BOSONIZATION OF Z_N PARA-FERMIONS
AND $SU(2)_N$ KAC-MOODY ALGEBRA

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ABSTRACT

We extend Nemeschansky's bosonization to all parafermion currents of the Z_N theory and give an alternative bosonization for the $\hat{su}(2)_N$ Kac-Moody currents and the Z_N parafermions, based on Wakimoto's construction.

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1. Introduction

Bosonization is a powerful tool in two-dimensional conformal field theory. Since it was first noticed¹⁾ that soliton operators of a bosonic theory could behave as fermions, tremendous progress has been achieved, culminating with the vertex operator construction of level one simply-laced Kac-Moody algebras by Frenkel and Kac²⁾ and Segal³⁾. (For a review see ref.⁴⁾.) The bose-fermi equivalence^{1,5)} has proved extremely useful for understanding the covariant description of superstrings on the sphere⁶⁾ and on higher genus surfaces.⁷⁾

In general, a given bosonization procedure (by means of vertex type operators : $e^{i\alpha\phi}$:) involves a well-defined lattice of allowed zero-modes of the bosonic fields, corresponding to a compactification on the dual torus. The Frenkel-Kac construction²⁾ uses the root lattice of the simply-laced Lie algebra. Its basis vectors have length squared two. The bose-fermi equivalence uses lattices containing vectors of length one (which e.g. might be the three sets of vector, spinor and conjugate spinor weights of $so(8)$ as in Goddard et al⁶⁾). The operator product expansions (OPE) of these vertex operators then contain singularities which are double or simple poles as is appropriate for commuting or anticommuting operators, respectively. One can consider lattices which are not integer, leading to singularities in the OPE which are not just poles but involve branch cuts. These operators are non-local and known as parafermions⁸⁾.

During the last two years, parafermion theories have received a lot of interest, since they are related to Kac-Moody algebras at arbitrary level. The Z_N parafermions of Zamolodchikov and Fateev⁸⁾ are related to $\hat{su}(2)_N$. The corresponding partition functions and modular invariants have been studied by Gepner and Qiu⁹⁾. More general parafermion systems allow to construct any Kac-Moody algebra at any level.¹⁰⁾

Recently, Nemeschansky¹¹⁾ expressed the basic parafermion fields ψ and $\psi^\dagger$ of the Z_N theory in terms of two bosons associated with a Lorentzian lattice, the lengths of its basic vectors being $1/\sqrt{2N}$ and $1/\sqrt{2(N+2)}$.^{*} Adding one more free boson, this allowed him to obtain a bosonic construction of the $su(2)$ Kac-Moody algebra at any level N . The Virasoro central charge being no longer integer (as is the case for the level one construction^{2,3)}) the bosonic theory necessarily contains a background charge and is of the Feigin-Fuchs type¹²⁾ as studied in detail by Dotsenko and Fateev¹³⁾ (for the case

* We use a normalisation for the bosonic fields which differs by a factor of $\sqrt{2}$ from the one of Nemeschansky. Hence the same is true for the radii.

$c < 1$) and by Gervais and Neveu¹⁴⁾ (for the case $c > 1$ in the context of the Liouville theory).

The aim of the present letter is to study in more detail the bosonic representation of the Z_N parafermions. After briefly reviewing some properties of the Z_N parafermion algebra in section 2, we generalize Nemeschansky's construction to all parafermion currents of the Z_N theory (section 3). In section 4, we obtain a different bosonization of the parafermions and of $\hat{su}(2)_N$ by bosonizing the Wakimoto construction¹⁵⁾ which expresses the $\hat{su}(2)$ level N currents in terms of a $\beta\gamma$ ghost system and a boson with background charge. Some conclusions and outlooks are presented in section 5.

2. The Z_N parafermion systems and $\hat{su}(2)_N$

The Z_N parafermion algebra of Zamolodchikov and Fateev⁸⁾ contains $N - 1$ parafermion currents ψ_m , $m = 1, 2, \dots, N - 1$ and the identity operator $I \equiv \psi_0$. One also writes $\psi_m^\dagger = \psi_{N-m} \equiv \psi_{-m}$, all indices being *mod* N . These parafermion currents are primary fields of the Virasoro algebra with conformal dimensions

$$\Delta_m = \frac{m(N-m)}{N} = \Delta_{N-m} \quad (1)$$

Hence, ψ_m and $\psi_m^\dagger$ have the same dimension. The two-point functions are normalized as

$$\langle \psi_m(z) \psi_{m'}^\dagger(w) \rangle = \delta_{mm'} (z-w)^{-2\Delta_m} \quad (2)$$

The OPEs of the ψ_m 's among themselves and with the energy-momentum tensor T_ψ are⁸⁾

$$\psi_m(z) \psi_{m'}(w) = c_{m,m'} (z-w)^{-2mm'/N} (\psi_{m+m'}(w) + O(z-w)) \quad m+m' < N \quad (3.a)$$

$$\psi_m(z) \psi_{m'}^\dagger(w) = c_{m,N-m'} (z-w)^{-2m(N-m')/N} (\psi_{m-m'}(w) + O(z-w)) \quad m' < m \quad (3.b)$$

$$\psi_m(z) \psi_m^\dagger(w) = (z-w)^{-2m(N-m)/N} \left(I + \frac{2\Delta_m}{c_\psi} (z-w)^2 T_\psi(w) + O((z-w)^3) \right) \quad (3.c)$$

$$T_\psi(z) \psi_m(w) = \frac{\Delta_m}{(z-w)^2} \psi_m(w) + \frac{1}{z-w} \partial_w \psi_m(w) + O(1) \quad (4)$$

where c_ψ is the central charge of the Virasoro algebra,

$$c_\psi = \frac{2(N-1)}{N+2} \quad (5)$$

and the structure constants $c_{m,m'}$ are given by⁸⁾

$$c_{m,m'}^2 = \frac{\Gamma(m+m'+1)\Gamma(N-m+1)\Gamma(N-m'+1)}{\Gamma(m+1)\Gamma(m'+1)\Gamma(N-m-m'+1)\Gamma(N+1)} \quad (6)$$

By adding a free bosonic field ϕ_3 , compactified on a circle of radius $\sqrt{2N}$, with a two-point function $\langle \phi_3(z)\phi_3(w) \rangle = -\ln(z-w)$ and energy-momentum tensor $T_{\phi_3} = -\frac{1}{2}(\partial\phi_3)^2$ one constructs the $su(2)$ level N Kac-Moody currents as

$$J^\pm(z) = \sqrt{N}\psi_{\pm 1}(z)\exp(\pm i\sqrt{2/N}\phi_3(z)), \quad J_3(z) = i\sqrt{N/2} \partial_z\phi_3(z) \quad (7)$$

which, by virtue of (3.c), satisfy the OPE's characteristic for the $su(2)$ level N Kac-Moody algebra:

$$\begin{aligned} J_3(z)J^\pm(w) &\sim \pm \frac{2}{z-w}J^\pm(w) + O(1), & J_3(z)J_3(w) &\sim \frac{N/2}{(z-w)^2} + O(1) \\ J^+(z)J^-(w) &\sim \frac{N}{(z-w)^2} + \frac{2}{z-w}J_3(w) + O(1) \end{aligned} \quad (8)$$

The energy-momentum tensor is obtained by the Sugawara construction⁴⁾ as the regular part of these OPEs:

$$\begin{aligned} T_{Sugawara}(w) &= \frac{1}{N+2} \lim_{z \rightarrow w} \left(J_3(z)J_3(w) + \frac{1}{2}J^+(z)J^-(w) + \frac{1}{2}J^-(z)J^+(w) - \frac{3N/2}{(z-w)^2} \right) \\ &= T_\psi + T_{\phi_3} \end{aligned} \quad (9)$$

Let us mention that more general Z_N parafermion systems exist⁸⁾. They have e.g. conformal dimensions and central charge

$$\Delta_m = \frac{2m(N-m)}{N}, \quad c_\psi = \frac{4(N-1)(N+\lambda-1)\lambda}{(N+2\lambda)(N+2\lambda-2)} \quad (10)$$

with an arbitrary parameter λ . For $\lambda = \frac{1}{2}$, this coincides with the even Z_{2N} algebra (the subalgebra of the ψ_{2m} 's), and for $\lambda = 1$ with the tensor product of two copies of the Z_N algebra. Apart from a brief remark later on, we will not consider these more general parafermion systems in the following.

3. Bosonization of the Z_N parafermion currents

Nemeschansky¹¹⁾ found the following construction of the parafermions $\psi \equiv \psi_1$ and $\psi^\dagger \equiv \psi_1^\dagger \equiv \psi_{-1}$ in terms of two bosons ϕ_1 and ϕ_2 ^{*}:

$$\psi_{\pm 1}(z) = \frac{1}{\sqrt{2}} \left(\partial\phi_2 \mp \sqrt{\frac{N+2}{N}} \partial\phi_1 \right) e^{\pm i\sqrt{\frac{2}{N}}\phi_2} \quad (11)$$

ϕ_1 and ϕ_2 are compactified on a Lorentzian lattice with signature $(+, -)$, the compactification radii being $\sqrt{2(N+2)}$ and $\sqrt{2N}$, respectively. Their two-point functions are

$$\langle \phi_1(z)\phi_1(w) \rangle = - \langle \phi_2(z)\phi_2(w) \rangle = -\ln(z-w) \quad (12)$$

The field ϕ_1 has a background charge $\frac{1}{\sqrt{2(N+2)}}$ leading to a bosonic energy-momentum tensor

$$T_\psi(z) = -\frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 + \frac{i}{\sqrt{2(N+2)}}\partial^2\phi_1 \quad (13)$$

and a central charge $c_\psi = 2 - \frac{12}{2(N+2)} = \frac{2(N-1)}{(N+2)}$ in agreement with eq.(5). Using (12), it is straightforward to verify that the parafermions (11) satisfy eqs.(3.c) and (4).

We can now use the operator product algebra (3.a) to compute the other parafermion currents ψ_m by recursion. A priori the OPE of $\psi_{\pm 1}(z)$ with $\psi_{\pm 1}(w)$ contains terms as singular as $(z-w)^{-2-2/N}$ and $(z-w)^{-1-2/N}$. However all these terms cancel and one is left with[†]

$$\psi_{\pm 1}(z)\psi_{\pm 1}(w) \sim (z-w)^{-2/N} \left(\chi_\pm^2(w) \mp \frac{i}{\sqrt{N}}\partial_w\chi_\pm(w) \right) e^{\pm 2i\sqrt{\frac{2}{N}}\phi_2(w)} \quad (14)$$

where we have introduced the convenient notation

$$\chi_\pm = \frac{1}{\sqrt{2}} \left(\partial\phi_2 \mp \sqrt{\frac{N+2}{N}} \partial\phi_1 \right) \quad (15)$$

Comparing eqs.(14) and (3.a) we find, using eq.(6), that the field on the r.h.s. of (14) is $\sqrt{2(N-1)/N}$ times $\psi_{\pm 2}(w)$. Next, taking the OPE of $\psi_{\pm 1}$ with $\psi_{\pm 2}$ we get $\psi_{\pm 3}$, and so

^{*} see previous footnote

[†] eq.(14) also shows that $J^\pm(z)J^\pm(w)$ is non singular.

on. The parafermion currents so constructed explicitly read

$$\psi_{\pm 2} = d_2 \left(\chi_{\pm}^2 \mp \frac{i}{\sqrt{N}} \partial \chi_{\pm} \right) e^{\pm 2i \sqrt{\frac{2}{N}} \phi_2} \quad (16a)$$

$$\psi_{\pm 3} = d_3 \left(\chi_{\pm}^3 \mp \frac{3i}{\sqrt{N}} \chi_{\pm} \partial \chi_{\pm} - \frac{1}{N} \partial^2 \chi_{\pm} \right) e^{\pm 3i \sqrt{\frac{2}{N}} \phi_2} \quad (16b)$$

$$\psi_{\pm m} = d_m P_m(i\sqrt{N}; \chi_{\pm}, \partial \chi_{\pm}, \dots, \partial^{m-1} \chi_{\pm}) e^{\pm mi \sqrt{\frac{2}{N}} \phi_2} \quad (16c)$$

where

$$d_m = \left(\frac{N^{m-1} (N-m)!}{m! (N-1)!} \right)^{1/2} \quad (17a)$$

and P_m is a polynomial of $\chi_{\pm}$ and its derivatives, defined by

$$P_m(a; f', f'', \dots, f^{(m)}) = a^{-m} e^{-af} \partial^m e^{af} \quad (17b)$$

for any function f of one variable. As a check of eqs.(16) we have explicitly verified that the $\psi_{\pm 2}$ satisfy eqs.(3) and (4) with $\Delta_2 = 2(N-2)/N$.

One point which merits to be mentioned is the apparent difference between ψ_{-m} and ψ_{N-m} . This is already the case for Nemeschansky's formula (11) if we consider the case $N = 2$ (the Ising model). Then ψ_1 and ψ_{-1} are ordinary free Majorana fermions and should both represent the $\Delta = \frac{1}{2}$ field of the $c = \frac{1}{2}$ minimal theory. It is however a well-known feature that, in the presence of a background charge¹³⁾, the same field is represented by two apparently different vertex operators of the bosonic theory, e.g. for $c = \frac{1}{2}$ the $\Delta = \frac{1}{2}$ field corresponds to $\psi_{(2,1)}$ and $\psi_{(1,3)}$ (using the notation of ref.¹³⁾), and the identity operator ($\Delta = 0$) corresponds to $\psi_{(1,1)} \equiv 1$ and to the non-trivial field $\psi_{(2,3)}$.

We can use eqs.(16) to gain some insight into the more general Z_N parafermion theories (10). As already mentioned, for $\lambda = \frac{1}{2}$ they reduce to the even part of the Z_{2N} algebra. Thus we immediatly conclude:

$$\psi_{\pm m}^{(\lambda=1/2)} = \psi_{\pm 2m} |_{N \rightarrow 2N} \quad (18a)$$

and in particular

$$\begin{aligned} \psi_{\pm 1}^{(\lambda=1/2)} &= \left(\frac{N}{2N-1} \right)^{1/2} \left(\frac{1}{2} \left(\partial \phi_2 \mp \sqrt{\frac{N+1}{N}} \partial \phi_1 \right)^2 \mp \frac{i}{2\sqrt{N}} \left(\partial^2 \phi_2 \mp \sqrt{\frac{N+1}{N}} \partial^2 \phi_1 \right) \right) \\ &\quad \times \exp \left(\pm \frac{2i}{\sqrt{N}} \phi_2 \right) \end{aligned} \quad (18b)$$

On the other hand, for $\lambda = 1$, the $\psi_{\pm m}^{(\lambda=1)}$ are just tensor products of two copies of the $\psi_{\pm m}$, and hence involve four bosonic fields. It would be interesting to obtain a general bosonization formula for any λ . Work in this direction is in progress¹⁶⁾.

4. Bosonization of the Wakimoto construction

The Wakimoto construction¹⁵⁾ of the $su(2)$ level N Kac-Moody algebra with Virasoro central charge $c = \frac{3N}{N+2}$ involves a $\beta\gamma$ ghost system of central charge $c_{\beta\gamma} = 2$ and a free boson Φ with a background charge such that $c_\Phi = 1 - \frac{6}{N+2}$. The two-point functions are

$$\langle \Phi(z)\Phi(w) \rangle = -\ln(z-w), \quad \langle \beta(z)\gamma(w) \rangle = -\langle \gamma(z)\beta(w) \rangle = \frac{1}{z-w} \quad (19)$$

The Kac-Moody currents are then constructed as follows

$$\begin{aligned} J^- &= -\beta, \quad J^3 = \beta\gamma + i\sqrt{\frac{N+2}{N}}\partial\Phi \\ J^+ &= \gamma^2\beta + i\sqrt{2(N+2)}\gamma\partial\Phi - N\partial\gamma \end{aligned} \quad (20)$$

It is straightforward to verify that these currents satisfy the $\hat{su}(2)_N$ algebra (8), and moreover that the Sugawara energy-momentum tensor as given by (9) is

$$T_{\text{Sugawara}} = \beta\partial\gamma - \frac{1}{2}(\partial\Phi)^2 + \frac{i}{\sqrt{2(N+2)}}\partial^2\Phi = T_{\beta\gamma} + T_\Phi \quad (21)$$

It is convenient to bosonize the $\beta\gamma$ -system¹⁷⁾:

$$\begin{aligned} \beta &= \xi e^{i\phi}, \quad \gamma = \partial\eta e^{-i\phi}, \quad \langle \phi(z)\phi(w) \rangle = \ln(z-w) \\ \xi &= e^{i\chi}, \quad \eta = e^{-i\chi}, \quad \langle \chi(z)\chi(w) \rangle = -\ln(z-w) \end{aligned} \quad (22)$$

Then

$$T_{\beta\gamma} = -\xi\partial\eta + \frac{1}{2}(\partial\phi)^2 + \frac{i}{2}\partial^2\phi = -\frac{1}{2}(\partial\chi)^2 - \frac{i}{2}\partial^2\chi + \frac{1}{2}(\partial\phi)^2 + \frac{i}{2}\partial^2\phi \quad (23)$$

and β and γ have conformal weight 1 and 0, respectively, ξ has weight 1, $e^{i\phi}$ weight 0, η

has weight 0 and $e^{-i\phi}$ weight -1 . The central charges are $c_\chi = 1 - 12(-\frac{1}{2})^2 = -2$ and $c_\phi = 1 + 12(\frac{1}{2})^2 = 4$ adding up to $c_{\beta\gamma} = 2$.

Now we can proceed to bosonize the currents J^a of eq.(20). This is straightforward, except for the $\gamma^2\beta$ term which requires some care. The result is

$$J^- = -e^{i\chi+i\phi}, \quad J^3 = i\partial\phi + i\sqrt{\frac{N+2}{2}}\partial\Phi$$

$$J^+ = \left((N+1)((\partial\chi)^2 + i\partial^2\chi) + (N+2)\partial\chi\partial\phi + \sqrt{2(N+2)}\partial\chi\partial\Phi \right) e^{-i\chi-i\phi} \quad (24)$$

Comparing with eq.(7), we decompose the currents into their parafermion and their ϕ_3 part. Looking at J^3 immediatly tells us that

$$\sqrt{\frac{2}{N}}\phi + \sqrt{\frac{N+2}{N}}\Phi = \phi_3 \quad (25)$$

The parafermion currents then are identified as

$$\psi_{-1} = -\frac{1}{\sqrt{N}} \exp i \left(\chi + \frac{N+2}{N}\phi + \frac{\sqrt{2(N+2)}}{N}\Phi \right)$$

$$\psi_1 = \frac{1}{\sqrt{N}} \left((N+1)((\partial\chi)^2 + i\partial^2\chi) + (N+2)\partial\chi\partial\phi + \sqrt{2(N+2)}\partial\chi\partial\Phi \right)$$

$$\times \exp -i \left(\chi + \frac{N+2}{N}\phi + \frac{\sqrt{2(N+2)}}{N}\Phi \right) \quad (26)$$

It is convenient to change the basis of bosonic fields from χ, ϕ, Φ to ϕ_3, σ, ν by eq.(25) and

$$\sigma = \sqrt{\frac{N}{2}}\chi + \frac{N+2}{\sqrt{2N}}\phi + \sqrt{\frac{N+2}{N}}\Phi$$

$$\nu = \sqrt{\frac{N+2}{2}}\chi + \sqrt{\frac{N+2}{2}}\phi + \Phi \quad (27a)$$

$$\langle \nu(z)\nu(w) \rangle = - \langle \sigma(z)\sigma(w) \rangle = -\ln(z-w) \quad (27b)$$

Then the parafermion currents become

$$\psi_{-1} = -\frac{1}{\sqrt{N}} e^{i\sqrt{\frac{2}{N}}\sigma}$$

$$\psi_1 = \frac{1}{\sqrt{N}} \left(\frac{N+2}{2} (\partial\nu)^2 - \frac{N}{2} (\partial\sigma)^2 + i(N+1) \sqrt{\frac{N+2}{2}} \partial^2\nu - i(N+1) \sqrt{\frac{N}{2}} \partial^2\sigma \right) e^{-i\sqrt{\frac{2}{N}}\sigma} \quad (28)$$

and the energy-momentum tensor reads

$$T_\psi = T_{\beta\gamma} + T_\Phi - T_{\phi_3} = -\frac{1}{2} (\partial\nu)^2 + \frac{1}{2} (\partial\sigma)^2 - i \frac{N+1}{\sqrt{2(N+2)}} \partial^2\nu + i \sqrt{\frac{N}{2}} \partial^2\sigma \quad (29)$$

with central charge

$$c_\psi = 2 - 12 \left(-\frac{N+1}{\sqrt{2(N+2)}} \right)^2 + 12 \left(\sqrt{\frac{N}{2}} \right)^2 = \frac{2(N-1)}{N+2}$$

in agreement with eq.(5).

It is straightforward to check that both ψ_1 and ψ_{-1} as given by eq.(28) have conformal dimension $\Delta_1 = 1 - \frac{1}{N}$ in agreement with eq.(1), and that the OPE of $\psi_{-1}(z)$ and $\psi_1(w)$ yields $(z-w)^{-2+2/N} (I + \frac{N+2}{N} T_\psi(w)(z-w)^2 + \dots)$. To check that the OPE of $\psi_1(z)$ and $\psi_{-1}(w)$ gives the same result, it is convenient to exploit the remarkable fact that

$$\psi_1 = -\frac{1}{\sqrt{N}} \left((N+2)T_\psi + \frac{N}{2} \partial^2 \right) e^{-i\sqrt{\frac{2}{N}}\sigma} \quad (30)$$

The other parafermion currents $\psi_{\pm m}$ can be computed as in section 3 by repeatedly taking OPE's of ψ_{-1} 's or ψ_1 's. The result is

$$\psi_{-m} = (-)^m \left(\frac{(N-m)!}{m!N!} \right)^{1/2} e^{mi\sqrt{\frac{2}{N}}\sigma} \quad (31)$$

The expression for ψ_m is more complicated and will not be displayed here.

5. Conclusion and Outlook

In this letter we have completed Nemeschansky's bosonization of the Z_N parafermion current algebra¹¹⁾ in the sense that we computed all parafermion currents $\psi_{\pm m}$, not only $\psi_{\pm 1}$, in terms of two bosons with a background charge. This also gives the regular part of the OPE's of several $\hat{su}(2)_N$ Kac-Moody currents J^+ or J^- , which might prove useful for the study of $\hat{su}(2)_N$ representation theory in this bosonized language.

By bosonizing the Wakimoto construction¹⁵⁾ we obtained an alternative bosonic expression for the Z_N parafermions and of the $\hat{su}(2)$ level N Kac-Moody currents. The hermiticity properties of the Kac-Moody and parafermion currents resulting from the Wakimoto construction are much more subtle than in the first bosonization scheme, but for different computational applications, one or the other bosonization might reveal itself more appropriate.

It would be interesting to obtain a bosonization of the more general Z_N parafermion systems depending on a parameter λ , as discussed briefly at the ends of sections 2 and 3, since these theories cover a wide range of central charges¹⁶⁾. Even more important would be a bosonization of the parafermions associated with Lie algebras other than $su(2)$ ¹⁰⁾, leading to a Feigin-Fuchs type bosonic representation of any Kac-Moody algebra at any level.

6. Note added

After completion of this work, we became aware of several related papers: Fucito¹⁸⁾ achieves the bosonization of parafermions and WZW models by constructing the N^{th} root of the canonical line bundle on the sphere. This approach is rather formal and comparison of the correlation functions of the bosonic and the parafermionic or WZW theories has not yet been achieved. Dunne, Halliday and Suranyi¹⁹⁾ obtain a bosonization of the currents of a simply-laced Kac-Moody algebra at arbitrary level N and of the corresponding parafermion algebra by tensoring N copies of the level 1 Frenkel-Kac construction. This leads to constructions which differ from level to level, in contrast with the approach of Nemeschansky or the one of the present letter where the level N appears as a parameter determining the background charge. Also, their bosonic energy-momentum tensor has not the form appropriate for a theory of free bosonic fields, but explicitly contains soliton operators. Finally, a very recent paper by Gerasimov et al²⁰⁾ also bosonizes the Wakimoto

construction and gets bosonic expressions for the Kac-Moody currents which look slightly different from ours but are essentially equivalent.

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