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Topological super Yang-Mills theories

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Abstract

We present a manifestly supersymmetric formulation of $N=1$ and $N=2$ topological super YM theories in four dimensions. In both cases, we construct the complete quantum action together with its invariance algebra. For $N=2$ in harmonic superspace this algebra formally coincides with the one of the bosonic theory in ordinary space-time.

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1 Introduction

Topological Yang-Mills theories [1]-[3] arise from gauge-fixing a topological invariant depending on the connection field. A natural question to ask is whether these theories can be generalized so as to include spinor fields in a supersymmetric way. This problem has already been addressed in reference [4] where the authors restrict themselves to the WZ supergauge and to vanishing auxiliary fields. Although this gauge choice is quite convenient to discuss some basic aspects of the theory (like the instanton condition), it is not appropriate for a complete and manifestly supersymmetric formulation of topological SYM. The development of such a description will be the main goal of the present letter.

Usually, supersymmetric theories are formulated in a Minkowskian set-up. However, the gauge-fixing procedure for topological (super) Yang-Mills theories makes use of instanton configurations and thereby consistency requires a transition from the Minkowskian to the Euclidean regime. In doing so, one encounters the well-known problem that there are no real Majorana spinors in Euclidean 4-space [5]. This problem can be overcome by doubling the number of fermionic degrees of freedom, i.e. considering $N=2$ instead of $N=1$ supersymmetry [5] [6]. However, one can also give a meaning to the superinstanton condition in $N=1$ Euclidean superspace by complexifying the structure group [6]. (This amounts to a doubling of the fermionic and bosonic fields contained in the gauge multiplet.) In both cases, the relevant equations in Minkowski and Euclidean space essentially differ by the interpretation.

In view of this situation, we will proceed as follows. In analogy to the bosonic theory, topological SYM will be realized through a BRS-invariant gauge-fixing of the supersymmetric generalization of $\int \text{Tr}(F \wedge F)$. We first present the approach in the framework of $N=1$ supersymmetry which is best-known and for which the coupling of SYM to supergravity has been worked out in detail [7]. Starting from $N=1$ rigid superspace, we derive the complete quantum Lagrangian together with its symmetry algebra. The latter is then cast into a very compact form by assembling superfields of various degrees into "hyper-superfields". Subsequently, we show that the algebra is not modified by the coupling to supergravity (as long as one doesn't include global zero modes). Finally, the previous results are generalized to $N=2$ harmonic superspace which represents the adequate arena for a geometric and unconstrained formulation of extended SYM theories [8]. Although this generalization involves some ideas of the $N=1$ theory, the resulting superspace BRS algebra has the same form as the one of the $N=0$ theory in ordinary space-time. A fortiori, the known results about the cohomology of the BRS operator hold in our supersymmetric case and are to be commented upon.

Since the Euclidean version of $N=1$ topological SYM relies on a complexified gauge group, our presentation of this theory should primarily be considered as an illustration (in the simplest and best-known framework) of the "true" theory, i.e. topological SYM in $N=2$ harmonic superspace. It should be noted that our procedure and notation closely follow the bosonic theory approach of reference [3].

2 N=1 theory

The starting point is rigid superspace based on $d = 4$ Minkowski space [7] [9]. (Shortly, we will pass over to the Euclidean regime.) We work in the antichiral representation where the gauge field is represented by a real Lie algebra valued superfield V transforming under the antichiral gauge group. The field strengths $W_\alpha \equiv e^V [-\bar{D}^2(e^{-V} D_\alpha e^V)] e^{-V}$ and $\bar{W}_{\dot{\alpha}} \equiv D^2(e^V \bar{D}_{\dot{\alpha}} e^{-V})$ define the super YM action¹

$$S_{YM} = \int d^4x \left\{ \int d^2\theta \operatorname{Tr} W^\alpha W_\alpha + \int d^2\bar{\theta} \operatorname{Tr} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right\} ,$$

as well as the supersymmetric extension of the topological invariant $\int \operatorname{Tr} F \wedge F$:

$$S_{inv} = \frac{1}{2i} \int d^4x \left[\int d^2\theta \operatorname{Tr} W^\alpha W_\alpha - \int d^2\bar{\theta} \operatorname{Tr} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right] . \quad (1)$$

Indeed, in the Abelian case, it is easy to verify that S_{inv} represents a space integral over a total derivative:

$$(S_{inv})_{abelian} = \frac{1}{2i} \int d^4x \int d^4\theta \left\{ D^\alpha [V \bar{D}^2 D_\alpha V] - \bar{D}_{\dot{\alpha}} [V D^2 \bar{D}^{\dot{\alpha}} V] \right\} .$$

Moreover, in the WZ-gauge the functional (1) reduces to

$$S_{inv} \propto \int d^4x \left\{ \operatorname{Tr} F_{ab} \tilde{F}^{ab} + 2i \partial_m \operatorname{Tr} (\lambda \sigma^m \bar{\lambda}) \right\} . \quad (2)$$

Here $\tilde{F}_{ab} \equiv i/2 \epsilon_{abcd} F^{cd}$ denotes the dual of the YM-curvature $F_{ab}(x)$ and $(\lambda_\alpha, \bar{\lambda}^{\dot{\alpha}})$ describes the gaugino. Both terms in (2) are total divergences, but contrary to the first one, the second contribution is a derivative of a quantity which is globally gauge invariant.² Thus, supersymmetry provides a topologically trivial extension of the YM Chern-Simons form (defined in ordinary space-time). The relation with the Chern-Simons 3-form in superspace will be discussed in the appendix.

To find the supersymmetric generalization of the instanton condition $F_{ab}(x) = \tilde{F}_{ab}(x)$, we decompose this tensor into its selfdual and anti-selfdual parts:

$$\begin{aligned} F_{\alpha\dot{\alpha},\beta\dot{\beta}} &\equiv \epsilon_{\dot{\alpha}\dot{\beta}} f_{(\alpha\beta)} - \epsilon_{\alpha\beta} \bar{f}_{(\dot{\alpha}\dot{\beta})} \\ f_{(\alpha\beta)} &= (\sigma^{ab}\epsilon)_{\alpha\beta} F_{ab} = (\sigma^{ab}\epsilon)_{\alpha\beta} \tilde{F}_{ab} \\ \bar{f}_{(\dot{\alpha}\dot{\beta})} &= (\bar{\epsilon}\sigma^{ab})_{\dot{\alpha}\dot{\beta}} F_{ab} = -(\bar{\epsilon}\sigma^{ab})_{\dot{\alpha}\dot{\beta}} \tilde{F}_{ab} . \end{aligned}$$

In the WZ-gauge of SYM we have $\bar{D}_{(\dot{\alpha}} \bar{W}_{\dot{\beta})} = 2i f_{(\dot{\alpha}\dot{\beta})}(x)$ and thereby the instanton condition can be implemented in superspace by the requirement $\bar{W}_{\dot{\alpha}} \equiv 0$. As advocated in the introduction, this identity leads to an empty theory in the Minkowskian set-up, because the representations $(1/2, 0)$ and $(0, 1/2)$ of $SL(2, C)$ are related by complex conjugation: $(f_{\alpha\beta})^\dagger = \bar{f}_{\dot{\alpha}\dot{\beta}}$ and $(\bar{W}_{\dot{\alpha}})^\dagger = e^{-V} W_\alpha e^V$. By contrast, in Euclidean 4-space

¹In the more familiar (and equivalent) chiral representation one has $W_\alpha = -\bar{D}^2(e^{-V} D_\alpha e^V)$ and $\bar{W}_{\dot{\alpha}} = e^{-V} [D^2(e^V \bar{D}_{\dot{\alpha}} e^{-V})] e^V$.

²Recall that λ transforms with the adjoint representation.

the geometric invariance group is $SO(4, R) \simeq SU(2, C) \otimes SU(2, C)$ and the spinor indices α and $\dot{\alpha}$ then represent $SU(2, C)$ -indices which are completely unrelated. In particular, the field strengths W_α and $\bar{W}_{\dot{\alpha}}$ still have the same form in this case, but they are now to be viewed as unrelated complex superfields. This corresponds to considering a complex internal symmetry group and a double number of fermionic and bosonic degrees of freedom [6].

All superfields considered in the following take their values in the Lie algebra of the (compact) structure group. The action (1) is invariant under arbitrary translations of the gauge field V by a scalar superfield Ψ (i.e. $V \rightarrow V + \Psi$). Clearly, this extends the translation invariance $\delta A_a(x) = \psi_a(x)$ of the bosonic theory. This symmetry can be gauge-fixed in the Lagrangian framework by choosing as gauge condition $0 = \mathcal{G}_{\dot{\alpha}}(V) \equiv \bar{W}_{\dot{\alpha}}$. Introducing a Lagrange multiplier $\bar{b}_{\dot{\alpha}}$ and an anti-ghost field $\bar{\psi}_{\dot{\alpha}}$, the differential algebra

$$\begin{aligned} s_1 V &= \Psi & , & & s_1 \bar{\psi}_{\dot{\alpha}} &= \bar{b}_{\dot{\alpha}} \\ s_1 \Psi &= 0 & , & & s_1 \bar{b}_{\dot{\alpha}} &= 0 \end{aligned} \quad (3)$$

obviously leaves invariant S_{inv} and the gauge-fixing action

$$\begin{aligned} S^{(1)} &= s_1 \int d^4x \int d^2\bar{\theta} \text{Tr} \bar{\psi}_{\dot{\alpha}} (\bar{b}^{\dot{\alpha}} - \bar{W}^{\dot{\alpha}}) \\ &= \int d^4x \int d^2\bar{\theta} \text{Tr} [\bar{b}_{\dot{\alpha}} \bar{b}^{\dot{\alpha}} - \bar{b}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} - \bar{\psi}_{\dot{\alpha}} (s_1 \bar{W}^{\dot{\alpha}})] \quad . \end{aligned} \quad (4)$$

(As usual, the superspace BRS operator acts as an antiderivation from the right [9].)

The functional $S_{inv} + S^{(1)}$ is also invariant under supergauge transformations (whose gauge-fixing is only to be considered in the end [1] [3]). In fact, the s_1 -variation can be extended to an operation

$$\begin{aligned} s_2 V &= \Psi - (c - \bar{c}) - \frac{1}{2} [V, c + \bar{c}] - \frac{1}{12} [V, [V, c - \bar{c}]] - \dots \\ &= \Psi + \sum_{n=0}^{\infty} h_n (\mathcal{L}_V)^n (c - (-1)^n \bar{c}) \quad (\mathcal{L}_V c \equiv [V, c]) \\ s_2 c &= -\frac{1}{2} [c, c] = -cc \\ s_2 \bar{c} &= -\frac{1}{2} [\bar{c}, \bar{c}] = -\bar{c}\bar{c} \quad . \end{aligned} \quad (5)$$

Here $\bar{c}$ (c) represents an (anti)chiral superfield of ghost number one and all commutators are assumed to be graded by the ghost number. The s_2 -variation of Ψ is determined from the requirement that $s_2(s_2 V) = 0$:

$$\begin{aligned} s_2 \Psi &= -\frac{1}{2} [\Psi, c + \bar{c}] - \frac{1}{12} \{ [\Psi, [V, c - \bar{c}]] + [V, [\Psi, c - \bar{c}]] \} - \dots \\ &= \sum_{n=1}^{\infty} h_n \sum_{i=1}^n (\mathcal{L}_V)^{i-1} \mathcal{L}_{\Psi} (\mathcal{L}_V)^{n-i} (c - (-1)^n \bar{c}) \quad . \end{aligned} \quad (6)$$

The other fields transform under the adjoint representation of the gauge group:

$$\begin{aligned} s_2 \bar{\psi}_{\dot{\alpha}} &= \bar{b}_{\dot{\alpha}} - [\bar{\psi}_{\dot{\alpha}}, \bar{c}] \\ s_2 \bar{b}_{\dot{\alpha}} &= -[\bar{b}_{\dot{\alpha}}, \bar{c}] \quad . \end{aligned} \quad (7)$$

To check the gauge invariance of $S_{inv} + S^{(1)}$, one notes that

$$\begin{aligned}
s_1 e^V &= (e^{V+\Psi} - e^V)_{lin \text{ in } \Psi} \\
&= \Psi + \frac{1}{2}(\Psi V + V \Psi) + \frac{1}{6}(\Psi V^2 + V \Psi V + V^2 \Psi) + \dots \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=1}^n (V)^{i-1} \Psi (V)^{n-i}
\end{aligned} \tag{8}$$

and

$$s_2(s_1 e^V) = -(s_1 e^V) c - \bar{c}(s_1 e^V) \quad , \tag{9}$$

henceforth

$$s_2(s_1 \bar{W}_{\dot{\alpha}}) = -[s_1 \bar{W}_{\dot{\alpha}}, \bar{c}] \quad . \tag{10}$$

In the bosonic theory [3], the action $S_{inv} + S^{(1)} - \int \text{Tr } b^2$ is characterized by a residual invariance which is very reminiscent of gauge transformations for which the gauge parameter is replaced by a new field $\varphi(x)$. In our supersymmetric set-up, the analogue of the gauge ghosts $\bar{c}$ (c) are (anti-)chiral fields $\bar{\Phi}$, Φ and the corresponding variations can be included into the BRS algebra with the result that

$$\begin{aligned}
sV &= \Psi - (c - \bar{c}) - \frac{1}{2}[V, c + \bar{c}] - \frac{1}{12}[V, [V, c - \bar{c}]] - \dots \\
s\Psi &= -\frac{1}{2}[\Psi, c + \bar{c}] - \frac{1}{12}\{[\Psi, [V, c - \bar{c}]] + [V, [\Psi, c - \bar{c}]]\} - \dots \\
&\quad + (\Phi - \bar{\Phi}) + \frac{1}{2}[V, \Phi + \bar{\Phi}] + \frac{1}{12}[V, [V, \Phi - \bar{\Phi}]] + \dots \\
sc &= -\frac{1}{2}[c, c] + \Phi & s\bar{c} &= -\frac{1}{2}[\bar{c}, \bar{c}] + \bar{\Phi} \\
s\bar{\psi}_{\dot{\alpha}} &= \bar{b}_{\dot{\alpha}} - [\bar{\psi}_{\dot{\alpha}}, \bar{c}] \\
s\bar{b}_{\dot{\alpha}} &= -[\bar{b}_{\dot{\alpha}}, \bar{c}] + [\bar{\psi}_{\dot{\alpha}}, \bar{\Phi}] \\
s\Phi &= -[\Phi, c] & s\bar{\Phi} &= -[\bar{\Phi}, \bar{c}] \quad .
\end{aligned} \tag{11}$$

The induced variations of $s_1 e^V$ and $s_1 \bar{W}_{\dot{\alpha}}$ read

$$\begin{aligned}
s(s_1 e^V) &= -(s_1 e^V) c - \bar{c}(s_1 e^V) + e^V \Phi - \bar{\Phi} e^V \\
s(s_1 \bar{W}_{\dot{\alpha}}) &= -[s_1 \bar{W}_{\dot{\alpha}}, \bar{c}] - [\bar{\Phi}, \bar{W}_{\dot{\alpha}}] \quad .
\end{aligned} \tag{12}$$

Note that the last transformation formally coincides with the one of $s_1 F_{ab}$ in the $N = 0$ theory where $s(s_1 F_{ab}) = -[s_1 F_{ab}, \omega] + [\varphi, F_{ab}]$.

The symmetry parameter Φ is a chiral superfield and therefore the residual invariance should be fixed by choosing a function of the same type:

$$0 = \mathcal{F}(\Psi, V) \equiv \bar{D}^2(e^{-V} \tilde{\Psi}) \tag{13}$$

where $\tilde{\Psi} \equiv s_1 e^V$. Similarly, we demand that $0 = \mathcal{F}(\Psi, V)^\dagger \equiv D^2(\tilde{\Psi} e^{-V})$. These constraints can be incorporated into the action by the inclusion of new fields. Since $\mathcal{F}(\Psi, V)$ represents a chiral superfield of degree one, we introduce chiral fields B and Λ of degree -1

and -2, respectively (analogously antichiral fields $\bar{B}, \bar{\Lambda}$ for the Hermitian conjugate function $\mathcal{F}(\Psi, V)^\dagger$). The Φ -symmetry fixing action then reads

$$\hat{S}^{(2)} \equiv s \int d^4x \left[\int d^2\theta \operatorname{Tr} \Lambda \mathcal{F}(\Psi, V) + \int d^2\bar{\theta} \operatorname{Tr} \bar{\Lambda} \mathcal{F}(\Psi, V)^\dagger \right] \quad (14)$$

where

$$\begin{aligned} s\Lambda &= B - [\Lambda, c] & \text{and c.c.} \\ sB &= -[c, B] - [\Phi, \Lambda] & \text{and c.c.} \end{aligned} \quad (15)$$

To conclude, the action $S_{inv} + \hat{S}^{(1)} + \hat{S}^{(2)}$ with

$$\hat{S}^{(1)} \equiv s \int d^4x \int d^2\bar{\theta} \operatorname{Tr} \bar{\psi}_\alpha (\bar{b}^\alpha - \bar{W}^\alpha) \quad (16)$$

is invariant under the s -operation defined by (11) (12) and (15). Moreover, it is independent of the Faddeev-Popov ghosts c and $\bar{c}$ which parametrize supergauge transformations. Indeed, under these variations the field $\tilde{\Psi}$ behaves in the same way as e^V (see equation (9)) which ensures that $\mathcal{F}$ and $\mathcal{F}^\dagger$ transform under the adjoint:

$$s_g \mathcal{F} = -[c, \mathcal{F}] \quad , \quad s_g \mathcal{F}^\dagger = -[\bar{c}, \mathcal{F}^\dagger] \quad .$$

An explicit expression for the total action immediately follows from the given definitions. Finally, the gauge-fixing of the invariance under the $c, \bar{c}$ -transformations proceeds along the usual lines [10].

3 "Hyper-superfields"

Following reference [3], we introduce an anticommuting variable θ of ghost number -1 and assemble all superfields into graded functions with values in the Grassmann algebra generated by 1 and θ :

$$\begin{aligned} \hat{V} &\equiv V + \Psi \theta \\ \hat{c} &\equiv c + \Phi \theta & \text{and c.c.} \\ \hat{\bar{\psi}}_\alpha &\equiv \bar{\psi}_\alpha + \bar{b}_\alpha \theta \\ \hat{\Lambda} &\equiv \Lambda + B \theta & \text{and c.c.} \end{aligned} \quad (17)$$

In this formalism the s -variation is a sum of a "supertranslation" σ and a supergauge transformation w (both acting from the right):

$$\begin{aligned} s &= \sigma + w & \sigma &= \frac{\partial}{\partial \theta} \\ w \hat{V} &= -(\hat{c} - \hat{\bar{c}}) - \frac{1}{2} [\hat{V}, \hat{c} + \hat{\bar{c}}] - \dots \\ w \hat{c} &= -\frac{1}{2} [\hat{c}, \hat{c}] & \text{and c.c.} \\ w \hat{\bar{\psi}}_\alpha &= -[\hat{\bar{\psi}}_\alpha, \hat{c}] \\ w \hat{\Lambda} &= -[\hat{\Lambda}, \hat{c}] & \text{and c.c.} \end{aligned} \quad (18)$$

4 Coupling to supergravity

The coupling of the previously described theory to N=1 old minimal supergravity can be achieved as for ordinary SYM [7]. The field strengths of the vector multiplet are now defined by

$$\begin{aligned} W_\alpha &= e^V [-(\bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} - 8R) e^{-V} \mathcal{D}_\alpha e^V] e^{-V} \\ \bar{W}_{\dot{\alpha}} &= (\mathcal{D}^\alpha \mathcal{D}_\alpha - 8R^\dagger) e^V \bar{\mathcal{D}}_{\dot{\alpha}} e^{-V} \end{aligned}$$

where $\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}$ denote Lorentz-covariant derivatives and where R is one of the field strengths of supergravity. Component fields are introduced by expanding the superfields with respect to covariant theta-variables ($\mathcal{D}_\alpha \Theta^\beta = \delta_\alpha^\beta$). The topological Lagrangian has the form

$$\frac{1}{2i} \left[\int d^2\Theta (2\mathcal{E}) \text{Tr} W^\alpha W_\alpha - h.c. \right] = e \text{Tr} \epsilon^{abcd} (\hat{\mathcal{D}}_a v_b) (\hat{\mathcal{D}}_c v_d) + \dots \quad (19)$$

where $\mathcal{E}(x, \Theta)$ denotes the chiral density associated to the supervielbein and where $\hat{\mathcal{D}}_a v_b(x)$ represents the supercovariant derivative of the YM field $v_a(x)$.

As in the bosonic theory, we now assume that the BRS operator does not affect the gravitational fields (contained in the covariant derivatives $\mathcal{D}_A$ and in the superfields R and $\mathcal{E}$). The differential algebra for topological SYM in curved superspace then coincides with the one in rigid superspace.

5 N = 2 theory

N = 2 SYM in harmonic superspace [8] is described by an unconstrained, Lie algebra valued superfield V^{++} transforming under the analytic gauge group:

$$\delta_\lambda V^{++} = -\mathcal{D}^{++}\lambda \equiv -D^{++}\lambda - i[V^{++}, \lambda] \quad (D_\alpha^+ \lambda = 0 = \bar{D}_{\dot{\alpha}}^+ \lambda, \quad \lambda = \bar{\lambda}^*) \quad .$$

It is convenient to express the field strength $\bar{W}$ in terms of the so-called bridge e^{iv} . The latter transforms as $\delta_\lambda e^{iv} = i\lambda e^{iv}$ and is related to the basic field V^{++} by the equation $V^{++} = -i e^{iv} D^{++} e^{-iv}$ or

$$(D^{++} + iV^{++}) e^{iv} = 0 \quad . \quad (20)$$

To be more explicit, in the analytic basis we have

$$\bar{W} \equiv (D^+)^2 [e^{iv} D^{--} e^{-iv}] \quad (D_\alpha^+ \bar{W} = 0) \quad .$$

Note that this quantity is related to the covariantly chiral superfield $\tilde{W}$ of reference [8] by $\bar{W} = e^{iv} \tilde{W} e^{-iv}$.

Considering the WZ-gauge

$$V^{++}(x, \theta, \bar{\theta}, u) = i(\theta^+ \sigma^m \bar{\theta}^+) A_m(x) + \text{terms in } M, \lambda_\alpha^j, D^{(jk)} \quad ,$$

it is straightforward to verify (at least in the Abelian case) that $\bar{D}_{(\dot{\alpha}}^- \bar{D}_{\dot{\beta}}^+ \bar{W} | = -(\epsilon \bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} F_{ab}(x)$. This motivates the N=2 superinstanton condition $0 = \mathcal{G}(V^{++}) \equiv \bar{W}$.

As a matter of fact [8], the field strength $\bar{W}$ (as defined in harmonic superspace) can be identified with the field $\bar{W}$ occurring in the solution of the constrained Bianchi identities in ordinary N=2 superspace [11]. For the latter the complete WZ-gauge expression is known [11] and thereby one can deduce the action for N=2 topological SYM in this gauge:

$$S_{inv} \equiv \int d\nu \text{Tr } \bar{W}\bar{W} - \text{h.c.} \quad (21)$$

$$\propto \int d^4x \{ \text{Tr } F_{ab}\bar{F}^{ab} + 2i \partial_m \text{Tr } [\lambda_j \sigma^m \bar{\lambda}^j] + 4 \partial_m \text{Tr } [M(\mathcal{D}_m \bar{M}) - \bar{M}(\mathcal{D}_m M)] \} .$$

Here $\int d\nu \equiv \int d\zeta_R du$ denotes the integral over the chiral subspace and the harmonic variables $u_i^\pm$ [8].

From the previous considerations, it is clear that we can proceed as before. The only subtle point consists in deriving the transformation laws of e^{iv} and $s_1 e^{iv}$ from those of V^{++} without solving the differential equation (20). Yet it will be sufficient for us to know the variation of $s_1 e^{iv}$ under gauge and under Φ -transformations and these can be anticipated from, and checked by, eq. (20). For instance, from $s_1 V^{++} = \Psi^{++}$ and application of s_1 to (20) it follows that

$$0 = (D^{++} + iV^{++}) s_1 e^{iv} + i\Psi^{++} e^{iv} .$$

If we now apply a gauge transformation $s_\lambda V^{++} = -\mathcal{D}^{++}\lambda$, $s_\lambda e^{iv} = i\lambda e^{iv}$ and $s_\lambda \Psi^{++} = -i[\Psi^{++}, \lambda]$, we find that $s_\lambda(s_1 e^{iv}) = -i\lambda(s_1 e^{iv})$.

To summarize the results, we first write the differential algebra in terms of "hyper-superfields"

$$\begin{aligned} \hat{V}^{++} &\equiv V^{++} + \Psi^{++} \theta \\ \hat{\lambda} &\equiv \lambda + \Phi \theta \\ \hat{\bar{\psi}} &\equiv \bar{\psi} + \bar{b} \theta \\ \hat{\Lambda} &\equiv \Lambda + B \theta \end{aligned} \quad (22)$$

for which it takes the form

$$\begin{aligned} s &= \sigma + w & \sigma &= \frac{\partial}{\partial \theta} \\ w \hat{V}^{++} &= -\hat{\mathcal{D}}^{++} \hat{\lambda} & w \hat{\lambda} &= -i \hat{\lambda} \hat{\lambda} \\ w \hat{\bar{\psi}} &= -i[\hat{\lambda}, \hat{\bar{\psi}}] & w \hat{\Lambda} &= i[\hat{\lambda}, \hat{\Lambda}] \end{aligned} \quad (23)$$

Like the gauge ghost λ , the field Φ which parametrizes the residual symmetry is a real analytic superfield (i.e. $D_\alpha^+ \Phi = 0 = \bar{D}_{\dot{\alpha}}^+ \Phi$ and $\Phi = \bar{\Phi}^*$). The last set of equations is equivalent to

$$\begin{aligned} sV^{++} &= \Psi^{++} - \mathcal{D}^{++}\lambda & s\Psi^{++} &= -i[\Psi^{++}, \lambda] + \mathcal{D}^{++}\Phi \\ s\lambda &= \Phi - i\lambda\lambda & s\Phi &= i[\lambda, \Phi] \\ s\bar{\psi} &= \bar{b} - i[\lambda, \bar{\psi}] & s\bar{b} &= -i[\Phi, \bar{\psi}] + i[\lambda, \bar{b}] \\ s\Lambda &= B + i[\lambda, \Lambda] & sB &= -i[\Phi, \Lambda] - i[\lambda, B] \end{aligned} \quad (24)$$

and it induces the following variations

$$\begin{aligned} s(s_1 e^{iv}) &= -i \Phi e^{iv} - i \lambda (s_1 e^{iv}) \\ s(s_1 \bar{W}) &= -i [\Phi, \bar{W}] - i [\lambda, s_1 \bar{W}] \quad . \end{aligned} \quad (25)$$

Finally, the gauge conditions which fix the translation and residual invariances are

$$\begin{aligned} 0 &= \mathcal{G}(V^{++}) \equiv \bar{W} \\ 0 &= \mathcal{F}(\Psi^{++}, V^{++}) \equiv \mathcal{D}^{++} \Psi^{++} \end{aligned} \quad (26)$$

and the complete quantum action reads $S_{inv} + \hat{S}^{(1)} + \hat{S}^{(2)}$ with

$$\hat{S}^{(1)} \equiv s \int d\nu \operatorname{Tr} \bar{\psi} (\bar{b} - \bar{W}) \quad (27)$$

$$\hat{S}^{(2)} \equiv s \int d\nu \operatorname{Tr} \Lambda (\mathcal{D}^{++} \Psi^{++}) \quad . \quad (28)$$

This action is supergauge invariant and does not depend on the ghost field λ .

The transition to the Euclidean regime does not pose any problems [6]. The coupling to supergravity should be realizable along the lines of the N=1 theory, although this program has not even been completed for the non-topological theory. (For the coupling of the N=2 Maxwell multiplet to conformal supergravity, we refer to [12]; the coupling of the non-Abelian theory to supergravity in ordinary N=2 superspace is discussed in reference [13].)

6 Cohomology algebra

The results about the cohomology of topological YM-theories [1]-[3] also hold for our superspace differential algebras.

Let us briefly summarize the results for the N=2 case and add some comments on the formalism presented in reference [3]. After appropriately changing generators in the differential algebra (24) it becomes obvious that the local cohomology of the s -operation is trivial. In fact, the relevant cohomology is the "basic" cohomology [14] [3] defined as follows. Let μ denote an element of the gauge algebra $\mathcal{A}$ (i.e. in our case the set of real, analytic, Lie algebra valued superfields). Now, consider the following gauge Lie algebra which is implicit in the differential algebra (24):

$$\begin{aligned} \delta_\mu V^{++} &= -\mathcal{D}^{++} \mu & \delta_\mu \Psi^{++} &= i [\mu, \Psi^{++}] \\ \delta_\mu \lambda &= -i [\mu, \lambda] & \delta_\mu \Phi &= i [\mu, \Phi] \\ \delta_\mu \bar{\psi} &= -i [\mu, \bar{\psi}] & \delta_\mu \bar{b} &= i [\mu, \bar{b}] \\ \delta_\mu B &= -i [\mu, B] & \delta_\mu \Lambda &= i [\mu, \Lambda] \quad . \end{aligned} \quad (29)$$

Furthermore, introduce a linear operator i_μ which annihilates all fields except the Faddeev-Popov ghost λ for which

$$i_\mu \lambda = \mu \quad . \quad (30)$$

(Thus $i_\mu S = 0$ for all functionals S which do not depend on the field λ .) Then the operations s, δ_μ and i_μ are related by classical commutation relations (in particular $\delta_\mu = [s, i_\mu]_+$) and the "basic" cohomology algebra of s is defined by

$$H_{basic}^*(s) = \{ S / sS = 0, \delta_\mu S = 0 = i_\mu S \text{ for } \mu \in \mathcal{A} \text{ and there does not exist a } W \text{ such that } S = sW \text{ and } \delta_\mu W = 0 = i_\mu W \text{ for } \mu \in \mathcal{A} \} \quad (31)$$

Here S and W are assumed to be functionals of the fields for which the s -operation is defined.

7 Concluding comments

We have shown that topological SYM can be formulated in an algebraic and manifestly supersymmetric way. The projection from superspace to component field formalism represents a straightforward, albeit tedious exercise. Just like the actions for ordinary SYM, those for the topological theories only take a non-exponential form in the WZ-gauge.

As pointed out by Witten in the pioneering work [1], topological YM theory is closely related to a twisted version of N=2 SYM in flat Euclidean 4-space (see also ref. [15]). In the present case where we started from a supersymmetric theory, it is not clear if a similar relationship to another theory exists and, if it does, what it looks like.

Further developments along the lines of the bosonic theory [1]-[2] should also be possible in our supersymmetric case. For instance, it should be of interest to determine the supersymmetric extension of Donaldson polynomials which might provide some new insights into knot and superconformal field theories. Work in this direction is currently in progress.

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8 Appendix: Chern-Simons superforms

YM Chern-Simons forms can be introduced in $N = 1$ rigid superspace in complete analogy to ordinary space-time [16]:

$$\text{Tr } FF = d \text{Tr} [AF - \frac{1}{3} AAA] \quad . \quad (32)$$

Here, $A \equiv E^C A_C$ denotes the connection super 1-form, $F \equiv 1/2 E^C E^B F_{BC} = dA + AA$ its curvature. Substituting the well-known solution of the constrained Bianchi identities [7], one finds that

$$\text{Tr } FF = E^a E^b \left\{ (d\bar{\theta}) \bar{\sigma}_{ab} (d\bar{\theta}) \text{Tr } W^\alpha W_\alpha - (d\theta) \sigma_{ab} (d\theta) \text{Tr } \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right\} + \text{other terms} \quad . \quad (33)$$

Here, $d\theta_\alpha$ and $d\bar{\theta}^{\dot{\alpha}}$ are commuting quantities, since they correspond to 1-forms with Grassmann parity one. Clearly, there is a formal analogy between the given terms of the super 4-form (33) and the topological action (1) (in which $d\theta_\alpha, d\bar{\theta}^{\dot{\alpha}}$ are to be interpreted as the anticommuting duals of $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$).

The Chern-Simons super 3-form is invariant under local, but not under global supergauge transformations. As noted in section 2, the only physical implications of the latter correspond to the action of ordinary global gauge transformations on the ordinary Chern-Simons 3-form.

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