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MULTILOOP NO-RENORMALIZATION THEOREMS
FOR THE TEN-DIMENSIONAL HETEROTIC STRING

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A B S T R A C T

We demonstrate that the Chamseddine-Chapline-Manton ten-dimensional supergravity theory obtained as the low-energy field-theoretic limit of the heterotic string is not renormalized by the genus-two contribution to the effective action. We discuss how this result could be extended to higher genus. We give explicit formulae for genus-two four-point amplitudes of massless particles.

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1. - INTRODUCTION

The ten-dimensional heterotic string [1] played a key historical rôle in hastening the acceptance of string as a Theory of Everything, by opening the way to the construction of models with chiral fermions and realistic gauge groups [2]. Although it has been largely superseded by formulations of consistent string theories directly in four dimensions [3], the original heterotic string is still a useful theoretical laboratory and may have direct phenomenological relevance. Many of the tricks used to analyse the ten-dimensional heterotic string can also be applied to four-dimensional string models, and many of the latter can usefully be regarded as compactifications from ten dimensions.

Much string phenomenology can be discussed in the low-energy field-theoretical limit obtained by taking the zero-slope limit ($\alpha' \rightarrow 0$) of string amplitudes. The ten-dimensional heterotic string has long been known [1] to reduce to the Chamseddine-Chapline-Manton (CCM) [4] Einstein-Yang-Mills action in the $\alpha' \rightarrow 0$ limit of the tree-level (genus-zero) amplitudes. Many phenomenological conclusions have been based on the form of this action. Since there is no reason to believe that the string loop expansion has a small expansion parameter, it is important to check that the form of the CCM action is unaltered by higher-loop (higher-genus) amplitudes.

First steps in this direction have been taken [5], showing that one-loop (genus-one) corrections do not renormalize the CCM action. This was shown independently for the purely gravitational sector - including the antisymmetric tensor field and the dilaton, for gauge-boson/graviton couplings, and for the purely Yang-Mills interactions. The only corrections to the effective action that were found [6] are of higher order in the space-time derivatives of the fields, and do not renormalize the coefficients of the terms in the CCM action.

In this paper we extend the proof of this result to genus-two terms in the loop expansion for the ten-dimensional heterotic string, and indicate possible lines for an extension of the proof to arbitrary genus. Since much of our two-loop proof builds on our previous analysis of the one-loop effective action, in Section 2 we review the proof [5] that the CCM action is not renormalized at genus 1, stressing features that carry over to genus 2. Then, in Section 3 we show that two-loop effects do not renormalize the effective CCM action, but only produce terms of higher order in the space-time derivatives of the fields. Finally, in Section 4 we comment on the possibility of extending these results to arbitrary genus, stressing that it would be sufficient to prove that there are no higher-genus contributions to one-, two- and three-point functions. Three appendices are devoted to deriving the complete expressions for two-loop amplitudes for the emission of four massless particles in the heterotic string in terms of Jacobi θ -functions. Appendix A

presents the general formalism of higher loop calculations including explicit expressions for Green functions and zero mode wave functions. Appendix B describes the sum over spin structures in the hyperelliptic representation, and Appendix C obtains explicit expressions for the amplitudes.

2. - NO-RENORMALIZATION AT GENUS ONE

We start with a review of the non-renormalization of the CCM action by genus-one effects [5], stressing aspects shared with our subsequent genus-two analysis. The tree-level effective low-energy theory of the heterotic string was first derived in Ref. [1] where the tree-level two- and three-point bosonic amplitudes were studied. This analysis has been further developed for tree-level four-point couplings in Refs. [7-11]. As a result of these studies the CCM [4] action was found with the addition of terms containing up to six-derivative couplings of the gravitational field and two-derivative couplings for the Yang-Mills potentials:

$$I = \int d^{10}x e \left[-\frac{R}{2\kappa^2} - \frac{1}{8} e^{-\kappa D/\sqrt{2}} \left(\text{tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{g_{10}^2} \text{tr} R_{\mu\nu} R^{\mu\nu} \right) \right. \\ \left. - \frac{1}{2} \partial_\mu D \partial^\mu D - \frac{3}{2} e^{-\kappa D/\sqrt{2}} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{g_{10}^2}{g_{10}^2} R^{\mu\nu\rho\sigma} \bar{H}_{\rho\sigma\alpha} \bar{H}^\alpha_{\mu} \right] \quad (1)$$

where

$$H_{\mu\nu\rho} = \frac{1}{3} \left[\partial_\mu B_{\nu\rho} - \frac{1}{4} \kappa (\omega_{3Y})_{\mu\nu\rho} + \frac{\kappa}{4g_{10}^2} (\omega_{3L})_{\mu\nu\rho} + \text{cycl. perm.} \right] \quad (2)$$

$$(\omega_{3Y})_{\mu\nu\rho} = \text{tr} \left[A_\mu F_{\nu\rho} - \frac{2}{3} g_{10} A_\mu A_\nu A_\rho \right] \quad (3a)$$

$$(\omega_{3L})_{\mu\nu\rho} = \text{tr} \left[\omega_\mu R_{\nu\rho} - \frac{2}{3} \omega_\mu \omega_\nu \omega_\rho \right] \quad (3b)$$

and

$$\bar{H}_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \quad (4)$$

In (1), $R_{\mu\nu\rho\sigma}$ is the curvature tensor, $F_{\mu\nu}$ is the field strength of the Yang-Mills potentials, D is the dilaton field and $B_{\mu\nu}$ is the antisymmetric tensor field, and $(D, B_{\mu\nu})$ belong to the gravitational multiplet. Also, κ is the gravitational coupling and g_{10} is the gauge coupling in ten dimensions given by [1]

$$\kappa = (2\alpha')^2 g \quad (5a)$$

$$g_{10} = (2\alpha')^{3/2} g \quad , \quad (5b)$$

where α' is the string tension and g is a coupling constant.

The vertex operators for the emission of gravitons (including the dilaton and the antisymmetric tensor field) and gauge bosons in the heterotic string involve, as well as the usual exponential factor, the left-moving space-time and internal momenta and the right-moving supersymmetrized space-time momenta. This supersymmetrized momentum can be written as:

$$B^\mu = \partial X^\mu + \frac{1}{2} k_\tau \psi^\tau \psi^\mu \quad , \quad (6)$$

where ∂X^μ is the momentum operator in the right-moving sector, k_μ is the momentum of the particle emitted ($k^2=0$) and ψ^μ ($\mu=0, \dots, 9$) are the ten-dimensional right-moving space-time fermions. In calculating the n -point amplitudes we need to take into account all the possible two-point correlation functions of the bosonic and fermionic fields occurring in the product of n vertex operators. The fermionic two-point correlators for an even spin structure α on the torus are given by

$$\langle \psi^\mu(z_1) \psi^\nu(z_2) \rangle_\alpha = \frac{i}{2\pi} g^{\mu\nu} \frac{\theta_\alpha(z_{12}, \tau)}{\theta_1(z_{12}, \tau)} \frac{\theta'_1(0, \tau)}{\theta'_1(0, \tau)} \quad \alpha=2,3,4 \quad (7)$$

where

$$z_{12} = z_1 - z_2 \quad (8a)$$

$$\theta'_1(0, \tau) = \frac{d}{dz} \theta_1(z, \tau) \Big|_{z=0} \quad (8b)$$

The partition function for the right-moving fermions involves the sum over all spin structures α with weight $\theta_\alpha^4(0, \tau)$ and with phases which are determined by modular invariance. To get the amplitudes we need to add the correlation functions (7). Using now the Riemann identity

$$\sum_{a=1}^4 (-1)^{a+1} \frac{1}{\pi} \theta_\alpha(x_a, \tau) = 2 \frac{1}{\pi} \theta_1(x'_a, \tau) \quad (9)$$

where

$$x'_{1,2} = \frac{1}{2} (x_1 + x_2 \pm x_3 \pm x_4) \quad , \quad x'_{3,4} = \frac{1}{2} (x_1 - x_2 \pm x_3 \pm x_4)$$

we find that the n -point functions with $n < 3$ vanish as at least one of the x_a on the left-hand side and one of the x'_a in the right-hand side of (9) vanish, and we

have

$$\theta_1(0, \tau) = 0 \quad (10)$$

for the odd spin structure. Note that (10) also implies that we need not consider the contribution of the odd spin-structure (Ramond sector) when calculating up to four-point amplitudes, as those contain at most eight fermionic fields which cannot soak up all the ten zero modes which we have in this sector. We note that the vanishing of the n -point functions ($n \leq 3$) can also be proved by using Green-Schwarz fermions on the torus and a simple counting of fermionic zero-modes. However, this proof does not have as yet an extension to higher genus.

In addition to this vanishing of the one-, two- and three-point functions, it follows that any n -point function ($n \geq 4$) must contain at least four powers of momenta resulting from the factor $(\frac{1}{2} k_\tau \psi^\tau \psi^\mu)^4$ implied by (9), as 8 is the minimal number of fermionic fields (corresponding to four θ -functions with non-zero arguments) needed in order to get a non-zero contribution. Taking all the possible two-point correlators (7) and summing over the spin structures with the aid of the Riemann identity (9), we find that all the four-point one-loop amplitudes for the emission of massless particles in the heterotic string are proportional to $t_{\mu_1 \tau_1 \mu_2 \tau_2 \mu_3 \tau_3 \mu_4 \tau_4} \prod_{a=1}^4 k_{\tau_a}$, where the eight-index tensor t can be found for example in Ref. [12], and was first found in the light-cone gauge of superstring theory. Having these four powers of momenta in the amplitudes, it follows immediately that there can be no renormalization of the coefficients of R , $\partial_\mu D \partial^\mu D$, $\text{tr} F_{\mu\nu} F^{\mu\nu}$, $\omega_{3Y} \omega_{3Y}$ or $\partial_\mu B_{\nu\rho} [\partial^\mu B^{\nu\rho} - \omega_{3Y}^{\mu\nu\rho}]$ in the effective action (1), because they do not contain sufficient numbers of derivatives to be generated by the one-loop amplitudes.

To analyze the higher-loop effects on $R_{\mu\nu} R^{\mu\nu} e^{-\kappa D/\sqrt{2}}$, $\partial_\mu B_{\nu\rho} \omega_{3L}^{\mu\nu\rho} e^{-\kappa D/\sqrt{2}}$ and $\omega_{3L} \omega_{3L} e^{-\kappa D/\sqrt{2}}$ we need more detailed information on the four-graviton amplitude^{*)}. The vertex operator for the emission of a graviton in the heterotic string is given by

$$V_G = 2\alpha' g B_\mu \bar{\partial} X_\nu \rho^{\mu\nu}(k) e^{ik \cdot x} \quad (11)$$

where $\bar{\partial} X_\nu$ is the left-moving momentum operator, $\rho_{\mu\nu}(k)$ is the polarization tensor (symmetric for the graviton, antisymmetric for the antisymmetric tensor field and

*) Complete expressions for this and other one-loop amplitudes for the emission of four massless particles in the heterotic string can be found in Ref. [13].

traced for the dilaton) satisfying the gauge condition $k_\mu \rho^{\mu\nu}(k) = k_\nu \rho^{\mu\nu}(k) = 0$. As mentioned above, the only non-zero contribution from the right-moving sector to the correlation function of a product of four vertex operators is from the fermionic part of B_μ [Eq. (6)]; the bosonic part does not contribute due to the Riemann identity (9). From the left-moving sector we have the correlator

$$\begin{aligned} \langle \frac{4}{\pi} \bar{\partial} X_{\lambda_a} e^{ik_a \cdot X} \rangle &= \left[\bar{A}_{\lambda_1}^{(1)} \bar{A}_{\lambda_2}^{(1)} \bar{A}_{\lambda_3}^{(1)} \bar{A}_{\lambda_4}^{(1)} + \frac{1}{2\alpha'} \bar{A}_{\lambda_1}^{(1)} \bar{A}_{\lambda_2}^{(1)} \bar{R}_{34}^{(1)} g_{\lambda_3 \lambda_4} \right. \\ &\quad \left. + \left(\frac{1}{2\alpha'} \right)^2 \bar{R}_{12}^{(1)} \bar{R}_{34}^{(1)} g_{\lambda_1 \lambda_2} g_{\lambda_3 \lambda_4} + \text{permutations} \right] \pi e^{\alpha' k_a \cdot k_b} G_{ab}^{(1)} \end{aligned} \quad (12)$$

where $\bar{A}_{\rho_a}^{(1)}$ is the correlator between $\bar{\partial} X_{\rho_a}(z_a)$ and the exponential factor $e^{ik \cdot X}$

$$\bar{A}_{\rho_a}^{(1)} = -\frac{1}{4\pi i} \sum_{b=1}^4 k_b^{\rho_a} \partial_{\bar{z}_a} G^{(1)}(z_{ab}) \quad (13)$$

$\bar{R}_{ab}^{(1)} g_{\rho_a \rho_b}$ is the correlator between $\bar{\partial} X_{\rho_a}(z_a)$ and $\bar{\partial} X_{\rho_b}(z_b)$:

$$\bar{R}_{ab}^{(1)} = -\frac{1}{2\pi^2} \partial_{\bar{z}_a} \partial_{\bar{z}_b} G^{(1)}(z_{ab}) \quad (14)$$

and $G_{ab}^{(1)}$ is the bosonic Green function on the surface, which for the torus is given by

$$G_{ab}^{(1)}(z_{ab}) = \ln \left[e^{-\pi (\text{Im } z_{ab})^2 / \text{Im } \tau} \left| \frac{\theta_1(z_{ab}, \tau)}{\theta_1'(0, \tau)} \right| \right] \quad (15)$$

To explore the limit $\alpha' \rightarrow 0$, we expand the exponential factor in (12) in powers of α' . Integrating now over the location of the vertex operators z_a ($a=1, \dots, 4$) we find that all the terms in (12) which are of order $(1/2\alpha')^2$ or $1/2\alpha'$ vanish. Here we use the two identities [5,14]

$$\int_{F_\tau} d^2 z_a \partial_{\bar{z}_a} G^{(1)}(z_{ab}) = 0 \quad (16a)$$

$$\int_{F_\tau} d^2 z_a \partial_{\bar{z}_a} \partial_{\bar{z}_b} G^{(1)}(z_{ab}) = 0 \quad , \quad (16b)$$

where F_τ is the region in the complex plane defining the torus

$$F_\tau = \left\{ z_a / -\frac{1}{2} \leq \text{Re } z_a \leq \frac{1}{2}, 0 \leq \text{Im } z_a \leq \text{Im } \tau \right\}. \quad (17)$$

The two identities (16) are valid because contributions to the integrals can only

come from the boundary of F_τ , and the Green function is periodic

$$G^{(n)}(z+\tau, \tau) = G^{(n)}(z+\tau, \tau) = G^{(n)}(z, \tau). \quad (18)$$

Thus the leading order contribution to the four-graviton amplitude resulting from the expansion of (12) in powers of α' is of order one. Furthermore, all the terms of this type carry four powers of momenta. As a result, the leading order contributions of the four-graviton amplitude to the effective field theory carry eight powers of momenta, whereas $R_{\mu\nu} R^{\mu\nu} \sim k^4$, $\partial_\mu B_{\nu\rho} \omega_{3L}^{\mu\nu\rho} \sim k^4$ and $\omega_{3L} \omega_{3L} \sim k^6$. We thus conclude that the four-graviton one-loop amplitude cannot renormalize the coefficients of the CCM action.

For the terms involving the gauge boson couplings in (1), we need the vertex operator for the emission of a gauge boson. This is given by

$$V_g^A = \sqrt{2\alpha'} g K^A B_\mu \rho^\mu(k) e^{ik \cdot x} \quad , \quad (19)$$

where K^A is the contribution of the 16 left-moving compact co-ordinates, which has two equivalent representations: bosonic (Frenkel-Kac) or fermionic (current algebra), and $\rho_\mu(k)$ is the polarization tensor satisfying the gauge condition $k_\mu \rho^\mu(k) = 0$. Because of the term B_μ in (19) it is immediately seen that the amplitude for the emission of four gauge bosons is at least of fourth order in the momenta and therefore cannot renormalize any of the four gauge boson couplings of (1), which are at most of order two in the momenta. For the two-gauge-boson/two-graviton coupling we need a more detailed study of the amplitude [13]. As before, it is of at least fourth order in the momenta. The gravitons contribute the correlator

$$\langle \prod_{a=1}^4 \bar{\partial} x_{\lambda_a} \bar{\partial} x_{\lambda_2} e^{ik_a \cdot x} \rangle = \left[\bar{A}_{\lambda_1}^{(4)} \bar{A}_{\lambda_2}^{(2)} + \frac{1}{2\alpha'} \bar{R}_{12}^{(4)} g_{\lambda_1 \lambda_2} \right] \pi_{a < b} e^{\alpha' k_a \cdot k_b G_{ab}^{(2)}}. \quad (20)$$

Expanding now the exponential of (20) in powers of α' , we find that the term of order $1/2\alpha'$ in (20) does not contribute upon integration over the locations z_1, z_2 of the vertex operators, in view of the identity (16b). Thus the leading-order contribution to the two-gauge-boson/two-graviton amplitude carries at least six powers of momenta, whereas the term $\omega_{3Y} \omega_{3L} \sim k^4$, and it therefore cannot be renormalized by one-loop effects.

3. - NO-RENORMALIZATION AT GENUS TWO

The analysis of the higher-genus contributions to the CCM action goes along the same line. It is more complicated, though, as we need to insert $(2g-2)$ picture-changing operators (supercurrents), resulting from the integration over the $(2g-2)$ gravitino zero modes of a genus- g surface. These operators are coupled to the super-Beltrami differentials χ_a ($a=1,2,\dots,2g-2$). Taking these super-differentials to be independent of the supermoduli and located at the points y_a ($a=1,\dots,2g-2$), i.e., $\chi_a = \delta(z-y_a)$, we find that the points y_a where the supercurrents are inserted can be shifted around arbitrarily, thus changing the result by only a total derivative in moduli space [15] which vanishes upon integration over the moduli.

More specifically, the picture-changing operator J_s has two contributions

$$J_s = J_{gh} + J_m, \quad (21)$$

where the ghost part is

$$J_{gh} = -c \partial \beta + \frac{1}{2} \delta b - \frac{3}{2} \partial c \beta \quad (22)$$

and the matter part is

$$J_m = -\frac{1}{2} \psi^\rho \partial \chi_\rho \quad (23)$$

Here (b,c) are the ghosts and (β,γ) are the superghosts. As all the fields are free fields, J_{gh} does not correlate with the vertex operators (11) and (19) and the correlator of a product of n operators J_{gh} would factorize out of any correlation function. [We also need to insert $(3g-3)$ b fields to soak up the zero modes of the spin-two differential operator. These also do not correlate with the vertex operators of the particles emitted, and thus factorize out. [For details see Appendix A.] However, the matter part of the supercurrent J_m does correlate with the vertex operators and cannot a priori be factorized out. As a result, the proof of the non-renormalization theorem is not as immediate as for the genus-one case.

We specialize now to genus two, where some explicit results on n -point amplitudes ($n \leq 4$) are already known. We first note that it has already been proved that the n -point amplitudes with $n \leq 3$ vanish for genus two [16,17]. Furthermore, it has been shown in Ref. [18] that the four-point two-loop amplitudes in superstring models are of at least fourth order in the external momenta and proportional to $t^{\mu_1 \tau_1 \mu_2 \tau_2 \mu_3 \tau_3 \mu_4 \tau_4} \prod_{a=1}^4 k_{\tau_a}$ as in tree-level and one-loop amplitudes. The proof is based upon a special choice of points where the two supercurrents are inserted, namely the two zeroes of the holomorphic differential. For this special choice of

points, it was shown in Ref. [18] that the sum over spin structures does not allow correlations of the fermionic fields in J_m [see Eq. (23)] with those in the vertex operators (11) and (19). Thus there is an independent correlation of $J_s J_s$ (both for the matter and the ghost parts) which factorizes out. Taking now all the possible two-point correlators of the fermionic fields and summing over the spin structures we find eventually the tensor t . The reason is that the sum over spin structures yields a result which depends symmetrically on the points z_a ($a=1, \dots, 4$) where the vertex operators are inserted, as in tree-level and one-loop amplitudes. [For details see Appendix B]. We recall that the two-point correlator for an even spin structure α on a genus- g surface is given by [19]

$$\langle \psi^\mu(z_1) \psi^\rho(z_2) \rangle_\alpha = \frac{\theta_\alpha(\vec{u}_{12})}{\theta_\alpha(0)} \frac{1}{E(z_1, z_2)} g^{\mu\rho} \quad (24)$$

where $\vec{u}_{12}$ is the Jacobi map

$$u_{12}^i = \int_{z_1}^{z_2} \omega^i(z) dz \quad i=1, \dots, g, \quad (25)$$

$\omega^i(z)$ ($i=1, \dots, g$) are the g holomorphic differentials, and $E(z_1, z_2)$ is the prime form:

$$E(z_1, z_2) = \frac{\theta_*(\vec{u}_{12})}{h(z_1) h(z_2)} \quad (26)$$

In (26) θ_* is an odd θ -function and $h(z)$ is a holomorphic section of the spin bundle

$$h^2(z) = \omega^i(z) \partial_i \theta_*(\vec{u}) \Big|_{u=0}, \quad (\partial_i \equiv \partial_{u^i}). \quad (27)$$

The prime form in (26) does not depend on the choice of the odd characteristic θ_* . We note that the two-point correlators in (24) are similar to those of genus one [see Eq. (7)]. It is therefore not surprising that the sum over spin structures yields a result which is symmetric in the z_a ($a=1, \dots, 4$) at least up to genus two.

Knowing that the four-point amplitudes are of at least fourth order in external momenta also at genus two, we find that as before the coefficients of R , $\partial_\mu D \partial^\mu D$, $\text{tr} F_{\mu\nu} F^{\mu\nu}$, $\omega_{3Y} \omega_{3L}$ and $\partial_\mu B_{\nu\rho} [\partial^\mu B^{\nu\rho} - \omega_{3Y}^{\mu\nu\rho}]$ in the effective action (1) are not renormalized by two-loop effects. For the rest we need the correlators of the bosonic fields in the various amplitudes. Here we need to take account of the fact that J_m in (23) depends on the bosonic fields ∂X_μ .

We concentrate first on the two-gauge-boson/two-graviton amplitude which is

simpler to analyse^{*}). The gauge bosons contribute a factor $e^{ik \cdot X}$ to these correlators as is seen from (19). The gravitons contribute $\bar{\partial} X_\rho e^{ik \cdot X}$ each [see Eq. (11)]. We thus need to calculate two types of correlators:

$$(i) \quad \left\langle \prod_{a=1}^2 J_{gh}(y_a) \bar{\partial} X_{\lambda_a} \prod_{b=1}^2 e^{ik_b \cdot X} \right\rangle$$

$$(ii) \quad \left\langle \prod_{a=1}^2 J_m(y_a) \bar{\partial} X_{\lambda_a} \prod_{b=1}^2 e^{ik_b \cdot X} \right\rangle .$$

In the first, the $J_{gh}(y_a)$ ($a=1,2$) decouple and we find the correlator (20) for genus two. Here we need to replace $\bar{A}_{\lambda_a}^{(1)}$, $\bar{R}_{ab}^{(1)}$ and $G^{(1)}(z_{ab})$ by $\bar{A}_{\lambda_a}^{(2)}$, $\bar{R}_{ab}^{(2)}$ and $G^{(2)}(\vec{u}_{ab})$ respectively. Thus $\bar{A}_{\lambda_a}^{(2)}$ is the two-loop correlator between $\bar{\partial} X_{\lambda_a}$ and $e^{ik \cdot X}$ [compare with (13)],

$$\bar{A}_{\lambda_a}^{(2)} = \sum_{b=1}^4 k_b^{\lambda_a} \omega^i(\bar{z}_a) \bar{\partial}_i G^{(2)}(\vec{u}_{ab}) , \quad (28)$$

$\bar{R}_{ab}^{(2)}$ is the two-loop correlator between $\bar{\partial} X_{\lambda_a}(z_a)$ and $\bar{\partial} X_{\lambda_b}(z_b)$

$$\bar{R}_{ab}^{(2)} = \omega^i(\bar{z}_a) \bar{\partial}_i \bar{\partial}_j G^{(2)}(\vec{u}_{ab}) \omega^j(\bar{z}_b) \quad (29)$$

and $G^{(2)}$ is the genus-two Green function

$$G^{(2)}(\vec{u}_{ab}) = \ln \left[e^{-\pi \operatorname{Im} u_{ab} (\operatorname{Im} \tau)^{-1} \operatorname{Im} u_{ab}} |E(\bar{z}_a, \bar{z}_b)| \right] . \quad (30)$$

Here $E(z_a, z_b)$ is the prime form (26) for genus two, $\vec{u}_{ab}$ is the Jacobi map (25), the $\omega^i(z)$ are the holomorphic differentials, and τ is the 2×2 period matrix of genus two. The limit $\alpha' \rightarrow 0$ of the correlator (i) is of second order in the external momenta as in (20). Here we have the extension of the identities (16) for genus two [20]

$$\int d^2 z_a \omega(z_a) \omega^i(\bar{z}_a) \bar{\partial}_i G^{(2)}(\vec{u}_{ab}) = 0 \quad (31a)$$

$$\int d^2 z_a \omega(z_a) \omega^i(\bar{z}_a) \bar{\partial}_i \bar{\partial}_j G^{(2)}(\vec{u}_{ab}) \omega^j(\bar{z}_b) = 0 . \quad (31b)$$

These can be proven by going to the boundary and using the periodicity of the Green

^{*}) In Appendix C we present explicit expressions for this and other genus-two four-point amplitudes for the emission of massless particles in the heterotic string in terms of Jacobi θ -functions. Partial results on these amplitudes in terms of the hyperelliptic representation of genus two can be found in Ref. [18].

function. Thus all the terms proportional to $\bar{R}_{ab}^{(2)}$ in the correlator (i), which are independent of the external momenta, vanish upon integration over z_a due to (31b). As a result, the correlator (i) is at least quadratic in the external momenta.

In the correlator (ii) we have

$$\begin{aligned} & \left\langle \frac{2}{\pi} \partial X_{\rho_a}(y_a) \bar{\partial} X_{\lambda_b}(z_b) \frac{4}{\pi} e^{ik_b \cdot X} \right\rangle = \\ & = \left\{ g_{\rho_1 \lambda_1} \frac{1}{2\alpha'} D_{11} (A_{\rho_2}^{(2)} \bar{A}_{\lambda_2}^{(2)} + \frac{1}{2\alpha'} D_{22} g_{\rho_2 \lambda_2}) + (g_{\rho_1 \lambda_2} y_1 \leftrightarrow y_2) \right. \\ & \quad \left. + (A_{\rho_1}^{(2)} A_{\rho_2}^{(2)} + \frac{1}{2\alpha'} R_{12}^{(2)} g_{\rho_1 \rho_2}) (\bar{A}_{\lambda_1}^{(2)} \bar{A}_{\lambda_2}^{(2)} + \frac{1}{2\alpha'} \bar{R}_{12}^{(2)} g_{\lambda_1 \lambda_2}) \right\} \times \\ & \quad \times \frac{\pi}{\alpha' k_b} e_{\alpha' \rho} [\alpha' k_a \cdot k_b G^{(2)}(\bar{u}_{ab})] \quad , \end{aligned} \quad (32)$$

where $A_{\rho_a}^{(2)}$ is the two-loop correlator between $\partial X_{\rho_a}(y_a)$ and $e^{ik \cdot X}$, $R_{ab}^{(2)}$ is the correlator between $\partial X_{\rho_a}(y_a)$ and $\partial X_{\rho_b}(y_b)$ defined as in (28) and (29) respectively (for the left-movers). We also have the correlators D_{ab} of the left- and right-movers $[\partial X_{\rho_a}(y_a)$ and $\bar{\partial} X_{\lambda_b}(z_b)$, $a, b=1,2$]

$$D_{ab} g_{\rho_a \lambda_b} \equiv \langle \partial_{\rho_a} X(y_a) \bar{\partial}_{\lambda_b} X(z_b) \rangle = g_{\rho_a \lambda_b} \omega^i(y_a) \partial_i \bar{\partial}_j G(y_a, z_b) \omega^j(\bar{z}_b). \quad (33)$$

In the limit $\alpha' \rightarrow 0$ the terms not containing the external momenta in (32) are proportional to $\bar{R}_{12}^{(2)} R_{12}^{(2)}$ or $D_{11} D_{22}$ (and permutations). The first would vanish upon integration of $\bar{R}_{12}^{(2)}$ over z_a ($a=1,2$), which are the points where the vertex operators are inserted. Here we use identity (31b). [Recall that $R_{12}^{(2)}$ depends on y_a ($a=1,2$), the points where the supercurrents are inserted, and these are fixed.] The second also vanishes when integrated over z_a ($a=1,2$) as we have [20]

$$\int d^2 z_a \omega(z_a) \omega^i(\bar{z}_a) \bar{\partial}_i \partial_j G(z_a, y_b) \omega^j(y_b) = 0 \quad . \quad (34)$$

Once again this integral vanishes because of the periodicity of the Green function on the surface^{*}).

^{*}) The identities (31b) and (34) follow in fact from (31a). A stronger identity can be found in Ref. [20], namely that the integral of the first derivative of the Green function along any closed loop on the surface vanishes. Once again it follows from the periodicity requirement.

In conclusion, the leading behaviour of (32) in the limit $\alpha' \rightarrow 0$ is quadratic in the external momenta. We also found that the previous correlator (i) is quadratic in the external momenta in the limit $\alpha' \rightarrow 0$. Thus the two-gauge-boson/two-graviton amplitude as a whole is of sixth order in the external momenta and it cannot renormalize the coefficient of $\omega_{3Y}\omega_{3L} \sim k^4$ in the CCM action (1).

The analysis of the four-graviton amplitude is similar to the above. Here we need to consider two types of correlators resulting from the vertex operator (11). The first is as in (12) with $\bar{A}_{\lambda_a}^{(2)}$ and $\bar{R}_{ab}^{(2)}$ replacing $\bar{A}_{\lambda_a}^{(1)}$ and $\bar{R}_{ab}^{(1)}$ respectively. It involves $J_{gh} J_{gh}$ which factorizes out, and its field-theoretic limit ($\alpha' \rightarrow 0$) is fourth-order in the external momenta, as in the one-loop case, resulting from the identities (31) which would make all the terms having less than four external momenta vanish. The second correlator involves an insertion of $J_m J_m$ and we have

$$\begin{aligned} & \left\langle \frac{2}{\pi} \partial X_{\rho_a}(\gamma_a) \frac{4}{\pi} \bar{\partial} X_{\lambda_b}(z_b) e^{ik_b \cdot x} \right\rangle = \\ & = \left(A_{\rho_1}^{(2)} A_{\rho_2}^{(2)} + \frac{1}{2\alpha'} R_{12}^{(2)} g_{\rho_1 \rho_2} \right) \left\langle \frac{4}{\pi} \bar{\partial} X_{\lambda_a}(z_a) e^{ik_a \cdot x} \right\rangle + \\ & + \frac{1}{2\alpha'} A_{\rho_2}^{(2)} \langle \partial X_{\rho_1}(\gamma_1) \bar{\partial} X_{\lambda_1}(z_1) \rangle \left\langle \frac{4}{\pi} \bar{\partial} X_{\lambda_b}(z_b) \frac{4}{\pi} e^{ik_a \cdot x} \right\rangle \\ & + \left(\frac{1}{2\alpha'} \right)^2 \frac{2}{\pi} \langle \partial X_{\rho_a}(\gamma_a) \bar{\partial} X_{\lambda_a}(z_a) \rangle \left(\bar{A}_{\lambda_3}^{(2)} \bar{A}_{\lambda_4}^{(2)} + \frac{1}{2\alpha'} \bar{R}_{34}^{(2)} g_{\lambda_3 \lambda_4} \right) + \text{perm.} \end{aligned} \quad (35)$$

The correlator $\langle \partial X_{\rho_a} \bar{\partial} X_{\lambda_a} \rangle$ is defined in (33) and

$$\left\langle \frac{4}{\pi} \bar{\partial} X_{\lambda_b}(z_b) \frac{4}{\pi} e^{ik_a \cdot x} \right\rangle = \bar{A}_{\lambda_2}^{(2)} \bar{A}_{\lambda_3}^{(2)} \bar{A}_{\lambda_4}^{(2)} + \frac{1}{2\alpha'} \bar{A}_{\lambda_2}^{(2)} \bar{R}_{34}^{(2)} g_{\lambda_3 \lambda_4} + \text{perm.} \quad (36)$$

The limits as $\alpha' \rightarrow 0$ of the first two terms in (35) are of fourth order in the external momenta due to the identities (31). In the last expression of (35) we have terms of the type $D_{11} D_{22} \bar{R}_{34}^{(2)} g_{\rho_1 \lambda_1} g_{\rho_2 \lambda_2} g_{\lambda_3 \lambda_4}$, which do not depend on the external momenta. However, they vanish after integration over z_a ($a=1, \dots, 4$) and use of (31) and/or (34). In a similar way, terms quadratic in the external momenta vanish as well. As a result, the two-loop four-graviton amplitude is of eighth order in the external momenta (recall that there are four powers of the external momenta in the prefactor) and it cannot renormalize the terms $R_{\mu\nu} R^{\mu\nu} \sim k^4$, $\partial_\mu B_{\nu\rho} \omega_{3L}^{\mu\nu\rho} \sim k^4$ and $\omega_{3L} \omega_{3L} \sim k^6$ in the effective action (1). This completes the proof of the non-renormalization of the CCM up to two-loops.

4. - NO-RENORMALIZATION AT HIGHER GENUS

We would like to comment now on the possible extension of this proof to higher genus. We noticed in the above the resemblances of the proofs for genus one and two. The first element in this proof was to show the vanishing of the n -point functions with $n \leq 3$. This results from the sum over the spin structures. Once this is proved, all the higher-point functions are at least fourth order in the external momenta as a result of the factor $(\frac{1}{2}k_\tau \phi^\tau \psi^\mu)^4$ from the product of four vertex operators. For the rest, we need the bosonic correlators as in (12) for four gravitons or (20) for the two-graviton/two-gauge-boson amplitudes. We need, however, to insert in the correlators $(2g-2)$ supercurrents J_s . Taking all the possible two-point bosonic correlators both from J_m or the vertex operators we find expressions similar to the various terms in (35). In the limit $\alpha' \rightarrow 0$ the identities (31) and (34) for a genus- g surface [20] would make the four-graviton interactions resulting from the genus- g amplitude of at least eighth order in the external momenta, whereas the two-graviton/two-gauge-boson interaction would be of sixth order in the external momenta. Both interaction terms are not appropriate for renormalizing the coefficients in the CCM action. Thus, to complete the proof of the non-renormalization of the CCM action, we only need to show that the one-, two- and three-point amplitudes for any genus vanish [21].

There is an independent argument that the vanishing of the one-, two- and three-point amplitudes would be sufficient to guarantee the non-renormalization of the CCM action. The symmetries of the theory (supersymmetry, gauge invariance and general covariance) relate the higher-point functions of the CCM action to the one-, two- and three-point functions, and therefore corrections to the former would vanish if corrections to the latter vanish. This line of argument would have enabled us to omit some of the technical discussion in Section 3, provided we know that all the symmetries of the theory are respected. It is therefore reassuring that the detailed calculation does confirm this expectation. The detailed calculation is also for our understanding of the structure of possible two-loop corrections to the CCM action, enabling us to make a more complete analysis of the field-theoretic limit of genus-two amplitudes.

APPENDIX A

We present in this appendix general expressions for calculating N-point g-loop amplitudes in the heterotic string. We will be following mainly Refs. [15] and [20]. A general review on the geometry of string perturbation theory can be found in Ref. [22].

The N-point g-loop amplitude in the heterotic string can be written as a functional integral over the superfields $X^\mu(z, \theta)$ ($\mu = 0, \dots, 9$), representing the holomorphic and antiholomorphic bosonic fields X^μ and the holomorphic fields $\psi^\mu(z)$, and over the 16 antiholomorphic compact bosonic co-ordinates $X^I(\bar{z})$ ($I = 1, \dots, 32$) representing the gauge group $SO(32)$ or $E_8 \times E_8$. For $SO(32)$, a linear realization of the algebra in terms of 32 fermionic fields $\lambda^I(\bar{z})$ ($I = 1, \dots, 32$) is somewhat more useful. After gauge fixing, the expression for the amplitude becomes an integral over the supermoduli space S_g , which is the parameter space of all the inequivalent superconformal structures of the Riemann surface Σ_g described by $3g-3$ complex commuting co-ordinates $\{(m_i, \bar{m}_i)/i = 1, \dots, 3g-3\}$ and $2g-2$ Grassmann co-ordinates $\{\zeta_a/a = 1, \dots, 2g-2\}$ [23]. As a consequence of the splitting of the string modes into left- and right-movers, the integrand factorizes into a holomorphic times an antiholomorphic function on supermoduli space. Thus the N-point amplitude, given by the integral over the correlator of N vertex operators, can be written as

$$A_g^{WJ} = \int_{S_g} \mathcal{D}X^\mu \mathcal{D}\psi^\mu \mathcal{D}\lambda^I \mathcal{D}b \mathcal{D}c \mathcal{D}\beta \mathcal{D}\gamma \prod_{i=1}^{3g-3} d^2 m_i |\langle \eta_i | b \rangle|^2 \times \prod_{a=1}^{2g-2} d\zeta_a \delta(\langle \chi_a | \beta \rangle) \prod_{k=1}^N V(z_k) d^2 z_k e^{-S}, \quad (A.1)$$

where S is the total action

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma_g} d^2 z \sqrt{g} \left[\partial_z X^\mu \partial_{\bar{z}} X_\mu - \psi^\mu \partial_{\bar{z}} \bar{\psi}_\mu - \lambda^I \partial_z \lambda^I + 2b \partial_{\bar{z}} c + 2\bar{b} \partial_z \bar{c} + 2\beta \partial_{\bar{z}} \gamma \right] \quad (A.2)$$

and a sum over the spin structures both of ψ^μ and λ^I should eventually be added to get the final form of the amplitude. The integrals over (b, c) and (β, γ) in (A.1) generate the Faddeev-Popov superdeterminant, and the insertion of the operators $\langle \eta_i | b \rangle$ and $\delta(\langle \chi_a | \beta \rangle)$ is necessary to absorb the ghost zero modes, corresponding to the variations along the gauge slice. The η_i and χ_a are a set of super-Beltrami differentials dual to the one-forms dm_i , $d\zeta_a$ on S_g and have, as a consequence, vanishing Lie-brackets

$$\frac{d}{dm_i} \eta_j = \frac{d}{dm_j} \eta_i \quad ; \quad \frac{d}{d\zeta_a} \eta_i = \frac{d}{dm_i} \chi_a \quad ; \quad \frac{d}{d\zeta_a} \chi_a = - \frac{d}{d\zeta_b} \chi_b. \quad (A.3)$$

Different choices of the η_i and χ_a correspond to different co-ordinate systems on S_g . We take χ_a to be independent of ζ_a so that the total action can be expanded as

$$S = S_0 + \sum_{a=1}^{2g-2} J_a \langle \chi_a | J_S \rangle, \quad (A.4)$$

where J_S is the supercurrent defined in Eqs. (21)-(23), and $S_0 = S|_{\zeta_a=0}$.

Integrating now the Grassmann variables ζ_a we find for (A.1)

$$A_g^{[w]} = \int_{M_g} \mathcal{D}[\chi, \psi, \lambda, b, c, \beta, \gamma] e^{-S_0} \prod_{k=1}^N d^2 z_k V(z_k) \times \prod_{a=1}^{2g-2} [\delta(\langle \chi_a | \beta \rangle) + \frac{2}{J_S} J_a] \prod_{i=1}^{3g-3} d^2 m_i |\langle \eta_i | b \rangle|^2, \quad (A.5)$$

where M_g is the moduli space. We choose the special class of functions χ_a which are independent of the moduli and located at the points y_a , i.e., $\chi_a = \delta(z - y_a)$ $a = 1, \dots, 2g-2$. We then find from (A.3) that the η_i are independent of the ζ_a and the amplitude simplifies to the form

$$A_g^{[w]} = \int_{M_g} \mathcal{D}[\chi, \psi, \lambda, b, c, \beta, \gamma] e^{-S_0} \prod_{k=1}^N d^2 z_k V(z_k) \times \prod_{a=1}^{2g-2} \delta(\langle \chi_a | \beta \rangle) J_S(y_a) \prod_{i=1}^{3g-3} d^2 m_i |\langle \eta_i | b \rangle|^2. \quad (A.6)$$

Inserting now the supercurrents J_S [Eqs. (21)-(23)] in (A.6) we find that we need the various correlators of the ghost fields. These were calculated in Ref. [15] using the "bosonization" of the superghost system. Another way to calculate these correlators is to take all the combinations of two-point correlators (in the "fermionic" representation) after factorizing the contribution of the zero modes. It turns out that this representation is more useful for the sum over spin structures, as was shown in Refs. [16,18]. We will therefore use this representation in the following.

Using the general notation (B,C) for the ghost (b,c) or the superghost (β, γ) system we may write the correlators as [20,22]

$$\left\langle \prod_{i=1}^M B(z_i) \prod_{j=1}^N C(w_j) \right\rangle = \frac{(\det' \nabla_n)^\varepsilon}{\det \langle \phi_a | \phi_b \rangle^{\chi_2}} \sum_{i_1, \dots, i_I} (-1)^{\sum_{n=1}^I i_n} \det \phi_k(z_j) \det_{\ell \neq i_1, \dots, i_I} G_n(z_\ell, w_m) \quad (A.7)$$

with $I = M-N$, where I is the number of zero modes [$I = 2g-2$ for (β, γ) and $I = 3g-3$ for (b,c)]. The zero mode wave functions are denoted by ϕ_k and the Green functions for the spin- n fields [$n = 3/2$ for (β, γ) and $n = 2$ for (b,c)] are $G_n(z, w)$. $\det' \nabla_n$ in (A.7) is the determinant of the non-zero mode contribution of the spin- n system and $\varepsilon = 1, -1$ for the (b,c) and (β, γ) system respectively. Strictly speaking, the (β, γ) system is bosonic and the antisymmetrization in (A.7) is not justified a priori. However, these fields are coupled to the super-Beltrami differentials which are fermionic in nature, thereby justifying the eventual anti-

symmetrization of $\langle \chi_a | \beta \rangle$ and $\langle \chi_a | \gamma \rangle$. Explicit expressions for the two-point correlators and the zero mode wave functions of the various fields are found in Ref. [20] which we will now present in some detail.

The two-point correlator of spin- n system should satisfy the equations:

$$\frac{1}{\pi} \partial_{\bar{z}} G_n(z, w) = \delta^2(z, w) \quad (\text{A.8a})$$

$$\frac{1}{\pi} \partial_{\bar{w}} G_n(z, w) = -\delta^2(z, w) + \sum_{i=1}^S \varphi_n^i(z) \bar{\varphi}_n^i(w) (g_{w\bar{w}})^{1-n} \quad (\text{A.8b})$$

where ϕ_n^i is a basis for the zero mode wave functions. $G_n(z, w)$ is analytic in z with a single pole at $z = w$, but is not analytic in w . Any other meromorphic tensor $K_n(z, w)$ satisfying the equation

$$\frac{1}{\pi} \partial_{\bar{z}} K_n(z, w) = \delta^2(z, w) \quad (\text{A.9})$$

will differ from $G_n(z, w)$ by terms proportional to the zero modes

$$G_n(z, w) = K_n(z, w) + \sum_{i=1}^S \varphi_n^i(z) f_{1-n}^i(w) \quad (\text{A.10})$$

and can therefore be equivalently used in the correlator (A.7), as the second term of (A.10) would disappear in the process of the antisymmetrization. We will therefore make a convenient choice for the two-point correlators, which will facilitate the calculation of (A.7). We note in particular that $K_n(z, w)$ should have a single pole in $z = w$. It may have unphysical poles in w , but those will all be hidden in $f_{1-n}^i(w)$, and thus proportional to the zero mode wave functions $\phi_n^i(z)$.

For the spin-2 (b,c) system we choose $K_2(z, w)$ to be [20] $[G_2(z, w) \equiv \langle b(z)c(w) \rangle]$

$$K_2(z, w) = V_{zw} R^{(2)}(z, w) U_{ww}^{-1} \quad (\text{A.11})$$

where $R^{(2)}(z, w)$ is the two-point correlator of the spin-zero fields $\langle \partial X(z) \partial X(w) \rangle$ defined as in (29) (for the holomorphic contributions), V_{zw} is a holomorphic one-tensor in z and w with a single zero $z = w$

$$V_{zw} = \omega_1(z) \omega_2(w) - \omega_2(z) \omega_1(w) \quad (\text{A.12})$$

and

$$U_{ww} = \partial_z V_{zw} \Big|_{z=w} = \omega_1'(w) \omega_2(w) - \omega_2'(w) \omega_1(w) \quad (\text{A.13})$$

is a holomorphic three-tensor in w . The $\omega_{1,2}$ in (A.13) are two of the g holomorphic differentials of the genus- g surface. Their choice is arbitrary and they mix under modular transformations, however the correlation functions are modular-convariant. A basis for the zero mode wave functions $\phi^k(z)$ was found in Ref. [20] (see also the second reference of [15]), in terms of the holomorphic differentials $\omega_i(z)$ ($i = 1, \dots, g$) and the Abelian differentials of the third kind ω_{PQ} , which have poles at P, Q with residues $+1$ and -1 respectively [19]

$$\omega_{PQ} = \partial_z \ln \frac{E(z, P)}{E(z, Q)} \quad . \quad (A.14)$$

In addition to these differentials, there exists on the surface one holomorphic differential $\omega(z)$ which has $2g-2$ simple zeroes (see in particular the third reference of [19]). With these, the zero mode wave functions can be written as

$$\varphi^j(z) = \omega(z) \omega_{P_j Q}(z) \quad j=1, \dots, 2g-3 \quad (A.15a)$$

$$\varphi^k(z) = \omega(z) \omega^k(z) \quad k=1, \dots, g, \quad (A.15b)$$

where P_j, Q are fixed to be zeroes of $\omega(z)$, thereby cancelling the poles of $\omega_{P_j Q}$.

For the spin-3/2 (β, γ) system, we choose $K_{3/2}^\alpha(z, w)$ for spin structure α to be [20] ($G_{3/2}^\alpha(z, w) \equiv \langle \beta(z) \gamma(w) \rangle_\alpha$)

$$K_{3/2}^\alpha(z, w) = \omega(z) \langle \psi(z) \psi(w) \rangle_\alpha \omega^{-1}(w) \quad , \quad (A.16)$$

where the two-point correlator of the spin- $\frac{1}{2}$ fields $\psi(z)$ with even spin structure α is given in Eq. (24). For the four-point functions which will be constructed later, the odd spin structure contribution to the correlator would not be needed. In (A.16) $\omega(z)$ is an Abelian differential. The zero mode wave functions of the spin-3/2 fields were also constructed in Ref. [20], and for spin structure α can be written as

$$\varphi_e^{[\alpha]}(z) = \omega(z) \langle \psi(z) \psi(w) \rangle_\alpha \quad , \quad (A.17)$$

where $\omega(z)$ in (A.17) is the Abelian differential having $2g-2$ simple zeroes at z_λ ($\lambda = 1, \dots, 2g-2$). We note that if we use the same Abelian differential both in (A.16) and (A.17) the additional poles of $K_{3/2}^\alpha(z, w)$ in w are proportional to the zero mode wave functions (A.17) and those can be ignored in the correlator (A.7). However, we may also use different Abelian differentials in (A.16) and (A.17), in which case the additional poles of (A.16) in w would be a linear combination of the zero mode wave functions (A.17).

We use now these results to calculate the ghost contributions to the N-point amplitudes (A.6) for $g = 2$. Here we have two insertions of supercurrents J_s . From Eqs. (21)-(23) we find

$$J_s(y_1) J_s(y_2) = J_m(y_1) J_m(y_2) + J_{gh}(y_1) J_{gh}(y_2) . \quad (A.18)$$

The matter fields in J_m correlate with the matter fields in the vertex operators and those will be treated in Appendix B [see also (32) and (35)]. For the ghost part J_{gh} we have

$$\begin{aligned} I_{gh} &\equiv \left\langle \prod_{a=1}^2 \delta(\langle \chi_a | \beta \rangle) \langle \chi_a | J_{gh} \rangle \prod_{i=1}^3 \langle \eta_i | b \rangle \right\rangle \\ &= \frac{1}{2} R(y_1, y_2) \frac{\partial}{\partial y_2} P(y_1, y_2) + \frac{3}{4} \frac{\partial R(y_1, y_2)}{\partial y_2} P(y_1, y_2) \\ &\quad - (1 \leftrightarrow 2) , \end{aligned} \quad (A.19)$$

where

$$P(y_1, y_2) = \langle \gamma(y_1) \beta(y_2) \delta(\langle \chi_1 | \beta \rangle) \delta(\langle \chi_2 | \beta \rangle) \rangle \quad (A.20a)$$

$$R(y_1, y_2) = - \langle c(y_2) b(y_1) \prod_{i=1}^3 \langle \eta_i | b \rangle \rangle . \quad (A.20b)$$

To calculate the correlators in (A.20) we use $\delta(\langle \chi_a | \beta \rangle) = (\langle \chi_a | \beta \rangle)$ and to avoid singularities one may use the Gaussian representation of χ_a [16]

$$\chi_a(z) = \delta(z - y_a) = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \exp \left[- \frac{|z - y_a|^2}{\alpha} \right] . \quad (A.21)$$

Using now (A.7), we find that $P(y_1, y_2)$ in (A.20a) vanishes due to the antisymmetrizations. Its derivative, though, does not vanish, and we have [16]

$$\begin{aligned} I_{gh} &= -\frac{1}{2} \partial_2 K_{3/2}^\alpha(y_1, y_2) R(y_1, y_2) - \frac{1}{2} \left[\frac{1}{2} \partial_2 R(y_2, y_1) - \Lambda(y_2) R(y_2, y_1) \right] \frac{\partial \varphi_2^{[\alpha]}(y_1)}{\varphi_2^{[\alpha]}(y_2)} \\ &\quad - \frac{1}{2} K_{3/2}^\alpha(y_2, y_1) R(y_2, y_1) \frac{\partial \varphi_1^{[\alpha]}(y_1)}{\varphi_1^{[\alpha]}(y_1)} - (1 \leftrightarrow 2) , \end{aligned} \quad (A.22)$$

where $\phi_\lambda^{[\alpha]}$ ($\lambda = 1, 2$) are the zero mode wave functions (A.17), and $K_{3/2}^\alpha$ is the two-point correlator (A.16) for the spin-3/2 system. $\Lambda(y_2)$ is the finite part of $K_{3/2}^\alpha(z, w)$ when $z \rightarrow w$ and $\partial_2 R(y_2, y_1)$ in (A.22) results from the regularization. The correlator (A.20b) can be written in terms of the two-point correlator (A.11) and the zero mode wave functions (A.15) for spin-2

$$R(y_1, y_2) = -\det(\eta_i \varphi_j)^{-1} \int \prod_{k=1}^3 \eta_k(z_k) B(y_1, y_2, z_1, z_2, z_3) \quad (\text{A.23})$$

where [see (A.7)]

$$B(z, x_1, x_2, x_3, x_4) = \varepsilon^{ijk\ell} K_2(z, x_i) \varphi_1(x_j) \varphi_2(x_k) \varphi_3(x_\ell) . \quad (\text{A.24})$$

Inserting now (A.16), (A.17), (A.23) and (A.24) in (A.22), we may write I_{gh} as

$$I_{gh} = F_1 \langle \Psi(y_1) \Psi(y_2) \rangle_\alpha + F_2 \langle \partial \Psi(y_1) \Psi(y_2) \rangle_\alpha , \quad (\text{A.25})$$

where F_1, F_2 are independent of the spin structure α . They depend on the Abelian differentials $\omega(z)$ and the correlators $R(y_1, y_2)$.

APPENDIX B

We present in this appendix the hyperelliptic representation [24] which is valid for all genus-two surfaces, and the derivation of the kinematical factors of the amplitudes as found in Ref. [18] (see also Ref. [16]). The defining equation for the hyperelliptic surface is

$$S^2(z) = \prod_{i=1}^6 (z - a_i) \quad , \quad (B.1)$$

where the a_i ($i = 1, \dots, 6$) are six branch points defining three branch cuts in the lower and upper sheets for the function $S(z)$, which represents a generic genus-two Riemann surface. Three of these points can be fixed by Möbius invariance and the other three can be taken to be the moduli. The Abelian differentials in this representation can be written as [24]

$$\omega_1(z) = S^{-1}(z) \quad (B.2a)$$

$$\omega_2(z) = z S^{-1}(z) \quad , \quad (B.2b)$$

and the correlator of two fermionic fields for an even spin structure α is

$$\langle \Psi(z) \Psi(x) \rangle_\alpha = \frac{1}{z-x} \frac{U(z) + U(x)}{2\sqrt{U(z)U(x)}} \quad , \quad (B.3)$$

where

$$U(x) = \frac{3}{\pi} \sqrt{\frac{x-A_i}{x-B_i}} \quad . \quad (B.4)$$

Here $\{A_k\}$ and $\{B_k\}$ ($k = 1, 2, 3$) are a splitting of the branch points $\{a_i\}$ ($i = 1, \dots, 6$) into two non-intersecting sets. This splitting is equivalent to one of the ten possible even spin structures α and a permutation of the points a_i is equivalent to a change of the spin structure. Thus a modular-invariant combination would be found by symmetrization over the branch points. The non-zero mode contributions to the fermionic determinants for spin structure α are given by the Thomae formula [19]

$$Q_\alpha \equiv \theta_\alpha^4(0, \tau) = \pm \det^2 K \prod_{1 \leq i < j \leq 3} A_{ij} B_{ij} \quad , \quad (B.5)$$

where K is the period matrix along the A-cycle for the holomorphic differentials (B.2) and

$$A_{ij} = A_i - A_j \quad ; \quad B_{ij} = B_i - B_j \quad . \quad (B.6)$$

We fix now the points y_a ($a = 1, 2$), where the supercurrents are inserted, to be the

two simple zeroes of $\omega_1(z)$; i.e., at the points ∞ on the upper and lower sheet ($z_{1,2} = \infty \pm$). The correlation functions for the matter part of the supercurrent (23) then become [16]

$$\langle \Psi(y_1) \Psi(y_2) \rangle_\alpha = \frac{1}{2} \sum_{i=1}^3 (A_i - B_i) \quad , \quad (\text{B.7})$$

whereas the correlators with fields at an arbitrary point z on the Riemann surface are

$$\langle \Psi(z) \Psi(y_1) \rangle_\alpha = \frac{u(z) - 1}{2 \sqrt{u(z)}} \quad (\text{B.8a})$$

$$\langle \Psi(z) \Psi(y_2) \rangle_\alpha = \frac{u(z) + 1}{2 \sqrt{u(z)}} \quad . \quad (\text{B.8b})$$

To get these correlation functions, it should be remembered that the Riemann sphere is built out of two patches: the one around zero, for which we use the co-ordinate z , and the one around ∞ , for which we use the co-ordinate $t = 1/z$. We therefore need to transform (B.1)-(B.4) to the patch around infinity, remembering to add the transformation of the differentials $dz = -dt/t^2$ [the $\omega_a(z)$ in (B.2) are one-differentials and the correlator in (B.3) is half-differential in each variable].

For the ghost part of the supercurrent [Eq. (22)], we need the correlators of the spin-3/2 field. They can be expressed in terms of derivatives of spin- $\frac{1}{2}$ correlators [see (A.16), (A.17) and (A.22)], which for the above choice of y_a ($a = 1, 2$) can be written as

$$\langle \partial \Psi(y_1) \Psi(y_2) \rangle_\alpha = \frac{1}{8} \sum_{i=1}^3 (A_i^2 - B_i^2) \quad . \quad (\text{B.9})$$

Summing now over the spin structures α (i.e., permuting the branch points a_i) with phases which are uniquely determined by modular invariance, we find that the one- and two- and three-point functions vanish, and the four-point functions are proportional to the same kinematical factor as is found at genera zero and one. It results from independent correlations of the fermionic fields in the vertex operators multiplied by independent correlations of the fermionic fields in the supercurrents. All the mixed correlators of fermionic fields from the vertex operators and the supercurrents vanish after the sum over spin structures. To see this, we fix the phase η_α for the sum over spin structures as follows [16]

$$\eta_1 = -\eta_2 = \eta_3 = -\eta_4 = \eta_5 = -\eta_6 = \eta_7 = \eta_8 = -\eta_9 = \eta_{10} = 1 \quad , \quad (\text{B.10})$$

where by convention the spin structures Q_α [Eq. (B.5)] are given by

$$\begin{aligned}
 Q_1 &= 123/456, \quad Q_2 = 124/356, \quad Q_3 = 125/346, \quad Q_4 = 126/345, \\
 Q_5 &= 134/256, \quad Q_6 = 135/246, \quad Q_7 = 136/245, \\
 Q_8 &= 145/236, \quad Q_9 = 146/235, \quad Q_{10} = 156/234.
 \end{aligned}
 \tag{B.11}$$

Here ijk/lmn is a partition of a_1 into the sets A_i and B_i , and $a_1 = A_1$.

We use now (B.3)-(B.11) to calculate all the possible two-point correlators of the fermionic fields in a product of n vertex operators ($n \leq 4$) and two supercurrents inserted at the points y_a ($a = 1, 2$) [Eqs. (11), (19) and (21)]. The vanishing of the n -point functions for $n \leq 3$ follows from the following set of identities [16]

$$\sum_{\alpha=1}^{10} \eta_{\alpha} Q_{\alpha} \sum_{i=1}^3 (A_i^m - B_i^m) = 0 \quad m = 1, 2 \tag{B.12a}$$

$$\sum_{\alpha=1}^{10} \eta_{\alpha} Q_{\alpha} \left[\frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)} \right] = 0 \tag{B.12b}$$

$$\sum_{\alpha=1}^{10} \eta_{\alpha} Q_{\alpha} \left[\frac{u(z_1)}{u(z_2)u(z_3)} - \frac{u(z_2)u(z_3)}{u(z_1)} \right] = 0 \tag{B.12c}$$

$$\sum_{\alpha=1}^{10} \eta_{\alpha} Q_{\alpha} \left[\frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right] \sum_{i=1}^3 (A_i^m - B_i^m) = 0 \quad m = 1, 2 \tag{B.12d}$$

whereas the vanishing of the mixed correlators between fermionic fields in the vertex operators and those in the supercurrents, which are found in the four-point functions, follows from the identity [16]

$$\sum_{\alpha=1}^{10} \eta_{\alpha} Q_{\alpha} \left[\frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - \frac{u(z_2)u(z_4)}{u(z_1)u(z_3)} \right] = 0. \tag{B.13}$$

To prove these identities we use the fact that for the phases (B.10) the expressions in (B.12) and (B.13) when written in terms of the branch points a_1 are totally antisymmetric and should therefore be proportional to the totally antisymmetric polynomial

$$P(a) = \prod_{i < j} a_{ij} \tag{B.14a}$$

$$a_{ij} = a_i - a_j. \tag{B.14b}$$

This polynomial is of order 15 whereas the expressions in (B.12a), (B.12b) and (B.12d) can be expressed in terms of totally antisymmetric polynomials of order less than 15 and must therefore vanish. In (B.12c) we get an expression of order 15 which is proportional to $\prod_{i=1}^3 A_i B_i = \prod_{i=1}^6 a_i$. This term is independent of the

spin structure and can be factorized out of the sum, leaving a totally antisymmetric polynomial of order 9 which vanishes. In (B.13) we have terms of order up to 18. However, we can either factor out $\prod_{i=1}^3 A_i B_i$ leaving a totally antisymmetric polynomial of less than 15, or the expressions which are found do not have the correct structures to be proportional to $P(a)$.

We thus find that the n -point functions ($n < 3$) vanish, and that in four-point functions fermionic fields in the vertex operators do not correlate with those in the supercurrents. Taking now the independent correlations in the vertex operators and in the supercurrents and summing over the spin structures, we find the two identities [18]

$$\begin{aligned} \sum_{d=1}^{10} \gamma_d Q_d \left[\frac{U(z_1)U(z_2)}{U(z_3)U(z_4)} + \frac{U(z_2)U(z_4)}{U(z_1)U(z_3)} \right] \sum_{i=1}^3 (A_i^m - B_i^m) = \\ = \frac{2P(a)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)}{S(z_1)S(z_2)S(z_3)S(z_4)} \left\{ \sum_{i=1}^6 a_i - \sum_{k=1}^4 z_k \right\} \quad m=1,2 \quad (B.15) \end{aligned}$$

Once again the proof is rather simple. The expressions on the left-hand side of (B.15) are of order 19 or 20 (for $m = 1, 2$) and vanish for $z_1 = z_3$, $z_1 = z_4$, $z_2 = z_3$, or $z_2 = z_4$. They must therefore be proportional to the right-hand side. The proportionality factor is found by a more detailed analysis of the left-hand side. Thus the four-point functions carry at least four powers of momenta resulting from $(\frac{1}{2}k_\tau \psi^\tau \psi^\mu)^4$ [see Eq. (6)]. The tensor t results from the fact that the fermionic correlators are proportional to $g^{\mu\rho}$ [see Eq. (24)] and the expression in (B.15) is the same for all the possible combinations of two-point correlators. It therefore factorizes out, and we obtain the tensor t from all the possible antisymmetrizations of the fermionic fields.

The four-point two-loop amplitudes can now be summarized as

$$A_2^{[4]} = -\frac{1}{16} K(k, \rho) \frac{P(a)}{\prod_{i=1}^4 S(z_i)} \langle \tilde{\mathcal{V}}_S(y_1) \tilde{\mathcal{V}}_S(y_2) \prod_{j=1}^4 V_L(\bar{z}_j) e^{ik_j \cdot x} \rangle, \quad (B.16)$$

where the $V_L(\bar{z}_j)$ are the left-moving contributions to the vertex operators [see Eqs. (12) and (19)], and $K(k, \rho)$ is the kinematical factor

$$K(k, \rho) = t^{\mu_1 \tau_1 \mu_2 \tau_2 \mu_3 \tau_3 \mu_4 \tau_4} \prod_{a=1}^4 k_{\tau_a} \rho_{\mu_a}^a(k_a). \quad (B.17)$$

Its explicit expression in terms of the Mandelstam variables is [12]

$$\begin{aligned}
 K(k, p) = & -\frac{1}{4} \left[st p^1 p^3 p^2 p^4 + su p^2 p^3 p^1 p^4 + tu p^1 p^2 p^3 p^4 \right] \\
 & + \frac{1}{2} s \left[p^1 k_3 p^4 k_2 p^2 p^3 + p^1 k_4 p^3 k_2 p^2 p^4 + p^2 k_3 p^4 k_1 p^1 p^2 + p^2 k_4 p^3 k_1 p^1 p^4 \right] \\
 & + \frac{1}{2} t \left[p^3 k_1 p^4 k_2 p^2 p^1 + p^2 k_1 p^4 k_3 p^3 p^1 + p^3 k_4 p^1 k_2 p^2 p^4 \right. \\
 & \quad \left. + p^2 k_4 p^1 k_3 p^3 p^4 \right] \\
 & + \frac{1}{2} u \left[p^1 k_2 p^4 k_3 p^3 p^2 + p^3 k_2 p^4 k_1 p^1 p^2 + p^1 k_4 p^2 k_3 p^1 p^4 + p^3 k_4 p^2 k_1 p^1 p^4 \right]
 \end{aligned} \tag{B.18}$$

where

$$s = -2k_1 \cdot k_2 \quad ; \quad t = -2k_1 \cdot k_4 \quad ; \quad u = -2k_1 \cdot k_3 . \tag{B.19}$$

In (B.16) we use the notation that in $\tilde{J}_s(y_a)$ the fermionic contributions both in $J_m(y_a)$ and in $J_{gh}(y_a)$ were taken care of in the sum over spin structures. Thus

$$\tilde{J}_s(y_1) \tilde{J}_s(y_2) = \frac{1}{4} \partial x_\mu \partial x^\mu \prod_{i=1}^3 |\langle \eta_i | b \rangle|^2 + \tilde{I}_{gh} \prod_{i=1}^3 \langle \overline{\eta_i} | b \rangle , \tag{B.20}$$

where [see (A.23)]

$$\tilde{I}_{gh} = \tilde{F}_1 + \frac{1}{4} \tilde{F}_2 \left(\sum_{i=1}^6 a_i - \sum_{k=1}^4 \varepsilon_k \right) \tag{B.21}$$

and

$$\tilde{F}_1 = \frac{1}{2} \partial_2 R(y_1, y_2) - R(y_1, y_2) \sum_{i=1}^6 a_i \tag{B.22a}$$

$$\tilde{F}_2 = -R(y_1, y_2) . \tag{B.22b}$$

To get (B.22) we use (A.16), (A.17) and (A.22), where the points $y_a = \infty \pm$ are the zeroes of $\omega_1(z)$, and the Abelian differential $\omega(z)$ is taken to be $\omega_2(z)$ in (A.16) and $\omega_1(z)$ in (A.17).

APPENDIX C

We use now the results of the last two appendices to write the complete expressions for the four-point two-loop amplitudes for the emission of massless particles in the heterotic string. Partial results on these amplitudes, namely the leading contribution for the emission of four gauge bosons when $k \rightarrow 0$, can be found in Ref. [18] expressed in terms of the hyperelliptic representation of the genus-two surface. We will present here the amplitudes in terms of Jacobi θ -functions.

To get the amplitudes we need to insert the left-moving contributions of the vertex operators (11) and (19) in (B.16) and calculate the correlators. We first note that

$$\prod_{i=1}^4 S^{-1}(z_i) = \prod_{i=1}^4 \omega_i(z_i) \quad (C.1)$$

as is seen from (B.2a). Moreover, by taking the product over α of $\theta_\alpha^4(0, \tau)$ in (B.5) we find that

$$\prod_{\alpha=1}^{10} \theta_\alpha^4(0, \tau) = (\det k)^{20} p(a)^4 \quad (C.2)$$

The modular-invariant measure for genus-two can also be written in terms of the θ -functions [25]

$$\prod_{i=1}^3 d^2 m_i = \prod_{i \leq j=1}^2 \frac{d^2 \tau_{ij}}{\prod_{\alpha=1}^{10} \bar{\theta}_\alpha(0, \tau)^2 (\det \text{Im} \tau)^5} \quad (C.3)$$

We insert now (C.1)-(C.3) in (B.16) and integrate over τ_{ij} and z_k (the positions of the vertex operators). We therefore find for the four-point two-loop amplitudes

$$A_2^{[4]} = -\frac{1}{16} k(k, p) \int \frac{\prod_{i \leq j=1}^2 d^2 \tau_{ij}}{(\det \text{Im} \tau)^5 \prod_{\alpha=1}^{10} \bar{\theta}_\alpha(0, \tau)^2} \prod_{k=1}^4 d^2 z_k \omega_i(z_k) \\ \times \frac{\prod_{\alpha=1}^{10} \theta_\alpha(0, \tau)}{(\det k)^5} \langle \tilde{J}_s(y_1) \tilde{J}_s(y_2) \prod_{m=1}^4 V_L(\bar{z}_m) e^{ik_m x} \rangle. \quad (C.4)$$

We specialize now to the different amplitudes.

(i) Four-gauge-boson amplitude

The left-moving contribution to the vertex operators of the gauge bosons can be expressed in terms of the fermionic or bosonic variables. We use here the fermionic representation and for simplicity we present the result for $SO(32)$ only. We may therefore write

$$V_L^g(\bar{z}) = \sqrt{2\alpha'} \frac{g}{2} \lambda(\bar{z}) T^A \lambda(\bar{z}) \quad , \quad (C.5)$$

where $\lambda^I(\bar{z})$ ($I = 1, \dots, 32$) transform under the fundamental representation of $SO(32)$, and the T^A represent the generators of the algebra. Since the $\lambda^I(\bar{z})$ in (C.5) do not correlate with the space-time bosonic co-ordinates $X_\mu(z)$ ($\mu = 0, \dots, 9$), we will have in (C.4) a product of two correlators

$$\begin{aligned} \langle \prod_{m=1}^4 V_L^g(\bar{z}_m) \rangle &= A \text{Tr}(T^{A_1} T^{A_2} T^{A_3} T^{A_4}) + \text{permutations} \\ &+ B \text{Tr}(T^{A_1} T^{A_2}) \text{Tr}(T^{A_3} T^{A_4}) + \text{permutations}, \quad (C.6) \end{aligned}$$

where

$$A = \frac{1}{2} (\alpha' g^2)^2 \sum_{\alpha=1}^{10} \bar{\theta}_\alpha^{12} \left[\frac{\bar{\theta}_\alpha(\vec{u}_{12}) \bar{\theta}_\alpha(\vec{u}_{34})}{\bar{E}(z_1, z_2) \bar{E}(z_3, z_4)} \right]^2 \quad (C.7a)$$

$$B = -\frac{1}{4} (\alpha' g^2)^2 \sum_{\alpha=1}^{10} \bar{\theta}_\alpha^{12} \frac{\bar{\theta}_\alpha(\vec{u}_{12}) \bar{\theta}_\alpha(\vec{u}_{23}) \bar{\theta}_\alpha(\vec{u}_{34}) \bar{\theta}_\alpha(\vec{u}_{41})}{\bar{E}(z_1, z_2) \bar{E}(z_2, z_3) \bar{E}(z_3, z_4) \bar{E}(z_4, z_1)} \quad (C.7b)$$

To get (C.7), we use (24) and the weight $\frac{1}{2} \bar{\theta}_\alpha^{16} \equiv \frac{1}{2} \bar{\theta}_\alpha^{16}(0, \tau)$ for spin structure α , which is the partition function of $SO(32)$. The second correlator which we have in (C.4) is the one involving the space-time bosonic fields

$$\langle \prod_{a=1}^2 \tilde{J}_s(y_a) \prod_{i=1}^4 e^{ik_i \cdot x} \rangle = \frac{1}{4} \langle \partial X_\mu \partial X^\mu \prod_{i=1}^4 e^{ik_i \cdot x} \rangle + \tilde{I}_{gh} \langle \prod_{i=1}^4 e^{ik_i \cdot x} \rangle, \quad (C.8)$$

where the first correlator in (C.8) can be written as in (20), with $A_\mu^{(2)}$, $R_{12}^{(2)}$ and $G^{(2)}(\vec{u}_{ab})$ replacing $\bar{A}_{12}^{(1)}$, $\bar{R}_{12}^{(1)}$ and $G_{ab}^{(1)}$ [see (28)-(30)]. The second correlator in (C.8) can be written as

$$\langle \prod_{a=1}^4 e^{ik_a \cdot x} \rangle = \prod_{a < b} e^{\alpha' k_a \cdot k_b} G^{(2)}(\vec{u}_{ab}) \quad , \quad (C.9)$$

and $\tilde{I}_{gh}$ is given in (B.21). To get everything in terms of Abelian differentials and Jacobi θ -functions we use (B.2) to write

$$\omega_1''(y_b) = 2\omega_2'(y_b) = (-1)^b \sum_{i=1}^6 a_i \quad (b=1,2) \quad (C.10a)$$

$$\omega_2(z) = z\omega_1(z) \quad (C.10b)$$

(ii) Two-gauge-boson/two-graviton amplitude

Here two of the left-moving vertex operators in (C.4) are those of the gauge bosons (C.5) and two are those of the gravitons (11). For the two gauge bosons (particles 3,4) we have the correlator

$$\left\langle \prod_{i=3}^4 V_L^g(\bar{z}_i) \right\rangle = A_1 \text{Tr}(\tau^{A_3} \tau^{A_4}) \quad , \quad (\text{C.11})$$

where

$$A_1 = \frac{\alpha' g^2}{2} \sum_{\alpha=1}^{10} \bar{\theta}_\alpha^{12} \left(\frac{\bar{\theta}_\alpha(\vec{u}_{34})}{\bar{E}(\bar{z}_3, \bar{z}_4)} \right)^2 \quad . \quad (\text{C.12})$$

For the gravitons (particles 1,2) we have [see (11)]

$$V_L^G(\bar{z}) = 2\alpha' g \bar{\partial} X_\mu \quad (\text{C.13})$$

and the matter part of $\tilde{J}_S(y_a)$ (B.20) correlates with V_L^G

$$\begin{aligned} \left\langle \prod_{a=1}^2 \tilde{J}_S(y_a) V_L^G(\bar{z}_a) \prod_{i=1}^4 e^{ik_i \cdot x} \right\rangle &= 2\alpha' g \left\langle \prod_{a=1}^2 \tilde{J}_m(y_a) \bar{\partial} X_{\lambda_a}(z_a) \prod_{i=1}^4 e^{ik_i \cdot x} \right\rangle \\ &+ 2\alpha' g \tilde{I}_{gh} \left\langle \prod_{a=1}^2 \bar{\partial} X_{\lambda_a}(z_a) \prod_{i=1}^4 e^{ik_i \cdot x} \right\rangle \quad . \quad (\text{C.14}) \end{aligned}$$

The first correlator in (C.14) can be read from (32) (for $\rho_a = \rho_b = \mu$) and the second can be written as in (20) with $\bar{A}_{\lambda_a}^{(2)}$, $\bar{R}_{ab}^{(2)}$ and $\bar{G}^{(2)}(\vec{u}_{ab})$ replacing $\bar{A}_{\lambda_a}^{(1)}$, $\bar{R}_{ab}^{(1)}$ and $G^{(1)}(z_{ab})$ respectively. We note that the indices λ_a ($a = 1,2$) in (C.14) would be contracted with the gravitons' polarization tensors appearing in $K(k,\rho)$.

(iii) Four-graviton amplitude

Here we need to correlate $V_L^G(\bar{z}_a)$ ($a = 1, \dots, 4$) with the matter part of $\tilde{J}_S(y_a)$ (B.20) and the exponential factor. We therefore find

$$\begin{aligned} \left\langle \prod_{a=1}^2 \tilde{J}_S(y_a) \prod_{b=1}^2 V_L^G(\bar{z}_b) e^{ik_b \cdot x} \right\rangle &= \\ &= (2\alpha' g)^4 \left[\frac{1}{4} \left\langle \partial X_\mu(y_1) \partial X^\mu(y_2) \prod_{a=1}^4 \bar{\partial} X_{\lambda_a}(z_a) e^{ik_a \cdot x} \right\rangle \right. \\ &\quad \left. + \tilde{I}_{gh} \left\langle \prod_{a=1}^4 \bar{\partial} X_{\lambda_a}(z_a) e^{ik_a \cdot x} \right\rangle \right] \quad . \quad (\text{C.15}) \end{aligned}$$

The first correlator of (C.15) can be read from (35) and (36) (for $\rho_a = \rho_b = \mu$),

and the second is as in (12) with the two-loop correlators $\bar{A}_{\lambda_a}^{(2)}$, $\bar{R}_{ab}^{(2)}$ and $G^{(2)}(\vec{u}_{ab})$ replacing those of one-loop [$\bar{A}_{\lambda_a}^{(1)}$, $\bar{R}_{ab}^{(1)}$ and $G^{(1)}(z_{ab})$]. The indices λ_a ($a = 1, \dots, 4$) in (C.15) would correlate with the gravitons' polarization tensors appearing in $K(k, \rho)$.

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