CERN-TH.5287/89

TOPICS IN CONFORMAL FIELD THEORY AND STRING THEORY

L. Alvarez-Gaumé

CERN -- Geneva

Lectures delivered at
Nato School in Mathematical Physics
Montreal, Canada, July 1988

and

C.I.M.E. (Centro Italiano Matematico Estivo)
Montecatini, Italy, August 1988

CERN-TH.5287/89
February 1989

scale invariant. Thus, close to criticality, the most convenient way of describing the physics is in terms of scaling fields. They are eigenvectors of the scale transformation.

In the language of CFT, these fields are the primary fields, and their dimensions (scaling dimensions) determine the critical exponents of a critical system, the heat capacity as a function of T : $C_V \sim (T-T_c)^{-\alpha}$, the spontaneous magnetization, etc.

Since close to the critical point the system has "forgotten" the original lattice structure, we can describe its correlation functions in terms of a local field theory. The order parameters are represented by some local fields ϕ_i , all with well-defined scaling dimensions and, at criticality, the interactions are supposed to be scale invariant:

$$\xi_i \rightarrow \lambda \xi_i \quad \phi(\xi_i) \rightarrow \lambda^{-\Delta_i} \phi_i(\lambda^{-1} \xi) \quad (1.1)$$

Scale invariance automatically implies conformal invariance in a local 2-d QFT. Our exposition of conformal field theory will follow closely the work of Belavin, Polyakov and Zamolodchikov [3].

Given an action S in the presence of external gravity, the energy-momentum tensor is defined by the response of the system to a gravitational field:

$$\Theta_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}} \Big|_{g=\delta} \quad (1.2)$$

$$2^{\mu} = \int d^{n-1} \theta^{\mu} \quad T^{\mu\nu} = \int d^{n-1} (x^{\mu} \theta^{\nu} - x^{\nu} \theta^{\mu}) \quad (1.3)$$

θ generates arbitrary co-ordinate transformations:

$$\int d^{\mu}(V) d\Sigma^{\mu} = \int V_{\mu} \theta^{\mu} d\Sigma^{\mu} \quad (1.4)$$

$j_{\mu}(V)$ is conserved if $\partial_{\mu} \theta^{\mu\nu} = 0$ and

$$\partial_{\mu} (V_{\nu} \theta^{\mu\nu}) = \frac{1}{2} (\partial_{\mu} V_{\nu} + \partial_{\nu} V_{\mu}) \theta^{\mu\nu} = 0 \quad (1.5)$$

hence

TOPICS IN CONFORMAL FIELD THEORY AND STRING THEORY

L. Alvarez-Gaumé
Theory Division
CERN, CH-1211 Geneva 23, Switzerland

LECTURE 1

The aim of these lectures is to review some recent developments in string theory [1]. In particular we will present a formalism which can deal in principle with conformally invariant field theories on arbitrary Riemann surfaces. As explicit examples, we will discuss the bosonic string in flat space and the superstring. Since we will not assume the audience to be acquainted with conformal field theories in $d = 2$ dimensions, this lecture and part of the next will review some of the relevant features of these theories.

Conformally invariant field theories describe the long-range behaviour of correlation functions in systems undergoing a second-order phase transition. From the pioneering work of Wilson, Kadanoff, Fisher, Polyakov, etc. [2], we have learned to appreciate the fundamental rôle of scale invariance in the renormalization group approach to critical phenomena. Briefly, we start with some microscopic Hamiltonian $H_0(\phi^{(0)}, a)$ defined on some lattice of lattice spacing a , with some fields $\phi_i^{(0)}$ and coupling constants $g_i^{(0)}$. To describe long-range phenomena, we want to thin out the fast degrees of freedom (integrate them out). For example, we can define "block-spin" variables on a lattice of spacing $2a$. Integrating out the fast modes gives a new Hamiltonian $H_1(\phi^{(1)}, g^{(1)}, 2a)$ with new fields and coupling constants. If we keep on performing the renormalization group transformation $RH_1 = H_{1+1}$, we get a picture very similar to a dynamical system whose configuration space is the space of Hamiltonians, and the flow is defined by the renormalization group transformation. Starting with some initial Hamiltonian, the renormalization group flow produces a trajectory, and at each point we obtain an effective Hamiltonian describing the physics at a particular length scale. One of the characteristic features of a critical system is that the correlation length is effectively infinite compared with the initial lattice spacing, implying that these systems are described by the fixed points of the renormalization group transformations $RH^* = H^*$. The critical system is

$$\partial_\mu V_\nu + \partial_\nu V_\mu = 0 \quad (\text{Killing conditions}) \quad (1.6)$$

For dilatations, $\psi^\mu \sim x^\mu$, and invariance implies

$$\theta_\mu{}^\mu = 0 \quad (1.7)$$

An infinitesimal conformal transformation satisfies

$$\partial_\mu V_\nu + \partial_\nu V_\mu - \frac{2}{d} g_{\mu\nu} \partial_\alpha V^\alpha = 0 \quad (1.8)$$

or its covariant version:

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu}(x') dx'^\mu dx'^\nu = \Omega^2(x) g_{\mu\nu} dx^\mu dx^\nu \quad (1.9)$$

If V_μ satisfies $\partial_\mu V^\mu + \frac{2}{d} g_{\mu\nu} \partial^\mu V^\nu = 0$, and $\theta_\mu{}^\mu = 0$, then we have full conformal invariance:

$$\partial_\mu (V_\nu \theta^{\nu\mu}) = \partial_\mu V_\nu \theta^{\nu\mu} =$$

$$= \frac{1}{2} \left(\partial_\mu V_\nu + \partial_\nu V_\mu - \frac{2}{d} g_{\mu\nu} \partial_\alpha V^\alpha \right) \theta^{\nu\mu} = 0 \quad (1.10)$$

hence $\theta_\mu{}^\mu = \partial_\mu \theta^{\mu\nu} = 0$ implies conformal symmetry. If $d > 2$, the conformal group for Minkowski space $M_{p,q}$ with signature $(+, \dots, +, -, \dots, -)$ is isomorphic to some covering of $SO(p+1, q+1)$. Two dimensions, however, give a richer structure. In light-cone co-ordinates

$$\partial_+ V_+ = \partial_- V_- = 0 \quad (1.11)$$

and the other condition is empty:

$$\partial_+ V_- + \partial_- V_+ - \frac{2}{2} (\partial_+ V_- + \partial_- V_+) \approx 0$$

with

$$dS^2 = dx^+ dx^-$$

$$g_{+-} = g_{-+} = 1/2$$

$$g^{+-} = g^{-+} = 2$$

$$x^\pm = x^0 \pm x^1 \quad \partial_\pm = \frac{1}{2} \left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right) \quad (1.12)$$

Then a general conformal transformation in two dimensions takes the form

$$x'^+ = f(x^+) \quad , \quad x'^- = g(x^-) \quad (1.13)$$

and the light cone does not change. To avoid infra-red problems, we will compactify space and work on $S^1 \times \mathbb{R}$. Under Wick rotation,

$$x^0 + x^1 \rightarrow -i\omega$$

$$x^0 - x^1 \rightarrow -i\bar{\omega} \quad (1.14)$$

Furthermore, if we conformally map the cylinder onto a plane minus the origin:

Fig. 1

Quantizing the theory on the z -plane is known as radial quantization. Circles centred at the origin correspond to equal-time surfaces. This is particularly useful because it makes scale invariance very explicit. The Möbius group

$$z \rightarrow \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1 \quad (1.15)$$

acts on $\mathbb{C} \cup \{\infty\}$, and scale transformations

$$z \rightarrow \lambda z \quad , \quad \bar{z} \rightarrow \bar{\lambda} \bar{z} \quad (1.16)$$

are equivalent to time translation in the original co-ordinates.

Since we have an infinite number of symmetries, we have an infinite number of conservation laws. These are neatly summarized by saying that θ_{zz} ($\bar{\theta}_{zz}$) is a holomorphic (antiholomorphic) field

$$\begin{aligned}\bar{\partial}_z \theta_{zz} &= 0 \\ \partial_z \bar{\theta}_{zz} &= 0\end{aligned}$$

We can expand $T(z) \equiv \theta_{zz}(z)$ (and similarly for $\bar{T}$) in a Laurent series:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad (1.17)$$

T has scaling dimension 2, and the conserved quantities are

$$\begin{aligned}L_n &= \oint_0 \frac{dz}{2\pi i} z^{n+1} T(z) \\ \psi(z) &= \sum_n \epsilon_n z^{n+1}\end{aligned} \quad (1.18)$$

Conformal invariance is manifest when the objects we write down depend only on the holomorphic structure of the surface where we perform the computations. The vector $v(z)$ behaves under conformal transformations as $\partial/\partial z$, since (naively) $T(z)$ behaves as dz^2 , ψT is a (1,0) holomorphic differential, and its integral over a closed contour only depends on the holomorphic structure of the surface and on the homology class of the contour.

The ∞ -dimensional symmetry algebra $\{L_n\}$ is a representation of the algebra of vector fields

$$\psi_n = -z^{n+1} \frac{d}{dz} \quad [\psi_n, \psi_m] = (n-m) \psi_{n+m} \quad (1.19)$$

$T(v) \equiv \oint v(z) T(z)$ gives a representation of this algebra on the Hilbert space of states of the field theory. This representation is projective (see below). Now r -ordering becomes radial ordering. To compute the (anti) commutators, we only need to know the operator product expansions (OPEs) of the operators involved. In $d=2$, these infinite dimensional algebras are very useful in statistical mechanics. They often determine completely the spectrum of anomalous dimensions. Since the scaling fields diagonalize L_0 , some among them can be chosen to behave like tensors of type $(h, \bar{h})$:

$$\phi_{h, \bar{h}}(z, \bar{z}) dz^h d\bar{z}^{\bar{h}} \quad (h, \bar{h} \in \mathbb{Z} \text{ or } \mathbb{Z} + 1/2) \quad (1.20)$$

Then

$$\phi_{h, \bar{h}}(f(z)) f'(z)^h dz^h = \phi_h(z) dz^h$$

(same for antiholomorphic parts which we omit). Then, for infinitesimal transformations:

$$\begin{aligned}z &\rightarrow z + \psi(z) \\ \delta_\epsilon \phi(z) &= [T(\epsilon), \phi(z)] = (v\partial + h\partial\psi) \phi(z)\end{aligned} \quad (1.21)$$

hence

$$\begin{aligned}\delta_n \phi(z) &= [L_n, \phi(z)] = \left(z^{n+1} \frac{d}{dz} + h(n+1) z^n \right) \phi(z) \\ \psi_n &= z^{n+1}\end{aligned} \quad (1.22)$$

Fields transforming as in (1.22) are known as primary fields. There is a distinguished subalgebra in (1.21) represented by the global holomorphic vector fields on the sphere $\{1, z, z^2\}$ (conformal Killing vectors). They are the generators of the Möbius group. Hence L_{+1}, L_0, L_{-1} represent the generators of this group on the Hilbert space of the theory. Since this group $[SL_2(\mathbb{C})]$ is the conformal symmetry of the compactified plane, we can postulate the existence of a unique state $|0\rangle$ in the Hilbert space $\mathcal{H}$ which is $SL_2(\mathbb{C})$ -invariant:

$$L_{\pm 1} |0\rangle = L_0 |0\rangle = 0 \quad (1.23)$$

In an operator language this is equivalent to saying that the operator algebra contains the unit operator. The equal-time commutators of the energy-momentum tensor can be computed using the OPE. This is based on the "contour swapping" argument familiar from the early days of string theory. Given two holomorphic operators $A(z), B(w)$, we can compute the commutator:

$$\begin{aligned}[A(f), B(g)]_{\text{E.T.}} \\ A(f) = \oint_{C_0} f(z) A(z) \quad B(g) = \oint_{C_0} g(z) B(z)\end{aligned} \quad (1.24)$$

(from now on the contour integral contains the factor $1/2\pi i$) with the contours taken as circles centred at the origin. In radial quantization, (1.24) can be represented as

$$[A(f), B(g)] = \oint_{C_1} f(z) A(z) \oint_{C_2} g(w) B(w) - \oint_{C_2} g(w) B(w) \oint_{C_1} f(z) A(z) \quad |z| > |w| \quad (1.24)$$

and the contours are represented in Fig. 2.

Fig. 2

Leaving z fixed in the first integral in (1.24), we can deform the C_2 contour past C_1 . This will produce two terms: one cancels the second double integral, and the other is a small contour of z around w followed by an integral of w around the origin, namely

$$[A(f), B(g)] = \oint_0 dw g(w) \oint A(z) B(w) f(z) \quad (1.25)$$

By analyticity of A and B , we can choose the z -contour in (1.25) as small as we please, and therefore the z -integral will be completely determined by the singular terms of the OPE between A and B (as long as $f(z)$ is holomorphic in a neighbourhood of w). This information, together with (1.21), implies that for primary fields:

$$T(z) \phi(w) = \frac{h}{(z-w)^2} \phi(w) + \frac{1}{z-w} \partial \phi(w) + \text{analytic} \quad (1.26)$$

For the energy-momentum tensor itself, we obtain (1.26) with $h = 2$ plus an extra term, because for unitary theories the two-point function of $T(z)$ cannot vanish. In the plane it is completely determined by dimensional arguments and the fact that we compute the correlation function on the SL_2 -invariant ground state

$$\langle T(z) T(w) \rangle = \frac{c/2}{(z-w)^4} \quad (1.27)$$

where c is a constant depending on the theory considered. This, together with (1.26), yields

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) + \dots \quad (1.28)$$

Using (1.18) and (1.25), we finally obtain the Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m, 0} \quad (1.29)$$

This is a central extension of the algebra of vector fields (1.19), and it justifies the claim made previously that $T(w)$ provides a projective representation of (1.19) on the Hilbert space $\mathcal{H}$. Notice first of all that for $n = \pm 1, 0$, the central term in (1.29) vanishes. The representation theory of (1.29) is very rich, and some of its highlights are the minimal models of Belavin, Polyakov and Zamolodchikov [4] and the unitary series of Friedan, Qiu and Shenker [3].

The highest-weight representations of (1.29) are characterized by a highest-weight state

$$L_n |h\rangle = 0 \quad n > 0 \quad L_0 |h\rangle = h |h\rangle \quad (1.30)$$

and all the states in the representation or Verma module $V(c, h)$ can be written in the form

$$L_{-n_1} \dots L_{-n_N} |h\rangle \quad (1.31)$$

with L_0 eigenvalue $h + \sum_{i=1}^N n_i$. The Verma module is completely characterized by (c, h) . We can equivalently characterize the states in $V(c, h)$ in terms of operators. The highest-weight conditions (1.30) are equivalent to having a primary field $\phi_h(z)$ in the operator algebra. The other states (1.31), known as descendants, can be represented as

$$\begin{aligned} (L_{-n_1} \dots L_{-n_N} \phi_h)(z) &= \oint_{C_1} \frac{T(z_1)}{(z_1 - z)^{n_1-1}} \oint_{C_2} \frac{T(z_2)}{(z_2 - z)^{n_2-1}} \dots \\ &\dots \oint_{C_N} \frac{T(z_N)}{(z_N - z)^{n_N-1}} \phi(z) \end{aligned} \quad (1.32)$$

$$C_1 \supset C_2 \supset \dots \supset C_N \supset z$$

and they have rather complicated conformal transformation. Using the analyticity properties of $T(z)$, it is easy to derive the conformal Ward identities corresponding to the insertion of one or more energy-momentum tensors in a correlation function involving primary fields. For example the correlator

$$\langle 0 | T(z) \phi_1(z_1) \dots \phi_n(z_n) | 0 \rangle$$

for fixed $z_1, \dots, z_n$ is a holomorphic function of z in $S^2 - \{z_1, \dots, z_n\}$. The pole structure of this function as $z \rightarrow z_i$ is determined by the OPE of $T(z)$ with $\phi_i(z_i)$. We thus conclude

$$\begin{aligned} \langle 0 | T(z) \phi_1(z_1) \dots \phi_n(z_n) | 0 \rangle &= \\ &= \sum_{i=1}^n \left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle 0 | \phi_1(z_1) \dots \phi_n(z_n) | 0 \rangle \end{aligned} \quad (1.33)$$

and similarly for more insertions of $T(z)$. A simple conclusion one obtains from (1.32) and (1.33) is that the correlation functions of descendant fields are completely determined by those of the primary fields, and therefore the latter are the fundamental quantities to evaluate in a Conformal Field Theory.

The central term in (1.29) also allows us to compute the transformation law of $T(z)$ under arbitrary conformal transformations

$$T(z) \rightarrow \left(\frac{dz}{dz'} \right)^2 T(z') + \frac{c}{12} \left\{ z, z' \right\} \quad (1.34)$$

where $\{z, z'\}$ is the Schwarzian derivative:

$$\left\{ z, z' \right\} = \frac{S''' }{S'} - \frac{3}{2} \left(\frac{S''}{S'} \right)^2 \quad (1.35)$$

hence in the transformation from the cylinder to the plane $\zeta = e^z$

$$\left\{ e^z, z' \right\} = -1/2 \quad (1.36)$$

and only L_0 changes:

$$(L_0)_{cyl.} = (L_0)_{plane} - \frac{c}{24} \quad (1.37)$$

This change is important in the modular properties of Virasoro characters. The derivation of (1.34) can be done in a number of ways. Perhaps the simplest appears in Ref. [5], where the properties of the energy-momentum tensor in a conformal theory are derived in the presence of an arbitrary geometry on the plane.

LECTURE 2

We have so far studied the "kinematics" of conformal invariance in two dimensions. We concluded that the operator algebra of the theory contains the identity and the energy-momentum tensor, and all other operators fall into conformal classes $[\phi_{h,\bar{h}}]$, each furnishing a representation of the Virasoro algebra fully characterized

by c and $(h, \bar{h})$. The existence of an SL_2 -invariant ground state completely determines the form of the two- and three-point functions in a conformal theory. More generally, for an n -point function of the primary fields $\phi_1, \dots, \phi_n$, it is easy to show

that the Green function has the general form

$$\begin{aligned} \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle &= \left(\prod_{i,j} z_{ij}^{-\gamma_{ij}} \right) f(\eta_a) \\ \sum_{j \neq i} \gamma_{ij} &= 2h_i \quad \gamma_{ij} = \gamma_{ji}, \quad z_{ij} = z_i - z_j \end{aligned} \quad (2.1)$$

and η_a are $n-3$ independent harmonic ratios formed with the n points $z_1, \dots, z_n$. For a four-point function there is only one such variable:

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

The two-point function becomes

$$\langle \phi_{h,\bar{h}}(z,\bar{z}) \phi_{h',\bar{h}'}(w,\bar{w}) \rangle = \delta_{h,h'} \delta_{\bar{h},\bar{h}'} (z-w)^{-2h} (\bar{z}-\bar{w})^{-2\bar{h}} \quad (2.2)$$

and only a constant needs be determined on the three-point function:

$$\begin{aligned} \langle \phi_i(z_i) \phi_j(z_j) \phi_k(z_k) \rangle &= C_{ijk} z_{ij}^{-\gamma_{ij}} z_{jk}^{-\gamma_{jk}} z_{ik}^{-\gamma_{ik}} \\ \gamma_{ij} &= h_i + h_j - h_k \quad \text{etc.} \end{aligned} \quad (2.3)$$

The coefficients C_{ijk} are the basic structure constants of the operator algebra, and together with the spectrum of dimensions $(h, \bar{h})$ and c they determine the theory.

Strong constraints on C_{ijk} are imposed by the requirement of associativity of the OPE [3]. The set of fields of the theory $\{A_i(x)\}$ fall into conformal families (Verma modules). The OPE between two primary fields ϕ_i, ϕ_j can be written as:

$$\phi_i(z, \bar{z}) \phi_j(0, 0) = \sum_p \sum_{\{h, \bar{h}\}} C_{ij}^p z^{h_p - h_i - h_j} + \sum_k \frac{z^{h_p - \bar{h}_i - \bar{h}_j + 2\bar{h}}}{\beta_{ij}^k} \bar{z}^{k\bar{h}} \phi_k(0, 0) \quad (2.4)$$

$$[L_{-h_1} \dots L_{-h_N} \bar{L}_{-\bar{h}_1} \dots \bar{L}_{-\bar{h}_N} \phi_p](0, 0)$$

where p runs over primary fields and the coefficients $\beta, \bar{\beta}$ can be determined from c, h_i, h_j, h_k using conformal invariance [4]. Defining

$$|h\rangle = \lim_{z, \bar{z} \rightarrow 0} \phi_{h, \bar{h}}(z, \bar{z}) |0\rangle$$

$$\langle h| = \lim_{z, \bar{z} \rightarrow \infty} \phi_{h, \bar{h}}(z, \bar{z}) z^{2h_0} \bar{z}^{2\bar{h}_0} \quad (2.5)$$

The implications of associativity of the OPE are best expressed in terms of the four-point function

$$\langle k | \phi_l(1) \phi_n(x, \bar{x}) | m \rangle \quad (2.6)$$

Substituting (2.4) in the operator product between ϕ_n and ϕ_m , and using (2.3), we obtain

$$\langle k | \phi_l(1) \phi_n(x, \bar{x}) | m \rangle = \sum_p C_{nm}^p C_{klp} \frac{\mathcal{F}_{nm}^{pk}(p|x)}{\mathcal{F}_{nm}^{pk}(p|x)} \quad (2.7)$$

where the multivalued analytic function $\mathcal{F}$ is known as a conformal block, normalized so that

$$\mathcal{F}_{nm}^{pk}(p|x) = x^{h_p - h_l - h_k} (1 + \dots) \quad x \rightarrow 0 \quad (2.8)$$

This block can be represented graphically as in Fig. 3.

Fig. 3

If we want to use associativity of the OPE, instead of performing first the OPE between n and m and then with l , we can use SL_2 invariance first and then do the OPE between l and n . Associativity then implies [4]:

$$G_{nm}^{lk}(x, \bar{x}) = G_{nl}^{mk}(1-x, 1-\bar{x}) \quad (2.9)$$

Graphically this is similar to crossing in ordinary field theory (Fig. 4):

Fig. 4

When (2.9) is expressed in terms of the conformal blocks (2.8), we obtain a nonlinear equation involving the structure constants C_{ij}^k , the dimensions $h, \bar{h}$ and the central term c . The classification of conformal field theories (or universality classes in statistical mechanics) is closely related to the classification of solutions to these equations. In some cases the structure constants C_{ij}^k and correlation functions of primary fields can be computed due to the presence of null-vectors on the Verma modules. Given some module $V(c, h)$, there is a null-vector at level N if there is a state $|\chi\rangle$ in $V(c, h)$ satisfying

$$L_n |X\rangle = 0 \quad n > 0, \quad L_0 |X\rangle = (h+N) |X\rangle \quad (2.10)$$

It is easy to show that $|X\rangle$ is orthogonal to all of $V(c,h)$ and to the states in any other module; therefore it can be safely set to zero, $X \approx 0$. This yields non-trivial differential equations for the correlation functions involving the primary field ϕ_h . For example, if h satisfies

$$h = \frac{1}{16} (5-c \pm \sqrt{(c-1)(c-25)}) \quad (2.11)$$

then the state

$$|X\rangle = \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right) |h\rangle \quad (2.12)$$

is null. From the conformal Ward identities we can derive the equation:

$$\left(\frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^n \frac{h_i}{(z-z_i)^2} - \sum_{i=1}^n \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_h(z) \phi_1 \dots \phi_n \rangle = 0 \quad (2.13)$$

Kac [6] has given a list of all reducible representations v_n . They are labelled by pairs of positive integers (n,m) . Writing

$$\begin{aligned} c &= 1 - 24\alpha_0^2 \\ \alpha_+ + \alpha_- &= 2\alpha_0 \\ \alpha_+ \alpha_- &= -1 \end{aligned} \quad (2.14)$$

the module generated by a field of dimension

$$h(n,m) = -\alpha_0^2 + \frac{1}{4} (n\alpha_+ + m\alpha_-)^2 \quad (2.15)$$

has a null-vector at level nm yielding a differential equation of order nm (because the null-vector will contain a term proportional to $L_{-1}^{nm} \phi$, and using the Ward identities, we can replace L_{-1} by $\partial/\partial z$) for correlation functions involving the field $\phi_{n,m} = \phi_{h(n,m)}$. Among all the theories satisfying (2.15), those with the extra requirement

$$\alpha_+ / \alpha_- = -q/p$$

q, p being relatively prime positive integers, have

$$c = 1 - 6(p-q)^2/pq$$

$$h(\eta, m) = \frac{1}{4pq} [(mp-nq)^2 - (p-q)^2] \quad (2.16)$$

and the operator algebra truncates to a finite number of conformal families $0 < n < p$, $0 < m < q$ [4]. If one requires the theory to be unitary, it was shown in Ref. [3] that $q = p+1$, $p > 2$. The differential equations induced by the null-vector in (2.16) can all be solved in terms of a Coulomb gas representation [7].

The OPE coefficients determine the fusion rules of the theory. They give the number of independent ways of coupling three conformal families:

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}^k [\phi_k] \quad (2.17)$$

Equivalently, N_{ij}^k counts the number of different three-point vertices in Fig. 5.

Fig. 5

The associativity of the OPE implies that the matrices $(N_i)_j^k \equiv N_{ij}^k$ commute:

$$\begin{aligned} N_i N_j &= N_j N_i \\ N_i N_j &= N_{ij}^k N_k \end{aligned} \quad (2.18)$$

hence we have a commutative associative algebra whose structure constants are positive integers. One of the more interesting recent developments in conformal field theory is the analysis by E. Verlinde of this algebra [8], which implies that the fusion rules (2.18) are diagonalized by the matrix S representing the modular transformation $\tau \rightarrow -1/\tau$ on the characters $\chi_i(\tau)$ of the modules appearing in the theory.

This claim was checked in many examples in Ref. [8], and it was proved by Moore and Seiberg [9] in a paper where they wrote down a set of polynomial equations for Rational Conformal Field Theories. A conformal theory is rational if it is characterized by a symmetry algebra containing 1 and the Virasoro algebra at least, and with a finite number of primary fields with respect to this algebra.

The fusion algebra (2.18) is useful in determining the number of conformal blocks contributing to a given correlation function. For instance, the number of blocks contributing to the four-point function in Fig. 6 is:

Fig. 6

$$N_{ijkl} = \sum_p N_{ij}^p N_{pk}^l \quad (2.19)$$

The associativity of (2.18) implies that N_{ijkl} is independent of what basis we choose to draw the blocks. We could instead have chosen the dual diagram in Fig. 7, and N_{ijkl} would still be the same. For modular invariant theories this count of conformal blocks extends also to diagrams containing loops.

Fig. 7

For higher point states (Fig. 8), we obtain:

Fig. 8

$$N_{ijklm} = \sum_{p,p'} N_{ij}^p N_{pk}^{p'} N_{p'l}^m \quad (2.20)$$

etc. For a more general block $\mathcal{F}_I(z_1, \dots)$, let I label the possible intermediate

channels in the diagram. Then the physical correlation function is obtained by combining holomorphic and antiholomorphic blocks:

$$\mathcal{G}(z_i, \bar{z}_j) = \sum_{I, J} \mathcal{F}_I(z_i) \overline{\mathcal{F}_J(\bar{z}_j)} \quad (2.21)$$

into a local monodromy- and duality-invariant combination. From a geometrical point of view, the duality and monodromy transformations act on the blocks $\mathcal{F}_I$ and they form a representation of the braid group on $n = 1$ strands (for the n-point function). This feature brings in some interesting connection of conformal theories with the representation theory of braid groups, knots, polynomials, the Yang-Baxter equation, etc. (for a review and references, see [10]).

We will approach string theory from the point of view of conformal field theory in $d = 2$. The two dimensions represent the parameter space necessary to trace out the string history. In the remaining part of this lecture, we will give an overview of the Polyakov approach to the covariant quantization of the string (for many more details and references to the original literature, see the review in [11]). For a string moving in some target space-time, the amplitude corresponding to the process described in Fig. 9 is:

$$\mathcal{G}(C_1, C_2, C_3) = \int_S \mathcal{D}S = \int_S \mathcal{D}C_2 \mathcal{D}C_3 \quad (2.22)$$

Fig. 9

The area in (2.22) is the area induced on the surface S by the target space metric.

Consider a bosonic string whose co-ordinates are represented by $X^\mu(\sigma, \tau)$. The induced metric is

$$G_{\mu\nu}(X) = \partial_\mu X^\alpha(\sigma, \tau) \partial_\nu X^\alpha(\sigma, \tau) = \gamma_{\alpha\beta}(\sigma, \tau) \quad (2.23)$$

For example, we can take $G_{\mu\nu} = \delta_{\mu\nu}$ (flat space). The quantization of this system is

fairly complicated. It contains second-class constraints. Another procedure to quantize it, introduced by Polyakov, is to consider alternatively

$$S[X, g] = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \quad (2.24)$$

Then:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \frac{\delta S}{\delta X^\mu} = \square X^\mu = 0 \quad (2.25)$$

The first equation is a second-class constraint indicating the vanishing of the two-dimensional energy-momentum tensor:

$$\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0 \quad (2.26)$$

Solving for g_{ab} gives the induced metric. S is invariant under $\text{Diff}(\Sigma)$ and Weyl rescalings. In the functional integral quantization, we have:

$$Q(c; c) = \sum_{\text{topologies}} \int \{g_{ab}\} \frac{1}{\text{Vol}(\text{Diff}) \times \text{Vol}(\text{Weyl})} \sum_{(X^\mu)} e^{-S[X, g]} \quad (2.27)$$

Consistency requires that the field theory be conformal invariant with $c = 26$. This follows from the Faddeev-Popov quantization prescription. After a laborious procedure (see [11]), the measure contains a new set of anticommuting fields $b_{zz}, c^z, \bar{b}_{\bar{z}\bar{z}}, \bar{c}^{\bar{z}}$.

Choosing a slice of $\text{Metric}/\text{Diff} \times \text{Weyl} \equiv \text{moduli space}$, where $\text{Metric} \equiv \text{space of metrics on a surface of genus } g$, the quantity we need to study is schematically

$$\sum_{\text{genera}} \int d\mu_g \int [dX] e^{-i \int \partial X \bar{\partial} X + \int (b \bar{\partial} c + \bar{b} \partial c)} \quad \mathcal{M}_g = \text{moduli space} \quad (2.28)$$

For two-dimensional surfaces, the space of metrics on a fixed topology modulo diffeomorphisms and conformal transformations is a finite-dimensional space. If we count degrees of freedom, a metric has three independent components. A diffeomorphism depends on two functions and a conformal transformation introduces one more

arbitrary function. For the sphere, it is possible to show (see, for example, [12]) that we can always transform any metric to the standard one using this freedom. For a genus- g surface, the complex structure depends on a set of $6g-6$ real parameters [12], and therefore the space $\mathcal{M}_g = \text{Metrics}/\text{Diff} \times \text{Weyl}$ is a finite-dimensional space (the moduli space of curves of genus g).

The requirement of conformal invariance is easy to implement. A free scalar field X generates a conformal theory. The basic two-point functions are:

$$\begin{aligned} \langle \partial X^\mu(z) \partial X^\nu(w) \rangle &= - \frac{\eta^{\mu\nu}}{(z-w)^2} \\ \langle X^\mu(z) X^\nu(w) \rangle &= - \eta^{\mu\nu} \ln |z-w|^2 \end{aligned} \quad (2.29)$$

The energy-momentum tensor is

$$T(z) = - \frac{1}{2} \partial X \cdot \partial X \quad (2.30)$$

Then

$$T(z) T(w) = \frac{d/2}{(z-w)^4} + \dots \quad (2.31)$$

hence $c^{(X)} = d$. What is the central extension for the b - c system? In general, if b has spin j , and c spin $1-j$:

$$\begin{aligned} b(z) &= \sum_{n \in \mathbb{Z}} b_n \bar{z}^{n-j} \quad c(z) = \sum_{n \in \mathbb{Z}} c_n \bar{z}^{n-1+j} \\ \{b_n, c_m\} &= \delta_{n+m, 0} \\ b(z) c(w) &= \frac{1}{z-w} + \dots = c(z) b(w) \end{aligned} \quad (2.32)$$

Under a co-ordinate transformation,

$$\begin{aligned} \delta_\psi b(z) &= \psi \partial b(z) + j \partial \psi b(z) \\ \delta_\psi c(z) &= \psi \partial c(z) + (1-j) \partial \psi c(z) \end{aligned} \quad (2.33)$$

This implies that the energy-momentum tensor is

$$T(z) = j \partial c b - (1-j) c \partial b \quad (2.34)$$

Now the central extension is easy to compute:

$$C_j = -2(6j^2 - 6j + 1) \quad (2.35)$$

hence for $j = 2$, $c_2 = -26$. Thus

$$c(\text{matter}) + c(\text{gh}) = d - 26 = 0 \Rightarrow d = 26$$

However, we can achieve $c = 26$, by considering four uncompactified dimensions, and at the same time some conformal theory with $c = 22$. In the case of $d = 26$, the Polyakov amplitude at genus g is obtained by integrating a measure of the form

$$\frac{\rho \wedge \bar{\rho}}{(\det \text{Im } \Omega)^{13}} \quad (2.36)$$

where ρ is the Mumford form in moduli space [13], and Ω is the period matrix of the surface. Next, we want to construct the Polyakov measure in a purely operatorial language. The advantage is that many of the manipulations can be done in the standard language of operators, and all the complications with gauge fixing are circumvented. Furthermore, in the formalism to be introduced in the next lectures, it is quite easy to obtain:

- 1) Differential equations for the measure.
- 2) Belavin-Knizhnik theorem.
- 3) Analysis of infinities.
- 4) Physical state conditions.
- 5) It is easy to show that the BRST operator Q_{BRST} acts as the exterior derivative d in moduli space $\mathcal{M}_g$, leading to the decoupling (up to boundary components in $\mathcal{M}_g$) of spurious BRST states.
- 6) It also clarifies the interpretation of the energy-momentum tensor as a connection on $\mathcal{M}_g$ and its action on moduli space.

LECTURE 3

In previous lectures, we worked directly in a two-dimensional space-time with the topology of a cylinder $S^1 \times \mathbb{R}$. In this case (and in some cases also for the torus), we have the standard operator formalism which often simplifies the computations, and from a physicist's point of view it is fairly intuitive. It is a bit more difficult to construct an operator formalism on higher genus Riemann surfaces. Several groups have pursued the development of this formalism emphasizing different aspects. One started by solving some overlap equations [14] and eventually developed into the

group theoretical approach to string amplitudes [15]; the Copenhagen group started with the N-reggeon vertex built in a BRST-invariant way mixed with the functional integral formulation [16,17]. In Ref. [18], a formalism inspired by string field theory is derived, and in Ref. [19] the starting point was inspired by possible models for the universal moduli space proposal of Friedan and Shenker [20], and exploits the more geometrical aspects of Riemann surface theory. I will report on work done in collaboration with Gomez, Reina, Moore, Nelson, Vafa and Sierra with regard to an operator formalism for strings and superstrings. In the original motivation, we wanted to have a formulation of CFT and in particular string theory on a higher genus surface, in such a way that all the information about the geometry and topology is coded on a particular state. More precisely, for the case of the plane, $z = 0$ ($\tau = -\infty$) and $z = \infty$ ($\tau = +\infty$) are the "in" and "out" regions. Any correlation function can be written as

$$\langle 0 | \phi_1(z_1) \dots \phi_N(z_N) | 0 \rangle \quad (3.1)$$

Fig. 10

We want to replace the "in" state $|0\rangle$ by some other state $|W\rangle$, characterizing what happens on the rest of the surface, and such that

$$\frac{\langle 0 | \phi_1 \dots \phi_N | W \rangle}{\langle 0 | W \rangle} \quad (3.2)$$

agrees with the standard functional integral. Thus if $\mathcal{H}$ is the Hilbert space of the theory, and we consider $\mathcal{P}(g,n) \equiv \{\text{moduli space of genus } g \text{ surfaces with } n \text{ parametrized boundaries, i.e., } n \text{ points } P_i, i = 1, \dots, n \text{ and } n \text{ local parameters } z_i(P_i) = 0\}$, we want to construct a map

$$\mathcal{P}(g,n) \rightarrow \mathcal{H}^{\otimes n} \quad (3.3)$$

for each conformal theory. In other words, for every conformal theory, we want to associate a ray with every point $P \in \mathcal{P}(g,n)$, $P = \{X_g, P_1, \dots, P_n, X_g, z_i(P_i) = 0\}$.

Furthermore, we would like to have some consistency conditions enumerated below. In the collection of spaces parametrized by $\mathcal{P}(g,n) \forall g,n$ we have some kind of semi-group structure:

$$\begin{aligned} \mathcal{P}(g_1, n_1) \times \mathcal{P}(g_2, n_2) &\longrightarrow \mathcal{P}(g_1 + g_2, n_1 + n_2 - 2) \\ \mathcal{P}_i \times_j \mathcal{Q} = \mathcal{R} \quad \mathcal{P} \in \mathcal{P}(g_1, n_1), \mathcal{Q} \in \mathcal{P}(g_2, n_2) &\quad (3.4) \end{aligned}$$

Choose two points, i in $\mathcal{P}$ and j in $\mathcal{Q}$, and defining $\mathcal{R}$ by $2, z_j = 1$, or by sticking a plumbing fixture. Similarly we have the map $\mathcal{P}(g_1, n_1) + \mathcal{P}(g_1 + 1, n_1 - 2)$, where we glue two points of the same surface. Notice that from this point of view $\mathcal{P}(g,n)$ is generated from $\mathcal{P}(0,3)$ and $\mathcal{P}(0,2)$. In pictures:

Fig. 11

It is also clear that Fig. 11b acts like a "unit" with respect to sewing (gluing). A representation of these objects for conformal theories contains the information about the OPE C_{ijk} (the three-point function) and the spectrum of the theory, the two-point functions. Let $\{|n\rangle\}$ be an orthonormal basis of states in $\mathcal{H}$. Then one associates to Fig. 11b the state

$$|S_{ij}\rangle = \sum_n |n\rangle_i |n_j\rangle \quad (3.5)$$

(we need a real basis, CPT invariant). Then the state associated to $\mathcal{R} = \mathcal{P}_i \times_j \mathcal{Q}$ should be

$$|\mathcal{R}\rangle = \langle S_{ij} | \mathcal{P} \rangle \otimes |Q\rangle \quad (3.6)$$

For theories defined via functional integrals, this is a simple consequence of the properties of the Feynman integral.

Comments

1) Notice that rays are associated to points in $\mathcal{P}(g,n)$ only for conformal theories. Only the normalization of the ray may depend on the metric chosen to represent the conformally invariant data parametrized by $\mathcal{P}(g,n)$. Thus, if we wanted to associate a state to $\mathcal{P}$ rather than a ray, we would have to give more information.

2) $\mathcal{P}(g,n)$ is a space more natural than $\mathcal{M}(g,n) = \{\text{moduli space of genus } g \text{ curves with } n \text{ distinguished points}\}$ for several reasons. First, to define oscillators, creation and annihilation operators, etc., in the neighbourhood of a point $\mathcal{P}$, we need a local parameter (and some trivializing sections) to define the Laurent expansion of various fields. Second, if $T_{g,n} \equiv$ mapping class group of X_g with n marked points, then $\pi_1 \mathcal{P}(g,n) = T_{g,n}$. In other words, $\mathcal{P}(g,n)$ resolves the orbifold singularities of $\mathcal{M}_{g,n}$. This is because together with a point $\mathcal{P} \in X$ we give a tangent vector at $\mathcal{P}$. Thus, even though X may have non-formal automorphisms fixing $\mathcal{P}$ (and the holomorphic structure), the tangent vector will change. In fact, the orbifold points always correspond to curves with automorphisms. For instance, the moduli space $\mathcal{M}_1$ (genus 1) is the shaded region in Fig. 12. The points $\tau = i$ and $\tau = \rho$, $\rho^3 = 1$ correspond to the

Fig. 12

square torus with a Z_4 symmetry group, and to a unit cell in the $SU(3)$ root lattice respectively. For higher genus surfaces the same conclusion holds, although the automorphism groups are more complicated. In fact, they are all invariant subgroups of the Hurwitz group with presentation

$$H = \langle E_1, E_2 \mid E_1^2 = E_2^2 = (E_1 E_2)^7 \rangle$$

For instance, the famous Klein subgroup of $SU(3)$ of order 168 appears first in genus 3. The modular group $T_{g,n}$ admits a presentation in terms of elements of finite order. This means that we can represent $T_{g,n}$ in terms of the curves left fixed by these generators.

3) Harer has shown that for $g > 2$, $H_1(\mathcal{P}(g,n)) = 0$. This is very useful in the characterization of the Polyakov measure, because it implies that any flat line bundle is necessarily trivial.

4) As we will see in more detail later, the "equations of motion" are given by the energy-momentum tensor. Let $v = v(z)(dz)$ [in $\mathcal{P}(g,1)$ for simplicity] represent a variation of the moduli in $\mathcal{P}(g,1)$ at P . Then the change in $|W\rangle$ as we change the moduli is given by

$$\delta_v |W\rangle = (T(v) + \overline{T}(\overline{v})) |W\rangle \quad (3.7)$$

$$T(v) = \oint_{\mathcal{C}} v(z) T(z) \quad (3.8)$$

We will prove this in some cases, but it can be taken as an axiom in the construction of CFT for genus > 1 . This is essentially the approach taken by Segal in his axiomatic approach to CFT [21].

5) Consistency of the construction of $|W\rangle$ for $P \in \mathcal{P}(g,n)$ requires that $L_0 - \overline{L}_0 \in \mathbb{Z}$. This is because we can make a Dehn twist around a point P .

Fig. 13

On the holomorphic co-ordinate z , this amounts to $z \rightarrow e^{2\pi i \theta} z$, $0 < \theta \leq 1$. On the state $|W\rangle$, this transformation is implemented by $e^{2\pi i \theta (L_0 - \overline{L}_0)}$. When $\theta = 1$, then invariance requires $L_0 - \overline{L}_0 = \text{integer}$. This is a necessary condition for modular invariance.

Let us now consider some examples to illustrate the general framework.

Free fermions: $\psi(z)$, $\bar{\psi}(z)$, $S = \int \bar{\psi} \psi$

In genus g there are 2^{2g} spin structures, $2^{g-1}(2^{g-1}-1)$ odd [$\dim \text{Ker } \bar{\partial} = 1 \pmod{2}$] and $2^{g-1}(2^{g-1}+1)$ even [$\dim \text{Ker } \bar{\partial} = 0 \pmod{2}$]. For the time being, choose an even non-singular spin structure for $\psi, \bar{\psi}$, i.e., $\dim \text{Ker } \bar{\partial} = 0$, $\psi = c(z)$, $\bar{\psi} = b(z)$.

Fig. 14

We want to associate a state to each side of the surface in Fig. 14 so that on the Fock space of the circle

$$Z = \int (dbdc) \exp - \int b \bar{\partial} c = \langle \Psi_2 | \Psi_1 \rangle \quad (3.9)$$

We can represent $\bar{\psi}_1$ as follows:

$$\Psi_1[f] = \int dbdc \exp \left(- \int_{S^1} c \bar{\partial} b + \oint_{S^1} c b \right) \quad (3.10)$$

b and c behave as holomorphic $\frac{1}{2}$ differentials under conformal transformations $\{c, b\} = \delta_{n+m,0}$, i.e., $b(z) \sim \delta/(\delta c(z))$; that is, b is the "translation" operator for c .

Let w_n be the boundary value at S^1 of a holomorphic spinor on X_1 . Then we want to show that

$$\Psi_1[f + w_n] = \Psi_1[f] \quad (3.11)$$

This simply follows from the definition of $\Psi_1[f]$ and making a change of variables $c \rightarrow c - w_n$. Since b is the translation operator for $c(z)$, we can write this equation by saying that

$$\varrho(w_n) = \oint_{S^1} b(z) w_n(z) ; \quad \varrho(w_n) \tilde{\Psi}_1 = 0 \quad (3.12)$$

Furthermore, since $w_n(z)$ is holomorphic, if $b(z)$ is also holomorphic, $Q(w_n)$ only depends on the homology class of the contour. This follows because in correlation functions $\bar{\partial}b = 0$, as follows by making a change $c \rightarrow c+h$, such that $h|_{S^1} = 0$. This implies that $\bar{\partial}b = 0$ inside correlation functions. We can do similarly with $\tilde{w}_n(z)$, to show that

$$\begin{aligned} \tilde{\varrho}_n &= \varrho(\tilde{w}_n) = \oint_{S^1} \tilde{w}_n(z) c(z) \\ \tilde{\varrho}_n |\tilde{\Psi}_1\rangle &= 0 \end{aligned} \quad (3.13)$$

and also

$$\{ \varrho_n, \tilde{\varrho}_m \} = \oint_{S^1} w_n(z) \tilde{w}_m(z) \quad (3.14)$$

Since $w_n, \tilde{w}_m$ are holomorphic on S^1 , $w_n \tilde{w}_m$ is a holomorphic $(1,0)$ form on X_1 , and a simple contour deformation argument shows that

$$\{ \varrho_n, \tilde{\varrho}_m \} = 0 \quad (3.15)$$

In fact, $Q_n, \tilde{Q}_m$ give the maximal set of conditions on $|\Psi_1\rangle$. For example, for $g = 0$ and one point P , $z(p) = 0$, the holomorphic section in $S^2 - \{P\}$ with poles at P has the form: $\tilde{z}^n d\tilde{z}^{1/2}$

$$b(z) = \sum_{n \in \mathbb{Z} + 1/2} b_n \tilde{z}^{-n-1/2} = \sum_{n \in \mathbb{Z}} b_{n-1/2} \tilde{z}^{n-1} \quad (3.16)$$

Then

$$\oint_{S^1} b(z) \tilde{z}^{-n} d\tilde{z} = b_{n-1/2} \quad n \geq 1 \quad (3.17)$$

hence

$$b_n |0\rangle = c_n |0\rangle = 0 \quad n > 0 \quad (3.18)$$

and we obtain the SL_2 -invariant vacuum.

The state can be constructed explicitly using prime forms and θ -functions. Since we have a local parameter, let $S(z,w)$ be the Szegő kernel for spin $\frac{1}{2}$ with the given spin structure (the two-point function for the fermion):

$$S(z,w) = \frac{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z-w | \Omega)}{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (0 | \Omega) E(z,w)} \quad (3.19)$$

$E(z,w)$ is the prime for on X . It is the unique $(-\frac{1}{2}, -\frac{1}{2})$ differential such that

- a) $E(z,w) = -E(w,z)$
- b) Its only zero is first order, and it happens along the diagonal: $E(z,w) \sim (z-w) + O((z-w)^3)$ as $z \rightarrow w$.

To construct it, let $\{ \frac{a}{b} \}$ be a non-singular, odd theta characteristic. Then:

$$h^2(z) = \sum_{j=1}^g \frac{\partial}{\partial u^j} \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \left(\int_{\gamma^j} \omega | \Omega \right) \Big|_{z=w} \omega_j(z) \quad (3.20)$$

has double zeroes at $g-1$ points $P_1, \dots, P_{g-1}$ as a consequence of the Riemann vanishing theorem. Then

$$E(z,w) = \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z-w | \Omega)}{h(z) h(w)} \quad (3.21)$$

$h(z)$ is the holomorphic section of $\{ \frac{a}{b} \}$, and the square root in (3.20) does not introduce cuts, because all its zeroes are second order. For z,w in the neighbourhood of $P \notin X$, we can write

$$\begin{aligned} w_n(z) &= \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial w^{n-1}} S(z,w) \Big|_{w=0} \\ &= \frac{1}{z^n} + \sum_{m=1}^{\infty} B_{nm} z^{m-1} \end{aligned}$$

$$B_{nm} = \frac{1}{(n-1)!(m-1)!} \frac{\partial^{m-1}}{\partial \bar{z}^{m-1}} \frac{\partial^{n-1}}{\partial w^{n-1}} \left(S(z, w) - \frac{1}{z-w} \right) \quad (3.22)$$

and now we can write down a differential equation for $|\bar{\Psi}\rangle$. Since $\{c_n, b_m\} = \delta_{n+m, 0}$, $n, m \in \mathbb{Z} + \frac{1}{2}$, we represent the operators as

$$\begin{aligned} c_n &= \frac{\partial}{\partial b_{-n}} & b_n &= \frac{\partial}{\partial c_{-n}} & n > 0 \\ c_{-n} &= c_n & b_{-n} &= b_{-n} & n > 0 \end{aligned} \quad (3.23)$$

Hence $\bar{\Psi} = \bar{\Psi}(c_{-n}, b_{-n})$, and $|0\rangle \rightarrow \bar{\Psi} = 1$. The conserved charges are

$$\begin{aligned} \oint_{S^1} b(z) \bar{w}_n(z) &= \oint_{S^1} \sum_k b_{k+\frac{1}{2}} z^k \left(\bar{z} + \sum_{m=1}^{\infty} B_{nm} z^{m-1} \right) \\ &= b_{n-1/2} + \sum_{m=1}^{\infty} B_{nm} b_{-m+1/2} \\ &= \partial/\partial c_{n+1/2} + \sum_{m=1}^{\infty} B_{nm} b_{-m+1/2} \end{aligned} \quad (3.24)$$

hence

$$|\bar{\Psi}\rangle = C \exp \left(- \sum_{n, m=1}^{\infty} B_{nm} c_{n+\frac{1}{2}} b_{-m+\frac{1}{2}} \right) |0\rangle \quad (3.25)$$

C is the (undetermined) normalization of the ray. $|\bar{\Psi}\rangle$ is a Bogoliubov transformation of $|0\rangle$. Next, consider $P \in \mathcal{P}(g, n)$. Now we have to consider holomorphic sections of $K^{\frac{1}{2}}$ in $X - \{P_1, \dots, P_n\}$ with poles only at $P_1, \dots, P_n$. In particular, we can construct $|S_{12}\rangle$:

$$z^n dz^{1/2}, \quad n \in \mathbb{Z}$$

$$Q(w_n) = \oint_{P_1} b^{(1)}(z_1) \bar{w}_n(z_1) + \oint_{P_2} b^{(2)}(z_2) \bar{w}_n(z_2) \quad (3.26)$$

In this case,

$$|S_{12}\rangle = \left(\prod_{m=1}^{\infty} e^{\sum_{n+\frac{1}{2}=-m+\frac{1}{2}} b^{(1)}_{n+\frac{1}{2}} b^{(2)}_{-m+\frac{1}{2}}} \right) |0\rangle_1 |0\rangle_2 \quad (3.27)$$

and

$$(Q_1 + Q_2) |S_{12}\rangle = 0 \quad (3.28)$$

For illustrative purposes, we now prove the charge transport argument (CTA) which implies the sewing rules for spin $\frac{1}{2}$. Let $|P\rangle$ be the state associated to $P \in \mathcal{P}(g_1, n_1)$ and $|Q\rangle$ the one associated to $Q \in \mathcal{P}(g_2, n_2)$. If we sew along the i, j punctures we obtain $R = P_1 \circ_j Q \in \mathcal{P}(g_1 \# g_2, n_1 + n_2 - 2)$. We wish to prove that

$$|R\rangle = \langle S_{ij} | P \rangle_i \otimes |Q\rangle_j \in \mathcal{H}^{\otimes (n_1 + n_2 - 2)} \quad (3.29)$$

$z_i \bar{z}_j = 1$

Let $P_1, \dots, P_{i-1}, P_{j+1}, \dots, P_{n_1+n_2}$ be the distinguished points in R . The conserved charge condition is

$$(Q_1 + \dots + Q_{i-1} + Q_{j+1} + \dots + Q_{n_1+n_2}) |R\rangle = 0 \quad (3.30)$$

providing the maximal set of conditions on the state $|R\rangle$. To prove that $\langle S_{ij} | P \rangle |Q\rangle$ satisfies these equations, we use

$$\langle S_{ij} | (Q_i + Q_j) \quad (3.31)$$

Fig. 15

In R we still have some remnants of the two discs around punctures i (in P) and j (in Q). If we expand any of the holomorphic spinors in R away from $P_1, \dots, P_{i-1}, P_{j+1}, \dots$, we can express them as functions of t_i, t_j . As such, they will have at most poles in the (non-existent) points $t_i = t_j = 0$. Therefore, we now have for these spinors $\langle S_{ij} | (Q_i + Q_j) = 0_j$ and

$$\begin{aligned} & (Q_1 + \dots + Q_{i-1} + Q_{j+1} + \dots + Q_{n_1+n_2}) \langle S_{ij} | P \rangle \otimes | Q \rangle \\ &= \langle S_{ij} | (Q_1 + \dots + Q_i) + (Q_j + Q_{j+1} + \dots + Q_{n_1+n_2}) | P \rangle \otimes | Q \rangle = 0 \end{aligned}$$

by the definition of $|P\rangle$ and $|Q\rangle$. Therefore the sewing condition works. One can similarly construct the state for the sphere with three points. This is left as an exercise to the reader.

LECTURE 4

We continue with the construction of states associated to Riemann surfaces. Now we would like to treat the case of a single-valued scalar field ϕ . This is just one component of the bosonic string moving in flat space. We can start by defining the state from the functional integral point of view on the surface in Fig. 16.

Fig. 16

The state is a functional:

$$\Psi[f] = \int_{\phi_{S^1} = f} d\phi \exp \left(-\frac{i}{2} \int_{X^1} \partial\phi \bar{\partial}\phi \right) \quad (4.1)$$

Since f is a real function, we can always construct a harmonic function on X_1 whose boundary value is f . We also call f this harmonic function. Then the functional integral can be computed explicitly. Writing $\phi = f + \phi'$, with $\phi'|_{S^1} = 0$, we have, with $*$ being the duality operator,

$$\begin{aligned} S &= \frac{i}{2} \int d\phi \wedge * d\phi = \frac{i}{2} \int d(f + \phi') \wedge * d(f + \phi') \\ &= \frac{i}{2} \int df \wedge * df + \frac{i}{2} \int d\phi' \wedge * d\phi' + \int d\phi' \wedge * df \end{aligned} \quad (4.2)$$

Since $d*df = 0$ (harmonic function), we obtain

$$S[f + \phi'] = S[f] + \frac{i}{2} \oint_{S^1} \phi' * df \quad (4.3)$$

hence

$$\Psi[f] = (\det' \Delta)^{-1/2} e^{-\frac{i}{2} \int_{S^1} f * df} \quad (4.4)$$

The conservation laws are easy to write down. We only need to use Green's theorem

$$Q(h_n) = \oint_{S^1} (h_n * d\phi - \phi * dh_n) \quad (4.5)$$

where h_n is a harmonic function analytic on X_1 . Since in correlation functions the field ϕ satisfies $\partial\bar{\partial}\phi = 0$, it is easy to see that as an operator, $Q(h_n)$ only depends on the homology class of S^1 , and not on its detailed shape. Writing locally $h = h_H + h_A$, $\bar{\partial}h_H = \partial h_A = 0$ (holomorphic and antiholomorphic pieces resp.), we can write the conserved current as

$$j(h_n) = \partial\phi h_n^H - \bar{\partial}\phi h_n^A \quad (4.6)$$

Here we meet a problem in trying to construct the h_n 's explicitly. For a generic

point P, we can construct functions holomorphic off P with poles of order higher than g at P. In general, between 1 and 2g, there are g gaps in the possible order of the pole at P; this is the Weierstrass gap theorem [12]. Hence we get "s-g" conditions on the state Ψ . However, we do not have to take purely holomorphic objects, i.e., we can consider holomorphic multivalued functions, and compensate the multivaluedness using antiholomorphic quantities. Explicitly, consider the differentials:

$$\eta_n(t) = \frac{1}{(n-1)!} \frac{\partial}{\partial t} \frac{\partial^n}{\partial y^n} \ln E(t, y) \Big|_{y=0} \quad (4.7)$$

then define

$$h_n(t) = \int_0^t [\eta_n(t) - A_n (\text{Im} \Omega)^{-1} (\omega - \bar{\omega})] \quad (4.8)$$

where w_i are the Abelian differentials, and

$$\omega_i(t) = \sum_{n=1}^{\infty} A_{ni} t^{n-1} dt \quad (4.9)$$

Moreover, we need to use the property of $\eta_n(t)$ that

$$\oint_{b_i} \eta_n(t) = A_{ni} \quad (4.10)$$

It is easy to check, using the properties of the prime form, that $h_n(t)$ is single-valued. For later purposes, define

$$Q_{nm} = \frac{1}{2(n-1)!(m-1)!} \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial y^m} \ln \frac{E(t, y)}{t-y} \quad (4.11)$$

and expand $\phi(t)$ in oscillators:

$$\phi(t) = q + ip \ln t + i \sum_{n \neq 0} \left(\frac{t^n a_n}{n} + c.c. \right) \quad (4.12)$$

We can realize the a_n 's by

$$a_n = \frac{\partial}{\partial x_n} \quad a_{-n} = n x_n \quad (4.13)$$

Writing

$$Q(h_n) |\bar{\Psi}\rangle = \bar{Q}(\bar{h}_n) |\Psi\rangle = 0 \quad (4.14)$$

as a differential equation, we obtain

$$\begin{aligned} \langle x, \bar{x} | \bar{\Psi} \rangle &= \exp \left(x, \bar{x} \right) M \left(\frac{x}{x} \right) \\ M &= \begin{pmatrix} Q_{nm} + \frac{\pi}{2} A_n \Omega_2^{-1} A_m & -\frac{\pi}{2} A_n \Omega_2^{-1} \bar{A}_m \\ -\frac{\pi}{2} \bar{A}_n \Omega_2^{-1} A_m & \bar{Q}_{nm} + \frac{\pi}{2} \bar{A}_n \Omega_2^{-1} \bar{A}_m \end{pmatrix} \end{aligned} \quad (4.15)$$

If we want to consider $|S_2\rangle$, then

$$|S_2\rangle = \int d\rho \prod_{n=1}^{\infty} \exp \left(a_n^{(1)+} a_m^{(2)+} - a_n^{(1)+} a_n^{(2)+} \right) |p\rangle_1 \otimes | -p\rangle_2 \quad (4.16)$$

and finally, for an arbitrary number of points, let

$$\begin{aligned} G(z, w) &= -g_n \left(\text{Im} \int_z^w \omega \right) (\text{Im} \Omega)^{-1} \left(\text{Im} \int_z^w \omega \right) - \\ &\quad - \ln |E(z, w)|^2 \end{aligned} \quad (4.17)$$

If z_i, w_j coincide on the same patch, then define

$$g(z_i, w_j) = G(z_i, w_j) + \ln |z_i - w_j|^2 \quad (4.18)$$

otherwise $g = G(z_i, w_j)$. Then, for n punctures,

$$\begin{aligned} |\Psi_g^a\rangle &= \int d\rho_1 \dots d\rho_n \delta(\sum \rho_i) \exp \left\{ - \right. \\ &\quad \left. - \frac{1}{(2\pi i)^2} \sum_{i,j} \oint_{\gamma_i} \oint_{\gamma_j} (\partial x^{(i)}, \bar{\partial} x^{(j)}) M(z_i, z_j) \left(\frac{\partial x^{(i)}}{\partial x^{(j)}} \right) \right\} |p_1\rangle \dots |p_n\rangle \end{aligned} \quad (4.19)$$

where $M(z_i, z_j)$ is constructed out of $g(z_i, z_j)$ in an obvious way. Once again, it

is not difficult to prove the CTA argument for the conserved charges.

Comments

- 1) For arbitrary conformal theories, if there is a conserved current of spin j , $\bar{\partial}_j(z) = 0$, we can construct conserved charges in the same way:

$$Q_n = \oint_{S^1} j(z) \omega_n(z)$$

where $\omega_n(z)$ are $(1-j)$ -differential holomorphic off P .

- 2) The construction can easily be extended to multivalued fields, and in this way we can describe scalar fields ϕ^i taking values on some d -dimensional torus $\mathbb{R}^d/\Lambda$ (Λ being a d -dimensional lattice).

- 3) Combining the results we have presented with remark 2), the computations of the last section and some amount of information on the Hilbert-Schmidt Grassmannian, it is possible to prove the bosonization formulae for higher genus Riemann surfaces. In particular, one of the consequences of the proof is Fay's trisecant identity, which plays a central rôle in the $K-P$ hierarchy and the proof of the Novikov conjecture.

The ghost system (spin $j = 2$):

Consider first the sphere with one singled-out point P , $z(P) = 0$. Since

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{n-2} \quad c(z) = \sum_n c_n z^{n+1}$$

$$\{b_n, c_m\} = \delta_{n+m,0} \quad (4.20)$$

we can construct the vector fields holomorphic off P . They are

$$\{z^2, z, 1, z^{-1}, z^{-2}, \dots\} \quad (4.21)$$

For quadratic differentials, those extending holomorphically off P are

$$\{z^{-4}, z^{-5}, z^{-6}, \dots\} \quad (4.22)$$

These results follow from the Riemann-Roch theorem on the sphere [12]. The conserved charges are then

$$k=2, 1, 0, -1, -2, \dots \quad \oint b_m z^{m-2} z^k = b_{-k+1} \quad \text{i.e. } b_{-1}, b_0, b_1, \dots$$

$$k=4, 5, \dots \quad \oint c_m z^{m+1} z^{-k} = c_{k-2} \quad \text{i.e. } c_2, c_3, c_4, \dots \quad (4.23)$$

Hence the state $|\phi_0\rangle$ is defined by

$$\begin{aligned} b_n |\phi_0\rangle &= 0 & n > -2 \\ c_n |\phi_0\rangle &= 0 & n > 1 \end{aligned} \quad (4.24)$$

It is easy to show that this state is $SL_2(C)$ -invariant, i.e., $L_0 |\phi_0\rangle = L_{-1} |\phi_0\rangle = 0$. Its charge assignment is a bit more delicate. We have a ghost number current $j(z) = :c(z)b(z):$, and we have to determine the vacuum charge. Since

$$\begin{aligned} [L_0, b_n] &= -n b_n \\ [L_0, c_n] &= -n c_n \end{aligned} \quad (4.25)$$

$|\phi_0\rangle$ is not a highest weight of the oscillator algebra (i.e., there are negative energy modes which do not annihilate $|\phi_0\rangle$). Similarly, we have a zero mode algebra $\{c, b\} = 1$. We introduce two highest-weight states $|\pm\rangle$:

$$\begin{aligned} c_n |\pm\rangle &= b_n |\mp\rangle = 0 & n \geq 0 \\ c_0 |\mp\rangle &= |\pm\rangle & b_0 |\pm\rangle = |\mp\rangle \end{aligned} \quad (4.26)$$

It is natural to assign $Q|\pm\rangle = \pm \frac{1}{2} |\pm\rangle$, $Q = \int j(z)$. These states are obtained from $|\phi_0\rangle$ as follows:

$$\begin{aligned} |\mp\rangle &= c_1 |\phi_0\rangle & |\pm\rangle &= c_0 c_1 |\phi_0\rangle \\ b_0 |\pm\rangle &= b_0 c_0 c_1 |\phi_0\rangle = c_1 |\phi_0\rangle - c_0 b_0 c_1 |\phi_0\rangle = c_1 |\phi_0\rangle \end{aligned} \quad (4.27)$$

Then

$$L_0 |\pm\rangle = -\frac{1}{2} |\pm\rangle \quad (4.28)$$

(This is the origin of the tachyon, as we will see.) A similar construction can be carried out for arbitrary spin, and it is left as an exercise. The next thing we

need is the sewing state. On a sphere with two points, the quadratic differentials and vectors are holomorphic off 0, and ∞ are

$$z^n dz^2 \quad z^{n+1} \frac{d}{dz} \quad \forall n \in \mathbb{Z} \quad (4.29)$$

Now we have two Hilbert spaces, one for each puncture. The conserved charges are

$$(b_n^{(1)} - b_{-n}^{(2)}) |S_{12}\rangle = 0 \quad n \in \mathbb{Z} \quad (4.30)$$

and similarly for the c's. Then

$$|S_{12}\rangle = \prod_{m=1}^{\infty} \exp \left(-c_m^{(1)} b_{-m}^{(2)} - c_{-m}^{(2)} b_m^{(1)} \right) (b_0^{(1)} - b_0^{(2)}) |H\rangle |H\rangle \quad (4.31)$$

The total charge is

$$Q |S_{12}\rangle = (Q^{(1)} + Q^{(2)}) |S_{12}\rangle = 0 \quad (4.32)$$

For higher genus surfaces, let $|\phi_g^n\rangle$ be the state corresponding to a genus g surfaces with n distinguished points. First, we know that for $g > 1$, there are $3g-3$ holomorphic quadratic differentials. Similarly, we can construct at every point $P \in X$ quadratic differentials with arbitrary order poles. At a particular point P with local parameter z , we can write the holomorphic quadratic differentials as

$$\psi_{n+1} = z^n + \sum_{m \geq 1} c_{nm}^{(2)} z^m \quad q = 3g-3 \quad (4.33)$$

$$n = 0, 1, \dots, 3g-3$$

and those with poles as

$$S_n = z^{-n} + \sum_{m \geq 1} b_{nm}^{(2)} z^m \quad (4.34)$$

and similarly for vectors v_n . The coefficients B can be written in terms of prime forms and θ -functions. Note that

$$\oint c(z) \psi_{n+1}(z) = \oint \sum_k c_k z^{k+1} \left(z^n + \sum_{m \geq 1} c_{nm}^{(2)} z^m \right) = c_{n-2} + \sum_{m \geq 1} c_{nm}^{(2)} c_{m-2} \quad (4.35)$$

i.e., it only involves creation operators. In the same way,

$$\oint c(z) s_1(z) = c_{-1} + \sum_{m \geq 1} b_{1m}^{(2)} c_{m-2} \quad (4.36)$$

hence the only way the state $|\phi_g^n\rangle$ can be annihilated by these charges is if they appear explicitly in the state (due to the fermionic character of b and c). For quadratic differentials with higher-order poles, we get both creation and annihilation operators. In this way, we get a differential equation if we represent

$$\{c_n, b_m\} = \delta_{n+m, 0} \quad (4.37)$$

as

$$c_n = \frac{\partial}{\partial b_{-n}} \quad n \geq 0 \quad b_n = \frac{\partial}{\partial c_{-n}} \quad n > 0 \quad (4.38)$$

so that the wave functionals become functions of b_{-n} , $n > 0$, c_{-n} , $n > 1$, and $|+\rangle = 1$. Hence, the state $|\phi_g^n\rangle$ takes the form:

$$|\phi_g^n\rangle = c_{-1} \dots c_{3g-3} A_1^{(1)} \dots A_1^{(n)} \exp \left(- \sum_{\substack{m \geq 1 \\ n \geq 1}} b_{nm}^{(2)} c_{-m-2}^{(1)} \right) |+\rangle, \quad \theta \dots \theta |+\rangle_n \quad (4.39)$$

where

$$A_1^{(i)} = \sum_j \oint \psi_j^{(i)}(z_j) c^{(j)}(z_j) \quad (4.40)$$

and $s_{-1}^{(i)}(z)$ is the quadratic differential with a single pole at P_i . Finally,

$$c_{i+1} = \sum_j \oint \psi_j^{(i)}(z_j) c^{(i)}(z_j) \quad (4.41)$$

The ghost charge of $|\phi_g^n\rangle$ is easy to count once we recall that b (c) has ghost number -1 ($+1$):

$$Q_{gh} |\phi_g^n\rangle = (3g-3 + 3n/2) |\phi_g^n\rangle \quad (4.42)$$

Notice that this assignment satisfies the sewing rules. If we sew a state in $\mathcal{P}_{(g_1, n_1)}$ and another in $\mathcal{P}_{(g_2, n_2)}$, then their charges add up correctly:

$$[3(q_1-1)+3n_1/2] + [3(q_2-1)+3n_2/2] = 3(q_1+q_2-1) + 3(n_1+n_2-2)/2 \quad (4.43)$$

Furthermore, the charge transport argument applies here without modifications.

LECTURE 5

Next we would like to show how the Virasoro algebra gives a connection over $\mathcal{P}(g,n)$. In other words, in the space of states $\mathcal{H}^{\otimes n}$, for an arbitrary CFT, we have obtained a ray $|\phi\rangle_P$ for each $\mathcal{P}(g,n)$. We would like to know how to transport the ray $|\phi\rangle_P$ along different curves in $\mathcal{P}(g,n)$. It is clear in the examples discussed that the ray $|\phi\rangle_P$ is only dependent on the conformal data at $P \in \mathcal{P}(g,n)$, hence $|\phi\rangle$ is uniquely defined once P and the CFT are given. This is important, because it immediately shows that $|\phi\rangle$ is invariant under the action of the modular group, leaving invariant the data parametrized by $\mathcal{P}(g,n)$. The action of the Virasoro algebra on the states $|\phi\rangle$ provides an "equation of motion" for the state which can be used to compute chiral determinants. To understand the action of Virasoro (Vir), consider the infinitesimal Kodaira-Spencer construction. Since in our data we have a curve X , a point P and a local parameter, consider the covering of X shown in Fig. 17.

$$\begin{aligned} D &= \{t, |t| \leq 1\} \\ A &= \{r < |t| < 1\} \end{aligned}$$

Fig. 17

Take $X_1 = X - \{t \mid t < r\}$. For a meromorphic vector $v(t) \in \mathcal{H}^{\otimes n}$, $v(t) = \sum_{n=1}^{\infty} t^n v_n$, $v(t)$ is holomorphic in $D-P$. For every t on the annulus A , implement the change $t \rightarrow t + \varepsilon v(t)$ (Fig. 18):

Fig. 18

Then identify the "deformed annulus" point by point with the original one on the curve X . There are now three possibilities:

- i) v extends to a holomorphic vector on D and $v(0) = 0$. This corresponds to a change in the local parameter. If $v(0) = 0$, this is equivalent to an infinitesimal change in the point P .
- ii) v extends holomorphically to X_1 . In this case, the data parametrized by $P \in \mathcal{P}(g,1)$ does not change. This vector is an infinitesimal conformal isometry of the rest of the surface.
- iii) v does not extend holomorphically to the disc or to the rest of the surface. Using Riemann-Roch, the vector fields with this property are:

$$\left\{ \frac{1}{z}, \frac{1}{z^2}, \dots, \frac{1}{z^{3g-3}} \right\} \quad (5.1)$$

and they represent the tangent space to moduli space $\mathcal{M}_g$ at X . This follows from two facts: (1) $H^1(X, K^{-1})$ is the holomorphic tangent space to $\mathcal{M}_g$ at X ; (2) use the sheaf sequence (for P not a Weierstrass point):

$$0 \rightarrow K^{-1} \rightarrow K^{-1}(3g-3)P \rightarrow S_P^{3g-3} \rightarrow 0 \quad (5.2)$$

Here S_P^{3g-3} is the skyscraper sheaf of length $3g-3$ at P , $K^{-1}((3g-3)P)$ is the sheaf of meromorphic vectors with at most order $3g-3$ poles at P , and K^{-1} is the sheaf of local holomorphic vectors. When P is generic, the line bundle $K^{-1}((3g-3)P)$ has degree $g-1$, and it is away for the θ -divisor.

Using the long exact sequence and $H^0(X, K^{-1}((3g-3)P)) = H^1(X, K^{-1}((3g-3)P)) = 0$, we obtain $H^1(K^{-1}) = S_P^{3g-3} \equiv$ (Laurent tails at P with $3g-3$ terms). The Kodaira-

Spencer construction can be exponentiated locally in a neighbourhood of X to provide vector fields passing through $p \in \mathcal{P}(g,1): V(v)$. Those vector fields satisfy the algebra

$$[V(v_1), V(v_2)] = V([v_1, v_2]) \quad (5.3)$$

In a more axiomatic approach to CFT; $|\phi\rangle$ is a well-defined ray at each $p \in \mathcal{P}(g,1)$, and therefore it should be possible to represent the vectors $V(v)$ in terms of operators acting on the Hilbert space. However, since $|\phi\rangle$ is a ray, all we can expect is that the commutation relations of the operator $O(v)$ representing $V(v)$ be:

$$[O(v_1), O(v_2)] = O([v_1, v_2]) + c\text{-number} \quad (5.4)$$

In CFT, the operator $O(v)$ is naturally the energy-momentum tensor. This can be checked for any of the examples of previous lectures, or taken as one of the axioms for CFT's on higher-genus surfaces. Explicitly, the change of $|\phi\rangle$ along v is given by:

$$\delta_v |\phi\rangle = (\tau(v) + \overline{\tau(v)}) |\phi\rangle \quad (5.5)$$

This implies that $T(v)$ provides a projective connection over $\mathcal{P}(g,1)$. To check that this holds in the simple examples we analyzed previously, all we need to show is that the new conserved charges are:

$$Q_i \rightarrow Q_i + \epsilon [\tau(v), Q_i] = Q_i + \delta Q_i \quad (5.6)$$

Then the ray $(1 + \epsilon T(v))|\phi\rangle$ is annihilated by $(Q_i + \delta Q_i)$:

$$\begin{aligned} (Q_i + \delta Q_i)(1 + \epsilon T(v))|\phi\rangle &= \\ &= \epsilon Q_i T(v)|\phi\rangle + \epsilon (\tau(v) Q_i - Q_i \tau(v))|\phi\rangle \\ &= \epsilon T(v) Q_i |\phi\rangle = 0 \end{aligned} \quad (5.7)$$

Following the Kodaira-Spencer construction in detail, the change of a holomorphic section s_i of some holomorphic line bundle with a pole at P is $s_i \rightarrow s_i + \epsilon \mathcal{L}_{v,1} s_i$. Since by standard operator algebra

$$[T(v), Q(s_i)] = Q(\mathcal{L}_v s_i) \quad (5.8)$$

we conclude that all the states defined in previous sections do satisfy the "equation of motion", i.e., $T(v)$ provides a projective connection.

Comments

- 1) If we want to construct a state rather than a ray, we need to fix a normalization $C|\phi\rangle$. Imposing the parallel transport equation gives a differential equation for C :

$$\delta_v \log C = \frac{\langle \phi | T(v) + c.c. | \phi \rangle}{\langle \phi | \phi \rangle} \quad (5.9)$$

In particular, for the spin- $\frac{1}{2}$ fermion on the torus, the moduli is generated by $v = (1/z)(d/dz)$. Then

$$\begin{aligned} \delta_v \log C &= B_{1,2} \\ \frac{\partial}{\partial \tau} \log C &= \frac{\partial}{\partial \bar{\tau}} \log \frac{\mathcal{G}[\frac{\tau}{\tau}](\phi/\tau)}{\eta(\tau)} \end{aligned} \quad (5.10)$$

In higher genus,

$$\delta_n \log C = \sum_{n=1}^N \left(n - \frac{n+1}{2}\right) B_{n-m+1, m} \quad (5.11)$$

where δ_n corresponds to motions along $z^{-n+1}(d/dz)$.

- 2) Hence, to summarize:

$$\begin{aligned} L_{-n} |\phi\rangle, n > 0 & \text{ changes local coordinates} \\ L_{+1} |\phi\rangle & \text{ moves the point } P \end{aligned} \quad (5.12)$$

$$L_k |\phi\rangle, k = 2, 3, \dots, 3g-3+1 \text{ changes the moduli}$$

- 3) If v extends holomorphically off P , what should $T(v)|\phi\rangle$ be? Since v does not change any data, all we can expect is that $T(v)|\phi\rangle \propto |\phi\rangle$. One might be tempted to set $T(v)|\phi\rangle = 0$, because $T(z)$ is conserved. However, the anomalous commutator

$$[T(v_1), T(v_2)] = T([v_1, v_2]) + \frac{c}{12} \oint_P v_1 v_2''' \quad (5.13)$$

does not allow this possibility. Another way of looking at the same problem is that $T(z)$ does not transform as a quadratic differential when we move from one patch to another. Under an infinitesimal co-ordinate transformation, $T(v)$ changes by $\int_P \varepsilon'''(z)v(z)$. However, if we choose $\varepsilon'''(z) \in H^0(X, K^2)$, it seems that we obtain $(3g-3)+3$ ways of patching the Riemann surface so that T behaves like a quadratic differential. This is related to the classification of projective structures on a Riemann surface.

The Polyakov Measure

So far, we have associated states to points in $\mathcal{P}(g, n)$. Let us now come back to the bosonic string. As we argued in Lecture 3, we need to construct an integration measure over $\mathcal{M}_g$ or $\mathcal{M}_{g, n}$ (resp. the moduli space of genus g curves, and the moduli space of genus g curves with n marked points). The first one allows the computation of partition functions, and the second one the computation of scattering measures and scattering amplitudes.

It is clear from our construction that there is a lot of redundant information in $\mathcal{P}(g, 1)$. There is a projection $\pi(X, \mathcal{P}, z) = X$, which simply means forgetting the point and the local parameter.

$$(X, \mathcal{P}, z) \in \mathcal{P}(g, 1) \xrightarrow{\pi} \mathcal{M}_g$$

Fig. 19

At a point $X \in \mathcal{M}_g$, choose a basis of tangent vectors $v_1, \dots, v_{3g-3}$ (and c.c.). We want to construct a measure at X :

$$\mu(X) (v_1, \dots, v_{3g-3}; \bar{v}_1, \dots, \bar{v}_{3g-3}) \quad (5.14)$$

Given v_i , we can find in $\mathcal{P}(g, 1)$ at some point in the fibre $\pi^{-1}(X)$ representatives $v_i(z)\partial/\partial z$ of v_i . The obvious candidate for the measure is

$$\mu(X) = \langle 0 | b(v_1) \dots b(v_{3g-3}) \overline{b(v_1)} \dots \overline{b(v_{3g-3})} | \phi \rangle_{(X, \mathcal{P}, z)} \quad (5.15)$$

where

$$b(v) = \oint_P b(z)v(z) \quad (5.16)$$

$|0\rangle$ is the SL_2 -invariant vacuum in $\mathcal{F}_{\text{matter}} \otimes \mathcal{F}_{bc} \otimes \mathcal{F}_{\overline{bc}}$, i.e., $|\phi\rangle$ also belongs to $\mathcal{F}_{\text{matter}} \otimes \mathcal{F}_{bc} \otimes \mathcal{F}_{\overline{bc}}$; it is the total state for the free bosonic string. We must show that

- i) $\mu(X)$ is independent of the choice of representative of v .
- ii) $\mu(X)$ is independent of the local parameter and the position of the point $P \in X$:

iii) $\mu(X)$ satisfies the Belavin-Knizhnik theorem:

$$\partial\bar{\partial} \ln \mu = -13 \partial\bar{\partial} \ln \text{Im } \Omega \quad (5.17)$$

i.e.,

$$\mu = \frac{\rho \wedge \bar{\rho}}{(\det \text{Im } \Omega)^{13}} \quad (5.18)$$

where ρ is a holomorphic half-volume, the Mumford form.

- iv) To prove the last part of iii), we have to show that on surfaces with nodes, ρ has a second-order pole with respect to the holomorphic co-ordinate q such that $q = 0$ defines the curve with a node.
- v) If we consider the total energy-momentum tensor

$$T(z) = T_{\text{matter}}(z) + T_g(z) \quad (5.19)$$

then $C^{\text{total}} = 0$. Hence, if v is a vector field extending holomorphically off P , we can normalize $|\phi\rangle$ so that $T(v)|\phi\rangle = 0$. Since T is the connection acting on $|\phi\rangle$ and defining parallel transport, we conclude that $|\phi\rangle$ is a flat bundle over $\mathcal{P}(g, 1)$. This means that locally we can define the normalization of the ray $|\phi\rangle$. It may happen, however, that since $\pi_1(\mathcal{P}(g, 1)) = \pi_{g, 1}$ there is non-trivial holonomy for the line bundle $|\phi\rangle$. Since $H_1(\mathcal{P}(g, 1)) = 0$, i.e., $T_{g, 1}/[T_{g, 1}, T_{g, 1}] = 0$, a flat line bundle is a trivial line bundle, and the normalization of $|\phi\rangle$ can be defined consistently over all of $\mathcal{P}(g, 1)$ up to a constant. This implies among other things that $T(v)|\phi\rangle = 0$ for any v extending holomorphically. If we now change a representative v of moduli change by $v + v_{\text{hol}}$ (v_{hol} extends holomorphically off P), then in the measure we have a change

$$\langle 0 | \dots b(v + v_{hol}) \dots | \phi \rangle \quad (5.20)$$

Since $b(v_{hol})$ is a conserved charge, and the b 's anticommute, then $b(v_{hol})|\phi\rangle = 0$, and the measure does not change. A change in the local parameter is accomplished via L_{-n} , $n > 0$. Since $\langle 0 | L_{-n} = 0$, $n > 0$, we conclude, using $[T(v), b(v')] = b([v, v'])$ that there is no dependence on the local parameter. Similarly, since $\langle 0 | L_1 = 0$, we can change the point $P \in X$ infinitesimally, and $\mu(X)$ will not vary.

To prove iii), we use the equation of motion and the fact that the matter state does not factorize holomorphically into left and right components. From the equation of motion:

$$\begin{aligned} \partial \bar{\partial} \log \mu(\dots v \dots) &= \frac{\langle 0 | \dots T(\cdot) \bar{T}(\cdot) | \phi \rangle}{\langle 0 | \phi \rangle} - \\ &= \frac{\langle 0 | \dots T(\cdot) | \phi \rangle \langle 0 | \dots \bar{T}(\cdot) | \phi \rangle}{\langle 0 | \phi \rangle^2} \end{aligned} \quad (5.21)$$

Since only the matter sector has factorization problems, we can compute the previous expression in the matter sector. Using Rohrlieh's formula

$$\delta \Omega_{ij} = - \int_P \omega_i \omega_j \nu \quad (5.22)$$

we obtain

$$\partial \bar{\partial} \log \mu(\dots v \dots) = -13 \partial \bar{\partial} \log \det \bar{\partial} \Omega \quad (5.23)$$

Finally, to show that the measure at the boundary has second-order poles, we use sewing. Using the plumbing fixture description of degenerating surfaces (Figs. 20 and 21):

Fig. 20

As $q \rightarrow 0$, we generate the curve with a node.

Fig. 21

With the sewing prescription, the measure can be written as

$$\frac{d\bar{q}dq}{q\bar{q}} \langle S_{ij} | b_0 \bar{b}_0 q^{\bar{L}_0} \bar{q}^{\bar{L}_0} | \Phi \rangle \quad (5.24)$$

Since the lowest eigenvalue of $L_0, \bar{L}_0$ is -1 (the tachyon), as $q \rightarrow 0$ the leading singularity comes from $L_0 = \bar{L}_0 = -1$, and we obtain the behaviour

$$\left| \frac{d\bar{q}}{q\bar{q}} \right|^2 \quad (5.25)$$

Hence, ρ has a second-order pole at the boundary of moduli space, generated by the equivalence classes of curves with one node (either splitting or non-splitting). This property uniquely identifies ρ in μ as the Mumford form, which is the desired result.

Similar arguments can be used to define scattering measures and physical state conditions, and we refer the reader to Ref. [16] for further details. The extension

of this construction to fermionic strings is also possible (see the last entry in Ref. [16]), but now we have to adapt the arguments to super-Riemann surfaces. Questions concerning the uniqueness and properties of the supermeasure are answered in a manner similar to the bosonic construction. The problem which still remains unresolved is the correct prescription to integrate over super-moduli space. Details can be found in Ref. [16] (last entry).

REFERENCES

- [1] For a thorough account of string theory with many references to the literature, see:
M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory", vols. I and II, Cambridge University Press (1986).
- [2] See, for instance:
J. Kogut and K. Wilson, Physics Reports 12 (1974) 75;
A.M. Polyakov, "Gauge Fields and Strings", Harwood Academic Press, N.Y. (1987);
J. Cardy, Lectures at the 1988 Les Houches Summer School, eds. E. Brezin and J. Zinn-Justin.
- [3] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575.
- [4] A.A. Belavin, A.M. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
- [5] D. Friedan, Lectures at the 1983 Les Houches Summer School, eds. J.B. Zuber and R. Stora.
- [6] V. Kac, in "Lecture Notes in Physics", Springer Verlag (1978).
- [7] V. Dotsenko and V.I. Fateev, Nucl. Phys. B240 (1984) 312; B251 (1985) 691; Phys. Lett. 154B (1985) 291.
- [8] E. Verlinde, Utrecht preprint THU-88/17 (1988).
- [9] G. Moore and N. Seiberg, Princeton preprint IASSNS-HEP-88/8 (1988).
- [10] J. Fröhlich, in Proceedings of the Cargèse Summer School on "Progress in Field Theory and Statistical Mechanics" (1986), eds. A. Jaffe et al., to be published in Plenum Press.
- [11] E. D'Hoker and P.H. Phong, Princeton University preprint PUPT-1039 (1988), to appear in Rev. Mod. Phys.
- [12] H. Farkas and I. Kra, "Riemann Surfaces", Springer Verlag (1982).
- [13] For a readable account of Mumford forms and references to the relevant literature, see:
P. Nelson, "Lectures on Supermanifolds and Strings", to appear in the Proceedings of the Theoretical Advanced Study Institute, Brown University, June 1988.
- [14] G. Segal, talk at the Rindberg Schloss Meeting (March 1987), and to appear in Comm. Math. Phys.

- [15] A. Neveu and P. West, Phys. Lett. 168B (1985) 192; Nucl. Phys. B278 (1986) 601; Phys. Lett. B193 (1987) 187; Comm. Math. Phys. 114 (1988) 613; Phys. Lett. B194 (1987) 200; B200 (1987) 275; see also:
P. West, "A Review of Duality, String Vertices, Overlap Identities and the Group Theoretical Approach to Strings", CERN preprint TH.4819/87, and references therein.
- [16] P. Di Vecchia, R. Nakayama, J.L. Petersen and S. Sciuto, Nucl. Phys. B282 (1987) 179;
P. Di Vecchia, R. Nakayama, J.L. Petersen, J. Sidenius and S. Sciuto, Phys. Lett. B182 (1986) 164;
P. Di Vecchia, M. Frau, A. Lerda and S. Sciuto, Nucl. Phys. B298 (1988) 526; Phys. Lett. B199 (1987) 49;
P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda and S. Sciuto, Phys. Lett. B211 (1988) 301;
P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and S. Sciuto, NORDITA preprint 88/47P (1988).
- [17] P. Di Vecchia, R. Nakayama, J.L. Petersen, J. Sidenius and S. Sciuto, Nucl. Phys. B287 (1987) 621;
J.L. Petersen and J. Sidenius, Nucl. Phys. B301 (1988) 247;
J.L. Petersen, K.O. Roland and J.R. Sidenius, Phys. Lett. B205 (1988) 262;
K.O. Roland, Niels Bohr Inst. preprint NBI-HE-88-21;
J.L. Petersen, J.R. Sidenius and A. Tollstén, NBI-HE-88-30.
- [18] A. Leclair, Nucl. Phys. B297 (1988) 607; B303 (1988) 189;
A. Leclair, M.E. Peskin and C.R. Preitschopf, SLAC-PUB-4306, 4307 and 4464 (1988).
- [19] N. Ishibashi, Y. Matsuo and H. Ooguri, Mod. Phys. Lett. A2 (1987) 119;
L. Alvarez-Gaumé, C. Gomez and C. Reina, Phys. Lett. B190 (1987) 55;
C. Vafa, Phys. Lett. B190 (1987) 47;
L. Alvarez-Gaumé, C. Gomez, G. Moore and C. Vafa, Nucl. Phys. B303 (1988) 455;
L. Alvarez-Gaumé, C. Gomez, P. Melançon, G. Sierra and C. Vafa, CERN preprint TH.5018/88, to appear in Nucl. Phys. B.
- [20] D. Friedan and S. Shenker, Phys. Lett. B175 (1986) 287; Nucl. Phys. B281 (1987) 509.
- [21] See Ref. [2].