OPERATOR FORMALISM FOR CHERN-SIMONS THEORIES

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ABSTRACT

The operator formalism for Chern-Simons theories with gauge group G and parameter k ($G = U(1), SU(2)$) on an arbitrary oriented compact three-dimensional manifold is constructed. The states of the Hilbert space are obtained in a wave-functional representation which corresponds to a generalized form of the exponential of a Wess-Zumino-Witten action. It is shown explicitly that the states of a basis of the Hilbert space are in one-to-one correspondence with the characters of a Wess-Zumino-Witten model with gauge group G at level k and have their same properties under modular transformations. In addition, it is also shown that the Wilson line operators with gauge field in some distinguished representations act as creation operators in the Hilbert space and verify the fusion algebra of the corresponding conformal field theory.

1. *Introduction.* Recently, Witten has shown the connection between three-dimensional topological field theories whose actions consist of a Chern-Simons term and conformal field theories in two dimensions [1]. This connection was made for simply connected compact Lie groups and is based on the fact that the physical Hilbert space obtained upon quantization of Chern-Simons theory on a compact surface can be interpreted as the space of conformal blocks in two dimensions. This relation has been made somehow more explicit recently in [2] where it has been extended to any compact Lie group. A very explicit realization of this connection has been presented in [3] for the case of an Abelian gauge group. In that work the construction of the Hilbert space is carried out and it is shown to correspond to the characters of a two-dimensional $c = 1$ rational conformal field theory. All these analysis, as well as the ones recently presented in [4], are based on the canonical quantization of Chern-Simons theories on a "space-time splitting" of the form $\Sigma \times R$ where Σ is a Riemann surface with or without boundary. In this letter we show the connection between Chern-Simons theories and two-dimensional conformal field theories by means of the construction of an operator formalism on arbitrary oriented three-dimensional surfaces without boundary. This construction is similar in spirit to the one in string theory [5]. First, the states of the Hilbert space are defined as wave functionals via a functional integration. Second, by making use of the symmetry of the theory these states are determined and a representation on such a Hilbert space is obtained for the observables of the theory (Wilson lines). The resulting formulation provides a direct connection with conformal field theory. For the case in which no Wilson line is cut (the one treated in this letter) a basis of the Hilbert space of states can be identified with the set of characters of the conformal field theory, *i.e.*, their elements are in equal number and share the same properties under modular transformations. The observables, which are in one-to-one correspondence with the elements of a basis of the Hilbert space behave like the Verlinde operators [6] of the corresponding conformal field theory. In this letter we will give a brief report of our results. An extended version containing the details of our work will be presented elsewhere [7].

2. *Operator formalism: Abelian case.* Let us consider an oriented compact three-dimensional surface without boundary M and an $U(1)$ bundle E (which may well be trivial) endowed with a connection A_μ (which can be viewed as a one-form). The Feynman path integral of the corresponding Chern-Simons theory is defined as

$$Z(M)_k = \int [DA_\mu] e^{ikS(A_\mu)}, \quad (1)$$

with

$$S(A_\mu) = \frac{1}{2\pi} \int_M A \wedge dA, \quad (2)$$

where $[DA_\mu]$ represents Feynman's path integral over gauge orbits and k is an arbitrary integer. The reason why we must integrate over gauge orbits, *i.e.*, under all equivalent classes of connections modulo gauge transformations, is that when k is integer the argument of the functional integral in (1) is invariant under arbitrary gauge transformations $A_\mu \rightarrow A_\mu + g^{-1} \partial_\mu g$ where g is an arbitrary continuous map $g : M \rightarrow U(1)$ [8]. This gauge invariance plays a fundamental role in our construction.

All three-dimensional manifolds of the type considered here admit a Heegaard splitting [9]. This means that we may cut M along a Riemann surface Σ of genus g in such a way that $M = M_1 \cup M_2$ being M_1 and M_2 homeomorphic to a solid ball with g handles. The joining of M_1 and M_2 to build M is carried out by identifying their surfaces ∂M_1 and ∂M_2 via an homeomorphism. Of course, a given M admits many different splittings. To fix ideas let us consider for example $g = 1$. Let M_1 and M_2 be two identical solid tori with modular parameter τ . If they are joined by identifying the points of their surfaces the manifold $S^2 \times S^1$ is constructed. However, if before the identification a modular transformation which transforms $\tau \rightarrow -1/\tau$ is made, the resulting manifold is S^3 . The manifold S^3 admits also a $g = 0$ Heegaard splitting just cutting it along S^2 .

The operator formalism which we are about to construct will consist of states defined as functional integrals over configurations of gauge fields on either M_1 or M_2 after cutting M via a Heegaard splitting. To define those states we must decide which are the arguments of the corresponding wave functionals. These arguments must correspond to field configurations on ∂M_1 or ∂M_2 . The partition function $Z(M)_k$ must be evaluated by a suitable inner product of these states. The most natural way of making this choice is by invoking the holomorphic representation of the canonical quantization of the theory. This has been recently discussed in [3]. We define:

$$\Psi(A_{\bar{z}}) = \int [DA_\mu]_{M_1} \exp \left(ikS(A_\mu) - \frac{k}{\pi} \int_{\partial M_1} d^2 \sigma A_z A_{\bar{z}} \right), \quad (3)$$

where $[DA_\mu]_{M_1}$ represents the Feynman's path integral measure over gauge orbits such that $A_{\bar{z}}$ is fixed at ∂M_1 , and, at the surface, $A_z = \frac{1}{2}(A_1 - iA_2)$, $A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$, A_0 being in the direction perpendicular to ∂M_1 . In (3) σ^i represent real local coordinates on the Riemann surface ∂M_1 . In cutting M along the surface ∂M_1 and choosing

local coordinates σ^i we must select a metric in ∂M_1 . Since the theory is topological, one would expect that the analysis of its Hilbert space is independent of this choice. However, this is not entirely true because of the conformal anomaly. Let us discuss this point in some detail. According to the general analysis in [1] the state defined by (3) corresponds to a section in a vector bundle on moduli space. This vector bundle has a canonical connection which is only projectively flat because of the conformal anomaly. Different choices of the scale of the metric in (3) correspond to different projective factors. In the definition (3) we have added to the Chern-Simons action a surface term which is conformal invariant and so we do not need to make a choice of scale at this point. However, the dependence on the scale is implicit in a proper definition of the functional integral and it will appear naturally in the formalism.

Similarly, we define the state corresponding to M_2 as:

$$\Phi(A_z) = \int [DA_\mu]_{M_2} \exp \left(ikS(A_\mu) + \frac{k}{\pi} \int_{\partial M_2} d^2\sigma A_z \dot{A}_{\bar{z}} \right), \quad (4)$$

so we may write $Z(M)_k$ as

$$Z(M)_k = \int [DA_z DA_{\bar{z}}] \exp \left(\frac{2k}{\pi} \int_{\Sigma} d^2\sigma A_z A_{\bar{z}} \right) \Phi(A_z) \Psi(A_{\bar{z}}), \quad (5)$$

where $\Sigma = \partial M_1$. Let us compare this expression with the one resulting from a canonical quantization in an axial-time gauge in the holomorphic representation [3]. On one hand the exponential factors are the same. On the other hand, in the situation where a Heegaard splitting is such that ∂M_1 and ∂M_2 are identified without previously making any homeomorphism, $\Phi(A_z) = \overline{\Psi(A_{\bar{z}})}$, as follows from (3) and (4). We have defined (3) and (4) based on our intuition from the canonical quantization of the theory. Alternatively, we could think that (3) and (4) are defined in that way because then they are rather simple to compute, and then from (5) we could read the quantization relations of the theory. For a linear theory, such as the Abelian case we are treating in this section, there is no advantage in taking one point of view or the other. However, for the non-Abelian case non-linearities play a fundamental role and the second point of view is much more fruitful. As we will discuss in the next section a suitable definition of the states allows us to determine the Hilbert space and this in turn permits to read the “full” commutation relations from an expression similar to (5).

Our next task is to determine $\Psi(A_{\bar{z}})$ in (3) exploiting the symmetry present in the theory. It is rather straightforward to verify that the functional integral in (3) has its

extremal at field configurations such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$ on M_1 , and that under a gauge transformation $A_\mu \rightarrow A_\mu + g^{-1} \partial_\mu g$, where g is a continuous mapping $g : M_1 \rightarrow U(1)$, it transforms as

$$\Psi(A_{\bar{z}}) \rightarrow e^{-2k(\gamma(g) + \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma A_z \partial_z g g^{-1})} \Psi(A_{\bar{z}}), \quad (6)$$

where

$$\gamma(g) = \frac{1}{2\pi} \int_{\partial M_1} d^2 \sigma g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g. \quad (7)$$

The first consequence of (6) is that the state defined by (3) satisfies the Gauss law $F_{z\bar{z}} \Psi(A_{\bar{z}}) = 0$. This is easily proved by considering the infinitesimal form of (6) and the commutation relations stemming from the holomorphic representation (5).

The mappings g on M_1 are classified according to the number of times that they wrap around non-contractible loops in M_1 . Let us label a canonical set of closed cycles on ∂M_1 which generate its first homology by $\alpha_i, \beta_j, i, j = 1, \dots, g$, and let us assume that our solid ball with g handles M_1 is such that all the α_i cycles are contractible. Dual to these cycles we have one-forms $a_i, b_j, i, j = 1, \dots, g$, defined on ∂M_1 which allow us to define holomorphic one-forms $\omega_i(z), i = 1, \dots, g$, with the help of the period matrix Ω_{ij} . These holomorphic one-forms or Abelian differentials satisfy $\int_{\alpha_i} \omega_j = \delta_{ij}$ and $\int_{\beta_i} \omega_j = \Omega_{ij}$, as well as $\int_{\partial M_1} d^2 \sigma \omega_i \bar{\omega}_j = \text{Im} \Omega_{ij}$.

Considering the analysis in [10] we choose the following parametrization of the gauge fields: $A_{\bar{z}} = (u_a u)^{-1} \partial_{\bar{z}} (u_a u)$ where u is a single-valued and connected to the identity map $u : \partial M_1 \rightarrow U(1)^c$ ($U(1)^c$ is the complexification of $U(1)$), and $u_a = \exp(\pi \int^z \bar{\omega}(z) (\text{Im} \Omega)^{-1} a - \pi \bar{a} (\text{Im} \Omega)^{-1} \int^z \omega(z))$. Notice that u_a is not single-valued on ∂M_1 and that our parametrization is such that u_a has well-defined holonomy phases around α_i and β_i cycles under shifts $a \rightarrow a + n + \Omega m, n, m \in \mathbb{Z}$. In this parametrization gauge transformations take the form $u \rightarrow ug$. A solution to (6) has the form

$$\Psi(A_{\bar{z}}) = e^{-\gamma_{2k}(u_a u)}, \quad (8)$$

where $e^{-\gamma_{2k}(u_a)}$ is defined as a function such that for any single-valued map g satisfies

$$e^{-\gamma_{2k}(u_a g)} = e^{-\gamma_{2k}(u_a)} e^{-2k(\gamma(g) + \langle u_a, g \rangle)}, \quad (9)$$

being $\langle u_a, g \rangle = \frac{1}{\pi} \int d^2 \sigma u_a^{-1} \partial_{\bar{z}} u_a \partial_z g g^{-1}$ and $\gamma(g)$ as given in (7). If g is a map connected to the identity $\langle u_a, g \rangle$ vanishes. This means that our solution (8) factorizes

in u_a and u . In this case (9) does not give any information about $e^{-\gamma_{2k}(u_a)}$. However, if g is a map which winds n_i times around β_i and m_j around α_j , i.e., $g = \exp(-\pi(n + m\bar{\Omega})(\text{Im}\Omega)^{-1} \int^z \omega(z) + \pi \int^{\bar{z}} \overline{\omega(z)}(\text{Im}\Omega)^{-1}(n + \Omega m))$ we obtain the condition:

$$e^{-\gamma_{2k}(u_{a+n+\Omega m})} = e^{-\gamma_{2k}(u_a)} e^{\pi k(2(n+m\bar{\Omega})(\text{Im}\Omega)^{-1}a + (n+m\bar{\Omega})(\text{Im}\Omega)^{-1}(n+\Omega m))}, \quad n, m \in \mathbb{Z}^g. \quad (10)$$

The most general entire function of a which verifies (10) is a linear combination of

$$\psi_p(a) = \xi e^{\pi k a (\text{Im}\Omega)^{-1} a} \vartheta \left[\begin{smallmatrix} p \\ 0 \end{smallmatrix} \right] (2ka | 2k\Omega), \quad p \in (\mathbb{Z}_{2k})^g, \quad (11)$$

where ϑ is the Jacobi theta function with characteristics [11], and ξ is a constant (independent of a).

Using symmetry arguments we have found $(2k)^g$ functions in our search for the form of the functional integral (3). This non-uniqueness is expected. The non-trivial gauge invariant operators of the theory are Wilson lines around non-contractible cycles. If we have had a Wilson line inserted in the solid ball with g handles in the definition (3) all our arguments based on the symmetries of the theory would be still valid. Therefore, one does not expect to find a unique state but the full set of states of the Hilbert space. To find a basis we must study the orthogonality relations of the states we have obtained. Let us define

$$\Psi_p(A_{\bar{z}}) = e^{-2k\gamma(u)} \psi_p(a), \quad (12)$$

and let us consider the case in which $M = S^2 \times S^1$ and M_1 is the solid torus. From (5) we have the inner product

$$(\Psi_q, \Psi_p) = \int [DA_z DA_{\bar{z}}] e^{\frac{2k}{\pi} \int d^2\sigma A_z A_{\bar{z}}} \overline{\Psi_q(A_{\bar{z}})} \Psi_p(A_{\bar{z}}). \quad (13)$$

The measure in (13) is defined by $\|\delta A\| = \int d^2\sigma \delta A_z \delta A_{\bar{z}}$. In computing this inner product one finds determinants of operators which must be regulated. Here is where the conformal anomaly enters in our formulation. Different choices of the scale of the metric lead to different projective factors. The choice which seems to lead to the simplest formulation consists of a conformally flat metric g with $\int_{\partial M_1} d^2\sigma \sqrt{g} = 2\text{Im}\tau$, where τ denotes the period matrix Ω in the case of the torus. One finds from (13), (11) and (12),

$$(\Psi_q, \Psi_p) = \delta_{p,q} |\xi|^2 |\eta(\tau)|^2, \quad (14)$$

where $\eta(\tau)$ is the Dedekind η function. Therefore, all the states defined in (12) are orthogonal and so they constitute a basis. Also notice that all these states can be made

orthonormal by defining $\xi = \eta(\tau)^{-1}$. Actually, there are general arguments [1] to show that it is in $S^2 \times S^1$ where the states are normalized to 1.

From (11), (12) and (14) we have obtained that an orthonormal basis of the Hilbert space of the theory consists of

$$\Psi_p(A_{\bar{z}}) = e^{-2k\gamma(u)} e^{\pi k a (\text{Im}\tau)^{-1} a} \frac{1}{\eta(\tau)} \vartheta \left[\begin{matrix} \frac{p}{2k} \\ 0 \end{matrix} \right] (2ka | 2k\tau), \quad p \in Z_{2k}. \quad (15)$$

These functionals transform as the states of a unitary representation of the modular group of the torus. Using standard properties of the theta functions and the Dedekind η function one finds that under the generating transformations of the modular group $S : a \rightarrow a/\tau, \tau \rightarrow -1/\tau$, and $T : a \rightarrow a, \tau \rightarrow \tau + 1$,

$$\begin{aligned} \Psi_p|_S &= \frac{1}{\sqrt{2k}} \sum_{q=0}^{2k-1} e^{-i\pi \frac{pq}{k}} \Psi_q, \\ \Psi_p|_T &= e^{2\pi i (\frac{p^2}{4k} - \frac{1}{24})} \Psi_p. \end{aligned} \quad (16)$$

From the form of the T transformation one can read the conformal dimensions $h_p = \frac{p^2}{4k}$ and the central charge $c = 1$ of the corresponding conformal field theory.

As we mentioned above, the states (15) or a linear combination of them correspond to Feynman's path integral of the type (3) where some Wilson lines have been inserted. Now we will compute the effect of introducing a new unknotted and unlinked Wilson line of charge n , $\phi_n = \exp(-n \int_{\beta} A)$. Since it is unknotted and unlinked we may translate it to the surface of the solid torus by a continuous deformation. As the system has zero Hamiltonian, this operation leaves invariant the value of the path integral. Once on the surface, using the holomorphic representation dictated by (5), *i.e.*, $\bar{a} = \frac{\text{Im}\tau}{2k\pi} \frac{\partial}{\partial a}$, the Wilson line can be written in operator form and we have

$$\phi_n \Psi_p = \exp\left(-n \int_{\beta} A\right) \Psi_p = e^{-\frac{\pi n^2}{4k} \bar{\tau} (\text{Im}\tau)^{-1} \tau - \pi n \bar{\tau} (\text{Im}\tau)^{-1} a} e^{\frac{n\tau}{2k} \frac{\partial}{\partial a}} \Psi_p = \Psi_{p+n}. \quad (17)$$

In view of this result the most natural interpretation is to associate the state Ψ_0 to the case in which there are no Wilson lines inside the solid torus and the state Ψ_p to the case in which there is an unknotted Wilson line of charge p . We observe that there are only k distinguished charges. The states (15) are in one-to-one correspondence with the characters of a rational Gaussian model [12] and share their same properties under modular transformations. The operators ϕ_p may be identified with the primary fields

of those models. Certainly, if one considers two unknotted unlinked Wilson lines with charges p and q , the effect of acting on the states (15) is commutative and one has the same effect as if one considers only a Wilson line with charge $p + q$. In other words, they satisfy the fusion rule $\phi_p \times \phi_q = \phi_{p+q}$.

Once the states (15) are constructed and the unitary representation of the modular group (16) is worked out it is rather simple to compute the partition function (5). As an example let us calculate $Z(S^3)_k$. Since S^3 has a Heegaard splitting which involves a homeomorphism corresponding to an S modular transformation, from (14), (15) and (16) one finds $Z(S^3)_k = 1/\sqrt{2k}$.

3. Operator formalism: non-Abelian case. We will consider a non-Abelian Chern-Simons theory for the concrete case of the group $SU(2)$. However, all our arguments can be generalized to an arbitrary Lie group. We will be dealing with a Feynman path integral like the one in (1), but now

$$S(A_\mu) = \frac{1}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (18)$$

where ‘Tr’ denotes the trace in the fundamental representation. Similarly to the Abelian case $\exp(ikS)$ is invariant under gauge transformations $A_\mu \rightarrow g^{-1} A_\mu g + g^{-1} \partial_\mu g$ where g is an arbitrary continuous map $g : M \rightarrow SU(2)$.

Let us consider first the case in which $M = S^3$ and let us perform a Heegaard splitting such that M_1 and M_2 are two solid balls with a two-sphere as a common boundary. Similarly to the Abelian case we define the state associated to M_1 by

$$\Psi(A_{\bar{z}}) = \int [DA_\mu]_{M_1} \exp(ikS(A_\mu) - \frac{k}{2\pi} \int_{\partial M_1} d^2\sigma \text{Tr}(A_z A_{\bar{z}})), \quad (19)$$

and accordingly (see (4)) for the state $\Phi(A_z)$ associated to M_2 . For the splitting considered, $\Phi(A_z) = \overline{\Psi(A_{\bar{z}})}$. In the Abelian case the measure $[DA_\mu]$ had only one subtlety related to the conformal anomaly that we succinctly carried it along until the computation of the norm of the states. In the non-Abelian case there is an additional subtlety which is fundamental in our discussion. In computing the path integral (19) we must perform a gauge fixing. Assume that we take a gauge in which the radial component of A_μ in the solid ball vanishes. The integration then reduces to A_μ configurations on a continuous set of spherical layers. There is still a residual gauge invariance in the components of A_μ tangent to each layer. Let us parametrize these components like the

ones on the surface of the sphere at the boundary. We choose a measure which is gauge invariant [13] on each layer and by continuity we must have the same on the surface:

$$[DA_z DA_{\bar{z}}] = e^{c_v \Gamma(u \bar{u}^{-1})} |\det \partial_z \partial_{\bar{z}}| du d\bar{u}, \quad (20)$$

where $A_{\bar{z}} = u^{-1} \partial_{\bar{z}} u$ and $A_z = \bar{u}^{-1} \partial_z \bar{u}$ with u valued in $SU(2)^c$ ($A_{\bar{z}} = -A_z^\dagger$ and $u^{-1} = \bar{u}^\dagger$), c_v is the quadratic Casimir operator in the adjoint representation, $\Gamma(w)$ is the Wess-Zumino-Witten (WZW) action [14]:

$$\Gamma(w) = \frac{1}{2\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(w^{-1} \partial_z w w^{-1} \partial_{\bar{z}} w) + \frac{ik}{12\pi} \int_{M_1} \epsilon^{\mu\nu\rho} \text{Tr}(w^{-1} \partial_\mu w w^{-1} \partial_\nu w w^{-1} \partial_\rho w), \quad (21)$$

and $du d\bar{u}$ is an infinite product of de Haar measures of $SU(2)^c$. The exponential factor in (20) is the Jacobian that appears in the change of variables from $A_z, A_{\bar{z}}$ to $u, \bar{u}$, which can be written as a fermionic determinant with the fermions in the adjoint representation [13]. The WZW action satisfies the Polyakov-Wiegmann (PW) condition:

$$\Gamma(vw) = \Gamma(v) + \Gamma(w) + \langle v, w \rangle, \quad (22)$$

where (from now on) $\langle v, w \rangle = \frac{1}{\pi} \int d^2 \sigma \text{Tr}(v^{-1} \partial_{\bar{z}} v \partial_z w w^{-1})$.

Since the functional integral in (19) is performed at fixed $A_{\bar{z}}$ at the surface of the solid ball the measure $[DA_z DA_{\bar{z}}]$ has an asymmetry. One should have to factorize the full measure (20) into two parts, one depending on $\bar{u}$ and the other on u . However, this is not possible because of the non-Abelian chiral anomaly [13] which manifests itself in the PW property (22). The best we can do in virtue of (22) is to take measures in $\Psi(A_{\bar{z}})$ and in $\Phi(A_z)$ containing $e^{c_v(\Gamma(\bar{u}^{-1}) + \langle u, \bar{u}^{-1} \rangle)}$ and $e^{c_v(\Gamma(u) + \langle u, \bar{u}^{-1} \rangle)}$ respectively. In doing this an extra factor $e^{c_v \langle u, \bar{u}^{-1} \rangle}$ has been introduced. This implies that a factor $e^{-c_v \langle u, \bar{u}^{-1} \rangle} = e^{\frac{c_v}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(A_z A_{\bar{z}})}$ must be added in the measure of the inner product to keep the full measure the same:

$$Z(S^3)_k = \int |\det \partial_z \partial_{\bar{z}}| du d\bar{u} e^{\frac{1}{\pi}(k+c_v) \int_{\partial M_1} d^2 \sigma \text{Tr}(A_z A_{\bar{z}})} \Phi(A_z) \Psi(A_{\bar{z}}) \quad (23)$$

To determine $\Psi(A_{\bar{z}})$ we will use symmetry arguments as in the Abelian case. In the case at hand we must take into account that the part of the measure $e^{c_v(\Gamma(u) + \langle u, \bar{u}^{-1} \rangle)}$ is not gauge invariant. Under a gauge transformation $A_{\bar{z}} \rightarrow g^{-1} A_{\bar{z}} g + g^{-1} \partial_{\bar{z}} g$ we find,

$$\Psi(A_{\bar{z}}) \rightarrow e^{-(k+c_v)(\Gamma(g) + \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(A_z \partial_z g g^{-1}))} \Psi(A_{\bar{z}}). \quad (24)$$

In addition, by considering the infinitesimal form of (24) one finds that $\Psi(A_{\bar{z}})$ satisfies the Gauss law $F_{z\bar{z}} \Psi(A_{\bar{z}}) = 0$ when the “full” commutation relations dictated by the

exponential in (23), *i.e.*, $[A_{\bar{z}}, A'_z] = \frac{\pi}{k+c_v} \delta^{(2)}(\sigma - \sigma')$, are taken into account. The solution to (24) is simply obtained by making use of the PW property (22):

$$\Psi(A_{\bar{z}}) = \xi e^{-(k+c_v)\Gamma(u)}, \quad (25)$$

where ξ is a constant (independent of u).

Let us consider now the case of a Heegaard splitting such that M_1 and M_2 are two solid tori. If A_z and $A_{\bar{z}}$ are the components of A_μ tangent to ∂M_1 at ∂M_1 we choose the parametrization [10]: $A_{\bar{z}} = (u_a u)^{-1} \partial_{\bar{z}}(u_a u)$ being u a single-valued map $u : \partial M_1 \rightarrow SU(2)^c$, and u_a non-single valued and in the centre of the group, parametrized as $u_a = \exp\left((- \pi \int^{\bar{z}} \bar{\omega}(z) (\text{Im} \tau)^{-1} a + \pi \bar{a} (\text{Im} \tau)^{-1} \int^z \omega(z) \tau_3\right)$, where $\tau_3 = \sigma_3/2$, being σ_3 a Pauli matrix. In the same way $A_z = (\bar{u}_a \bar{u})^{-1} \partial_z(\bar{u}_a \bar{u})$ with $u^{-1} = \bar{u}^\dagger$ and $u_a^{-1} = \bar{u}_a^\dagger$. The measure (20) in this case becomes [15,16]:

$$[DA_z DA_{\bar{z}}] = e^{c_v \Gamma(u \bar{u}^{-1}, C)} |\Pi(\tau, a)|^4 e^{-2c_v \langle u_a, \bar{u}_a^{-1} \rangle} \text{Im} \tau \, du d\bar{u} du_a d\bar{u}_a \quad (26)$$

where $C_{\bar{z}} = u_a^{-1} \partial_{\bar{z}} u_a$ and $C_z = \bar{u}_a^{-1} \partial_z \bar{u}_a$; $\Gamma(u \bar{u}^{-1}, C)$ is the gauged WZW action [17],

$$\begin{aligned} \Gamma(h, C) = \Gamma(h) &- \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(h^{-1} C_{\bar{z}} h C_z) + \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(C_{\bar{z}} \partial_z h h^{-1}) \\ &- \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(h^{-1} \partial_{\bar{z}} h C_z) + \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(C_z C_{\bar{z}}); \end{aligned} \quad (27)$$

and, for $SU(2)$,

$$\Pi(\tau, a) = e^{\frac{\pi}{2\text{Im} \tau} a^2} (\Theta_{1,2}(a, \tau, 0) - \Theta_{-1,2}(a, \tau, 0)), \quad (28)$$

where $\Theta_{i,l}(a, \tau, 0)$ are the Θ functions of level l . This measure is invariant under two types of transformations: a) gauge transformations $u \rightarrow ug$, $\bar{u} \rightarrow \bar{u}g$ and b) transformations which leave $A_{\bar{z}}$ and A_z untouched, $u_a \rightarrow u_a g$, $u \rightarrow g^{-1}u$, $\bar{u}_a \rightarrow \bar{u}_a g$, $\bar{u} \rightarrow g^{-1}\bar{u}$ with g in the centre of the group and single valued. As in the case of genus zero it is not possible to factorize the measure (26) into a part depending on u and another on $\bar{u}$. In fact, using (22) one finds,

$$\begin{aligned} \Gamma(u \bar{u}^{-1}, C) = & \Gamma(u) + \langle u_a, u \rangle + \Gamma(\bar{u}^{-1}) + \langle \bar{u}_a^{-1}, \bar{u}^{-1} \rangle \\ & - \langle u_a, \bar{u}_a^{-1} \rangle - \frac{1}{\pi} \int_{\partial M_1} d^2 \sigma \text{Tr}(A_z A_{\bar{z}}). \end{aligned} \quad (29)$$

Therefore, the measure of $\Psi(A_{\bar{z}})$ contains $e^{c_v(\Gamma(u \bar{u}^{-1}, C) - \Gamma(u) - \langle u_a, u \rangle)}$, while the one of $\Phi(A_z)$ has $e^{c_v(\Gamma(u \bar{u}^{-1}, C) - \Gamma(\bar{u}^{-1}) - \langle \bar{u}_a^{-1}, \bar{u}^{-1} \rangle)}$. This implies that we must add a factor

$e^{c_v(\frac{1}{\pi} \int_{\partial M_1} d^2\sigma \text{Tr}(A_z A_{\bar{z}}) + \langle u_a, \bar{u}_a^{-1} \rangle)}$ in the measure of the inner product so that

$$Z(M)_k = \int |\Pi(\tau, a)|^4 e^{-c_v \langle u_a, \bar{u}_a^{-1} \rangle} \text{Im} \tau \, du d\bar{u} du_a d\bar{u}_a \quad (30)$$

$$e^{\frac{1}{\pi}(k+c_v) \int_{\partial M_1} d^2\sigma \text{Tr}(A_z A_{\bar{z}})} \Phi(A_z) \Psi(A_{\bar{z}}).$$

Again, symmetry arguments will guide us to find the form of $\Psi(A_{\bar{z}})$. Under transformations of type a) one finds, taking into account the variation of the measure inside the path integral defining $\Psi(A_{\bar{z}})$,

$$\Psi(A_{\bar{z}}) \rightarrow e^{-(k+c_v)(\Gamma(g) + \langle u_a u, g \rangle)} \Psi(A_{\bar{z}}). \quad (31)$$

From this relation it follows that $\Psi(A_{\bar{z}})$ must have the form $\Psi(A_{\bar{z}}) = e^{-\Gamma_{k+c_v}(u_a u)} \Lambda(u_a)$ where $e^{-\Gamma_l(u_a)}$ is such that

$$e^{-\Gamma_l(u_a v)} = e^{-\Gamma_l(u_a)} e^{-l(\Gamma(v) + \langle u_a, v \rangle)} \quad (32)$$

for any single-valued map $v : M_1 \rightarrow SU(2)^c$. On the other hand, performing a transformation of type b), one finds

$$\Lambda(u_a g) \rightarrow e^{c_v(\Gamma(g) + \langle u_a, g \rangle)} \Lambda(u_a), \quad (33)$$

so the solution for $\Psi(A_{\bar{z}})$ is of the form

$$\Psi(A_{\bar{z}}) = \xi \frac{e^{-\Gamma_{k+c_v}(u_a u)}}{e^{-\Gamma_{c_v}(u_a)}}, \quad (34)$$

where ξ is a constant (independent of u and u_a). Before determining the form of $e^{-\Gamma_l(w)}$ from its property (32) let us write down the form of the inner product once u is integrated out. Taking (34) and (30), and using [15]:

$$\int du d\bar{u} e^{-(k+c_v)\Gamma(u \bar{u}^{-1}, C)} = (\text{Im} \tau)^{-\frac{1}{2}} e^{c_v \langle u_a, \bar{u}_a^{-1} \rangle} |\Pi(\tau, a)|^{-2}, \quad (35)$$

one obtains for the case in which $M = S^2 \times S^1$,

$$Z(S^2 \times S^1)_k = \int du_a d\bar{u}_a (\text{Im} \tau)^{\frac{1}{2}} |\Pi(\tau, a)|^2 e^{-(k+c_v) \langle u_a, \bar{u}_a^{-1} \rangle} |\xi|^2 \left| \frac{e^{-\Gamma_{k+c_v}(u_a)}}{e^{-\Gamma_{c_v}(u_a)}} \right|^2. \quad (36)$$

To determine $e^{-\Gamma_l(u_a)}$ from (32) let us consider the case in which the map v in (32) winds n times around the β cycle of the torus and m times under the α one, i.e.,

$v = \exp(2\pi\tau_3(\text{Im}\tau)^{-1}(n + m\bar{\tau}) \int^z \omega(z) - 2\pi\tau_3(\text{Im}\tau)^{-1}(n + m\tau) \int^{\bar{z}} \overline{\omega(z)})$. One finds from (32)

$$e^{-\Gamma_l(u_{a+2n+2m\tau})} = e^{-\Gamma_l(u_a)} e^{\pi l(n+m\bar{\tau})(\text{Im}\tau)^{-1}(n+m\tau) - \pi l(n+m\bar{\tau})(\text{Im}\tau)^{-1}a}. \quad (37)$$

The most general entire function satisfying this relation is a linear combination of functions of the form

$$\psi_{p,l}(a) = e^{\frac{\pi l}{4\text{Im}\tau}a^2} \Theta_{p,l}(a, \tau, 0), \quad (38)$$

with $0 \leq p < 2l$. Our wave function (34) contains a ratio of such linear combinations. The inner product appearing in (36) possesses the exponential factor $e^{-(k+c_v)\langle u_a, \bar{u}_a^{-1} \rangle}$ which gives rise to the standard inner product for level $k + c_v$ theta functions. We also have in (36) the factor $|\Pi(\tau, a)|^2$ which has a difference of theta functions of level 2. In the denominator of our wave function (34), theta functions of this same level (for $SU(2)$, $c_v = 2$) also appear. This implies that in order to have the wave functions orthonormal in the case in which $M = S^2 \times S^1$ [1] we must also have such a difference of theta functions in the denominator of (34). Let us define

$$\lambda_{j,l}(a) = \psi_{j+1,l}(a) - \psi_{-j-1,l}(a), \quad (39)$$

with $0 \leq j < l - 1$. The denominator of (34) is just $\lambda_{0,2}(a) = \Pi(a, \tau)$. Invariance under transformations of the Weyl group of $SU(2)$, $a \rightarrow -a$, forces then to have also differences in the numerator of (34). A basis of the states of the theory is, after using (34), (32), (38) and (39) (and taking $c_v = 2$):

$$\Psi_{j,k}(A_{\bar{z}}) = \xi e^{-(k+2)(\Gamma(u) + \langle u_a, u \rangle)} \chi_{j,k}(a), \quad (40)$$

with $0 \leq j \leq k$, where

$$\chi_{j,k}(a) = \frac{\lambda_{j,k+2}(a)}{\lambda_{0,2}(a)}. \quad (41)$$

The inner product (36) turns out to be, after using (34) and (40):

$$(\Psi_{i,k}, \Psi_{j,k}) = \int \frac{dad\bar{a}}{2\text{Im}\tau} (\text{Im}\tau)^{\frac{1}{2}} |\xi|^2 e^{-\frac{(k+2)\pi}{2\text{Im}\tau}a\bar{a}} \overline{\lambda_{i,k+2}(a)} \lambda_{j,k+2}(a). \quad (42)$$

Demanding orthonormality, $(\Psi_{i,k}, \Psi_{j,k}) = \delta_{i,j}$, one obtains $\xi = 1$.

The modular properties of our states are better displayed in terms of the functions $\chi_{j,k}(a)$ which have the same properties as the $SU(2)_k$ Kac-Moody characters [18]. The reason for this is that when expressing (36) or (42) in terms of an integral over them, the measure is modular invariant. Thus, in virtue of the orthonormality the $\chi_{j,k}(a)$ must transform according to a unitary representation of the modular group. Indeed, using standard properties of the theta functions one finds

$$\begin{aligned}\chi_{j,k}|_S &= \left(\frac{2}{k+2}\right)^{\frac{1}{2}} \sum_{m=0}^k \sin \frac{\pi(j+1)(m+1)}{k+2} \chi_{m,k}, \\ \chi_{j,k}|_T &= e^{2\pi i(h_j - \frac{c}{24})} \chi_{j,k},\end{aligned}\tag{43}$$

where $h_j = \frac{j(j+1)}{4(k+2)}$ and $c = \frac{3k}{k+2}$. The quantities c and h_j correspond to the central charge and conformal dimensions of the primary fields of an $SU(2)$ WZW model [19,18].

In our analysis we have found $k+1$ states compatible with the symmetries we have used in their determination. In fact, we should expect to find several states since we could have introduced Wilson lines in the Feynman path integral (19) defining the state. Let us analyze the effect of introducing an unlinked and unknotted Wilson line $\phi_j = \text{Tr}_j \text{P exp}(-\int_\beta A)$, where Tr_j denotes the trace taken in the representation of isospin $j/2$ of $SU(2)$. Since the Wilson line is unlinked and unknotted we can move it to the surface as we did in the Abelian case with no effect on the Feynman path integral and once there write it in operator form. As in the case of genus zero, the infinitesimal form of the gauge transformation (31) implies that our states (40) satisfy the Gauss law $F_{z\bar{z}} \Psi_{m,k} = 0$. Thus, the Wilson line operator can be deformed [20] in such a way that it depends only on a and $\bar{a}$. Once in this form it will appear inserted in integrations like (42), and thus we can obtain the action of ϕ_j on the functions $\lambda_{m,k+2}(a)$. The operator representation of a and $\bar{a}$ can be read from the exponent of the exponential in (42), i.e., $\bar{a} = \frac{2\text{Im}\tau}{(k+2)\pi} \frac{\partial}{\partial a}$. One finds that ϕ_j takes the form

$$\phi_j = \sum_{n=0}^j \exp \left(\frac{(j-2n)\pi\bar{\tau}a}{2\text{Im}\tau} - \frac{(j-2n)\tau}{k+2} \frac{\partial}{\partial a} \right),\tag{44}$$

and therefore, after taking (38) and (39),

$$\phi_j \lambda_{0,k+2}(a) = \lambda_{j,k+2}(a).\tag{45}$$

This result leads to the interpretation that the state $\Psi_{0,k}(A_{\bar{z}})$ must be associated to the case in which there are no Wilson lines inside the solid torus, and that the state $\Phi_{j,k}(A_{\bar{z}})$

is “created” by introducing a Wilson line in the representation of isospin $j/2$. Notice also that there are only k “distinguished” representations of $SU(2)$ in the language of [1]. Using (44), it is simple to prove that the Wilson lines have the same properties as the Verlinde operators. In fact, it follows from (44) that the effect of inserting first an unlinked and unknotted Wilson line in the representation of isospin $i/2$ and then another unlinked and unknotted Wilson line in the representation of isospin $j/2$ can be written as the sum of the action of several Wilson lines, *i.e.*, one has the fusion rule, $\phi_i \times \phi_j = \sum_{n=|i-j|}^{\min(i+j, 2k-i-j)} \phi_n$ with $n - |i - j|$ even.

Using our results for the torus it is simple to compute $Z(S^3)_k$. As we discussed at the end of sect. 2 we must perform an S modular transformation in one of the tori before identifying their surfaces. From (43) one obtains $Z(S^3)_k = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}$ [1]. This type of analysis has been performed for a wide class of Heegaard splittings in [21].

4. Concluding Remarks. In this letter we have given a brief summary of our results leading to an operator formalism for Chern-Simons theories. We have considered only the case in which no Wilson line is cut, obtaining a one-to-one correspondence between non-trivial operators and states of a basis of the Hilbert space. As stated in [1] Wilson lines are identified with Verlinde operators and states with characters. Wilson lines can be also interpreted as creation operators when acting on the state corresponding to the vacuum, *i.e.*, the state without Wilson lines. We have presented only the effect of Wilson lines associated to non-contractible loops. It is simple to verify that for a contractible loop one obtains only a phase when acting on the states of the Hilbert space. Our analysis has been restricted to the case in which the three-manifold is cut via a Heegaard splitting but possibly the result can be extended to arbitrary cuts. In [7] we will present a full account of the results reported here. Certainly, the case in which Wilson lines are cut must be studied to obtain a complete operator formalism. We expect to consider that situation in due course. In addition, the formalism must be extended to arbitrary groups taking into account the effect of all gauge invariant operators, *i.e.*, including the ones associated to graphs discussed in [22].

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