructed as a path-ordered product along C_Q starting at point Q of solutions

$\alpha_i L_{l_i}$ of eq.(10), namely :

$$\Phi[C_Q]_{\{\alpha_i\}} = P \prod_{C_Q} \alpha_i L_{l_i} \quad (18)$$

where P stands as usual for the path-ordered product prescription here applied to the matrix product between the $\alpha_i L_{l_i}$ matrices. This is just the analogue of a usual Wilson loop quantity in gauge theories but now depending on the additional indices α_i distributed all along the loop C_Q . Hence the quantity $\Phi[C_Q]_{\{\alpha_i\}}$ is to be considered as a kind of "barbed Wilson loop". It is then possible to rewrite eq.(10) as a system of linear equations on the space of functionals of loops $\Phi[C_Q]$ as :

$$\Phi[C'_{Q'}]_{\{\alpha_i\}} = \sum_{\{\beta_i\}} T[\Sigma_{C'_{Q'}, C_Q}]_{\{\alpha_i\}}^{\{\beta_i\}} \cdot \Phi[C_Q]_{\{\beta_i\}} \quad (19)$$

where the potential T stands for a *surface-ordered product* of the elementary plaquette potentials $K_{ll'}$ on a 2- d surface $\Sigma_{C'_{Q'}, C_Q}$ having C_Q and $C'_{Q'}$ as boundaries, the sets of indices $\{\beta_i\}$ and $\{\alpha_i\}$ being continuously distributed on C_Q and $C'_{Q'}$, respectively. Now the compatibility condition which ensures that eq.(19) has non-trivial solutions $\Phi[C_Q]$ depending only on the given loop C_Q (and on a boundary initial value condition given on a loop $C_{Q_0}^0$ by $\Phi[C_{Q_0}^0]$) is just the requirement that the potential T has to depend in eq.(19) on the loops C_Q and $C'_{Q'}$, only and not on the particular surface $\Sigma_{C'_{Q'}, C_Q}$ joining them. It can be shown [8] that this condition is satisfied just as a consequence of our generalized Lax equation (11) which is therefore to be reinterpreted as a zero curvature condition, not on the 3- d lattice we are looking at, but on the space of functionals of loops drawn on it.

So what we have constructed here gives in particular a *classical reinterpretation* and *gauging* of the *quantum* Yang-Baxter equations (and hence of quantum groups associated to them), in term of functionals of loops having continuously distributed vector indices α_i along them. Therefore what we now have at hand is also a natural interpretation for the "framing" of loops, used in particular in the context of knot theory and of quantum Chern-Simons gauge theories [10], here in the framework of classical 3- d Integrable Systems. Moreover, this provides us with a natural setting for the construction of knot invariant polynomials [8].

More generally, the extended Lax systems and equations that can be constructed in any dimension d , using the concepts we have explained in this note, may be reinterpreted as linear systems of equations written for functionals of extended objects of dimension $d - 2$, with indices continuously distributed on such a $d - 2$ hypersurface, the potentials for such linear systems of difference (differentials) equations being given as an ordered

product on a hypersurface of dimension $d - 1$ of the elementary potentials living on cells of dimension $d - 1$ of the d -dimensional lattice. The compatibility condition for such an extended Lax system leads to a gauged version of the $d - 1$ -simplex equation that is to be interpreted as a zero curvature condition on the above space of functionals of hypersurfaces of dimension $d - 2$. Conserved quantities for such extended Lax type systems may then be obtained from a procedure analogous to the one we described in section 4. A more detailed discussion of the ideas and results presented here will be given in a forthcoming publication [8].

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