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COHERENT PAIR PRODUCTION IN LINEAR COLLIDERS

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ABSTRACT

The production of an electron-positron pair by a photon impinging on a high-density bunch of electrons is calculated in the deep quantum regime, typically found in linear colliders in the multi-TeV range. We obtain separately a leading contribution proportional to the bunch length and logarithmic edge effects. Pair production is found to be considerable.

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1. - INTRODUCTION

Electron-positron linear colliders in the TeV range are presently considered with much attention as very powerful future machines [1]. The intense luminosity which such machines have to achieve in order to study processes with cross-sections which typically decrease like the square of the collision energy imposes very high bunch densities. When bunch collide and annihilation processes can be observed, a large amount of photon radiation is simultaneously produced because of the very high fields in the bunch. Such a bremsstrahlung at the level of a whole bunch is referred to as "beamstrahlung" [2].

As one reaches the TeV level and the luminosity necessary at such energies, the radiation regime is likely to change dramatically. One is indeed driven into the so-called deep quantum regime. It is characterized by $T \gg 1$, where T is a key parameter in synchrotron radiation, namely

$$T = \frac{\gamma^2}{m \rho_c} \quad (1)$$

Here γ is the Lorentz contraction factor $\gamma = E/m$, and ρ_c the radius of curvature for an electron of energy E . At very high energy and very high bending fields (bunch density), T is pushed to high values whereas, in the classical regime, it is merely the ratio between the typical energy radiated and the electron incident energy. In that case, of course, $T \ll 1$.

The study of beamstrahlung in the deep quantum regime has recently received much attention [3-5]. One finds that the energy losses are considerable and correspond to a rather hard photon spectrum. While this intense radiation raises serious problems for experimentation on such machines, it can in some cases be considered a bonus, an electron-positron linear collider being also an efficient photon-photon collider [6].

These energetic and copiously-produced photons should in turn produce electron-positron pairs in the intense field of the bunch, and this certainly raises further problems. Such energetic particles, bent over a wide angular spread by the high field in the bunch, will shower the detectors and focusing devices in an undesirable way. It is therefore important to determine the corresponding production rate and the properties of these secondaries.

In this paper we study the coherent pair-production process in the deep quantum regime. We extend to this process the Feynman graph approach which we developed in the case of beamstrahlung [3,5] and calculate the production rate associated with the diagram of Fig. 1a. This is only the first step in a calculation which should eventually involve the diagram of Fig. 1b, whereby the photon

originates from the radiation of an electron traversing the bunch. The first process is, however, already quite instructive and worth a special study.

In the deep quantum regime there is a strong coherent process and the pair-production rate per unit flux ($I/\pi R^2$) is found to include a leading term:

$$\frac{I(x)}{\pi R^2} = K_p \frac{\alpha}{\pi} \frac{L_b}{\ell_c} \frac{(1-x)^2 + x^2}{(x(1-x))^{1/3}} \quad (2)$$

where K_p is a numerical factor of order 1 [7]. This is the important term insofar as it is the one proportional to the bunch length L_b . The fraction of the photon energy taken by the electron (positron) is denoted by x . The quantity λ_c is the same coherent length as that used in our previous work on beamstrahlung. It is the coherent radiation length relevant to the deep quantum regime. One indeed has

$$\ell_c = (L_e L_c^2)^{1/3} \quad (3)$$

where L_c is the classical coherent radiation length and L_e is the radiative length, $L_e = \hbar\gamma/m$, over which an electron can freely split into an electron and a photon. It is clear that the deep quantum regime corresponds to $L_e \gg L_c$ or $\lambda_c \gg L_c$. More precisely, one finds [3]

$$\gamma = \left(\frac{\ell_c}{L_c} \right)^3 \quad (4)$$

In practice, while $\lambda_c \gg L_c$, one still has $\lambda_c \ll L_b$, and

$$\Lambda = \frac{L_b}{\ell_c} \quad (5)$$

remains a large quantity, typically of the order of 10 to 100. Our previous study of the edge effect and of varying bunch density in beamstrahlung corresponds to an expansion in powers of Λ [5].

One can easily express Λ in terms of machine parameters

$$\Lambda = (\varepsilon N \alpha)^{1/3} \quad (6)$$

where N is the number of particles in the bunch of length L_b and radius R , and where

$$\varepsilon = \frac{N\alpha}{R^2} \frac{L_b}{E} \quad (7)$$

determines the disruption parameter D for small bending [3].

Our main result on pair production (2) was reported in a recent letter [7]. It corresponds to the contribution proportional to L_b . In this paper we come back in some detail to that calculation [8], and we also discuss the front edge effect, which can be looked at as production before the photon enters the bunch. Such a separation is of course somewhat arbitrary, since the production amplitude is an integral over all space. Nevertheless, one may always separate a contribution linear in Λ , which can be associated with radiation "during" bunch crossing, from contributions proportional to $\ln L_e$, which are the leading part of what can be called radiation "before" and "after" bunch crossing. These different terms were calculated and separately isolated in the case of beamstrahlung [3,5]. They add incoherently insofar as $\lambda_c \ll L_b$.

In the case of pair production, it is interesting and, one may say, natural to obtain a leading term, proportional to Λ , which is very similar to the corresponding contribution to beamstrahlung [3]. However, while photon radiation and pair production are a priori simply related by crossing, this relation is hard to exploit directly. The coherent process is indeed largely determined by coherence conditions which take a very different form in the two cases! In the case of beamstrahlung there is a phase variation due to the field inside the bunch which affects the electron wave function before and after radiation in a largely constructive way in the radiation amplitude. In the case of pair production there is, on the contrary, a strong destructive effect due to the fact that one is dealing with an electron and a positron which both propagate along similar paths in the same strong field. Something remains only insofar as the electron and the positron are not bent in quite the same way by the bunch, whereas in beamstrahlung, the mere bending of the electron trajectory throughout the bunch provides the leading effect. One sees that the phase coherence conditions are therefore bound to be very different in the two cases. The phase space conditions are also different, since for beamstrahlung bending occurs throughout the bunch, whereas, in pair production, it takes place only after the pair is formed. It turns out that, while these two different inputs lead to quite different expressions for photon radiation and pair production during the respective calculations, some remarkable matching eventually occurs, to give results for rates which show a great similarity [7]. It is this subtle combination of new coherence conditions and new phase-space limitations which we wish to discuss here in more detail than previously [7]. We now also discuss the way they affect front-edge effects, a leading $\ln L_e$ term now occurring only in the long bunch limit ($L_b \gg L_e$).

Our calculations are accurate to first order in D . The special position dependence met in pair production at the different stages of the calculation shows formal similarities to that resulting from a varying bunch density in the case of beamstrahlung. We therefore make great use of a calculation procedure developed for that purpose [5].

To discuss these new interesting pair-production effects, we keep the simplifying assumption of a bunch of uniform density with sharp boundaries, namely

$$\rho = \frac{Ne}{\pi R^2 L_b} \quad , \quad |z| < \frac{L_b}{2} \quad ; \quad \rho = 0 \quad , \quad |z| > \frac{L_b}{2} \quad (8)$$

The z axis defines the bunch axis.

2. - THE PAIR-PRODUCTION AMPLITUDE

We consider the amplitude associated with the Feynman graph of Fig. 1a. In order to simplify the presentation, we deal with the spinless case (Klein-Gordon). In the high-energy deep quantum regime, it is an excellent approximation to assume that pair production occurs only with opposite helicities as the electron mass is negligible in front of transverse momenta. The rate obtained in the Dirac case is then merely proportional to the one more simply calculated in the Klein-Gordon case, the overall proportionality factor being [7]:

$$S(x) = \frac{(1-x)^2 + x^2}{x(1-x)} \quad (9)$$

The photon with incident momentum k_γ splits into an electron of momentum k (energy Xk_γ) and a positron of momentum k' [energy $(1-X)k_\gamma$], in the field of the bunch. Unless otherwise specified, we work in the system where the bunch is at rest. The kinematics is specified in Fig. 2.

The phase of the electron wave function is determined, as was previously done for beamstrahlung. It reads [3]

$$\phi_f(\vec{x}) = \vec{k} \cdot \vec{x} + S_f + \frac{1}{2k} \int_3^\infty \left\{ (\nabla_\perp S_f)^2 + 2(\vec{z}' \cdot \vec{z}) \vec{k}_\perp \cdot \vec{\nabla}_\perp e^{V(\vec{x})} \right\} dz' \quad (10)$$

where

$$S_f = \int_z^\infty e V(\vec{z}') dz' \quad (11)$$

As previously discussed, this is an excellent approximation for low disruption parameter D. We separate the momentum component perpendicular to the bunch axis $k_\perp$ from the component along the bunch axis k_L .

The phase of the positron wave function reads, similarly:

$$\phi_{f'}(\vec{x}) = \vec{k}' \cdot \vec{x} + S_{f'} + \frac{1}{2k'} \int_z^\infty \left\{ (\nabla_\perp S_{f'})^2 + 2(\vec{z} - \vec{z}') \cdot \vec{k}'_\perp \cdot \vec{\nabla}_\perp e V(\vec{z}') \right\} dz' \quad (12)$$

One merely changes the sign of the charge with, accordingly,

$$S_{f'} = -S_f \quad (13)$$

The phase of the production amplitude varies as a function of the location ($\vec{x}$) as:

$$\phi(\vec{x}) = \vec{k}_x \cdot \vec{x} - \phi_f(\vec{x}) - \phi_{f'}(\vec{x}) \quad (14)$$

We have

$$eV(\vec{x}) = \begin{cases} 0 & z < -\frac{L_e}{2} \\ \frac{N\alpha}{L_e} \frac{r^2}{R^2} & |z| < \frac{L_e}{2} \\ 0 & z > \frac{L_e}{2} \end{cases} \quad (15)$$

and, accordingly (11),

$$S_f(\vec{x}) = \begin{cases} N\alpha \frac{r^2}{R^2} & z < -\frac{L_e}{2} \\ \frac{N\alpha}{L_e} \frac{r^2}{R^2} \left(\frac{L_e}{2} - z \right) & |z| < \frac{L_e}{2} \\ 0 & z > \frac{L_e}{2} \end{cases} \quad (16)$$

Combining (10), (12) and (14) with (13) and (16), one sees that the leading term (of order $N\alpha$) in the phase, which remains in the beamstrahlung case and controls the phase coherence conditions, now cancels out identically! One finds instead:

$$\begin{aligned}
 \phi(\vec{x}) &= \vec{h}_y \cdot \vec{x} - \vec{h} \cdot \vec{x} - \vec{h}' \cdot \vec{x} + \frac{2}{3} \frac{r^2}{R^4} (N\alpha)^2 \left(\frac{L_e}{2} + 3\delta \right) \frac{1}{h_y x(1-x)} \\
 &\quad + 2N\alpha\delta \frac{1}{R^2} \left(\vec{r} \cdot \left(\frac{\vec{h}_\perp}{h} - \frac{\vec{h}'_\perp}{h'} \right) \right) \quad ; \quad \delta < -\frac{L_e}{2} \\
 \phi(\vec{x}) &= \vec{h}_y \cdot \vec{x} - \vec{h} \cdot \vec{x} - \vec{h}' \cdot \vec{x} - \frac{2}{3} \frac{r^2}{R^4} (N\alpha)^2 \left(\frac{L_e}{2} - \delta \right)^3 \frac{1}{L_e^2 h_y x(1-x)} \\
 &\quad + \frac{N\alpha}{L_e} \left(\frac{L_e}{2} - \delta \right)^2 \frac{1}{R^2} \left(\vec{r} \cdot \left(\frac{\vec{h}_\perp}{h} - \frac{\vec{h}'_\perp}{h'} \right) \right) \quad ; \quad |\delta| < \frac{L_e}{2}
 \end{aligned}
 \tag{17}$$

This is very different from the expression found for beamstrahlung, with its dominant z-independent $-N\alpha(r^2/R^2)$ term. This has of course to do with the fact that the electron and positron follow trajectories which only differ progressively due to different deflections inside the bunch. The bending momenta are not opposite. However, if one writes

$$\vec{h}'_\perp = -\vec{h}_\perp + \vec{K}_\perp
 \tag{18}$$

then $K_\perp \sim \epsilon k_\perp$ where ϵ is given by (7). One may thus neglect $K_\perp$ in front of $k_\perp$, since we remain in the low D (linear) approximation, and simplify (17) accordingly. this gives:

$$\begin{aligned}
 \phi(\vec{x}) &= \frac{h_\perp^2 + m^2}{h_y 2x(1-x)} \delta - \vec{K}_\perp \cdot \vec{r} + \frac{1}{h_y x(1-x)} \left\{ \frac{2}{3} \frac{r^2}{R^4} (N\alpha)^2 \left(\frac{L_e}{2} + 3\delta \right) \right. \\
 &\quad \left. + 2N\alpha\delta \frac{1}{R^2} (\vec{r} \cdot \vec{h}_\perp) \right\} \quad ; \quad \delta < -\frac{L_e}{2} \\
 \phi(\vec{x}) &= \frac{h_\perp^2 + m^2}{h_y 2x(1-x)} \delta - \vec{K}_\perp \cdot \vec{r} + \frac{1}{h_y x(1-x)} \left\{ -\frac{2}{3} \frac{r^2}{R^4} \left(\frac{N\alpha}{L_e} \right)^2 \left(\frac{L_e}{2} - \delta \right)^3 \right. \\
 &\quad \left. - \frac{N\alpha}{L_e} \left(\frac{L_e}{2} - \delta \right)^2 \frac{1}{R^2} (\vec{r} \cdot \vec{h}_\perp) \right\} \quad ; \quad |\delta| < \frac{L_e}{2}
 \end{aligned}
 \tag{19}$$

It is convenient to introduce scaled variables [3], and to write

$$\begin{aligned} \xi &= \tilde{\xi} \frac{L_b}{2} \\ \vec{k}_\perp &= \tilde{k}_\perp \frac{2N\alpha}{R} \quad \text{for transverse momenta} \end{aligned} \quad (20)$$

and also to use the variable $\varepsilon = (N\alpha/R^2)(L_b/k\gamma)$ (7).

The transverse stationary phase conditions are determined by $\nabla_\perp \phi = 0$, and they read

$$\begin{aligned} \frac{\vec{r}}{R} &= \frac{3x(1-x)}{1+3\tilde{\xi}} \frac{\tilde{k}_\perp}{\varepsilon} - \frac{3\tilde{\xi}}{1+3\tilde{\xi}} \tilde{k}_\perp \quad ; \quad \xi < \frac{L_b}{2} \\ \frac{\vec{r}}{R} &= -\frac{12x(1-x)}{(1-\tilde{\xi})^3} \frac{\tilde{k}_\perp}{\varepsilon} - \frac{3\tilde{k}_\perp}{1-\tilde{\xi}} \quad ; \quad |\xi| < \frac{L_b}{2} \end{aligned} \quad (21)$$

The conditions are now z -dependent, while they were not in the case of beamstrahlung! Since bending occurs only once the pair is formed, and is effective on the phase of the amplitude (14) only as the pair separates, this z dependence is, however, natural.

It disappears only in the limit of large $|\tilde{z}|$, i.e., when the bunch appears very short in comparison with the distance involved when considering the amplitude before bunch crossing ($\tilde{z} < -1$). In this ($L_b \rightarrow 0$) limit, the bunch merely provides opposite transverse kicks to the electron and the positron. In general, the z dependence is there and complicates the calculation significantly as compared to the case of beamstrahlung.

The partial derivative of the phase with respect to $\tilde{z}$ is (19)

$$\begin{aligned} \frac{\partial \phi}{\partial \tilde{z}} &= \frac{\Sigma N\alpha}{x(1-x)} \left\{ \left| \tilde{k}_\perp + \frac{\vec{r}}{R} \right|^2 + \tilde{m}^2 \right\} \quad ; \quad \tilde{z} < -1 \\ \frac{\partial \phi}{\partial \tilde{z}} &= \frac{\Sigma N\alpha}{x(1-x)} \left\{ \left| \tilde{k}_\perp + \frac{\vec{r}}{R} (1-\tilde{\xi}) \right|^2 + \tilde{m}^2 \right\} \quad ; \quad |\tilde{z}| < 1 \end{aligned} \quad (22)$$

with $m = (2N\alpha/R)\tilde{m}$ (20). One then substitutes (21) into (22) to obtain

$$\begin{aligned}\frac{\partial \phi}{\partial \tilde{z}} &= \frac{\Sigma N \alpha}{x(1-x)} \left\{ \left| \frac{3x(1-x)}{1+3\tilde{z}} \frac{\tilde{k}_\perp}{\Sigma} + \frac{1}{1+3\tilde{z}} \tilde{k}_\perp \right|^2 + \tilde{m}^2 \right\} ; \tilde{z} < -1 \\ \frac{\partial \phi}{\partial \tilde{z}} &= \frac{\Sigma N \alpha}{x(1-x)} \left\{ \frac{1}{4} \left| \frac{12x(1-x)}{(1-\tilde{z})^2} \frac{\tilde{k}_\perp}{\Sigma} + \tilde{k}_\perp \right|^2 + \tilde{m}^2 \right\} ; |\tilde{z}| < 1\end{aligned}\quad (23)$$

Although ϵ is small, $\epsilon N \alpha$ is large. One has

$$(\Sigma N \alpha)^{1/3} = \frac{L_c}{\ell_c} = \Lambda \quad (24)$$

using the definition of ℓ_c [3], namely

$$\ell_c = \left(\frac{R^2 L_c^2 k_r}{(N \alpha)^2} \right)^{1/3} \quad (25)$$

As in the case of beamstrahlung, there is no real point with a stationary phase. Since one needs the phase only up to a constant term, one may integrate (23) over $\tilde{z}$, matching the values at $\tilde{z} = -1$ where the two derivatives are clearly equal. As in beamstrahlung, one finds that $\Lambda^{-1}(\ell_c)$ sets the scale for the variation of the phase according to $\tilde{z}(z)$ in the stationary-phase zone determined by the kinematics. It is the relevant coherent length.

One may now proceed with the calculation of the production matrix element. It is a two-component vector $\mathbf{m}_\perp$ which should eventually be contracted with the photon polarization vector. We evaluate it at the point of the stationary phase with respect to the transverse co-ordinates. In the Klein-Gordon case it is readily obtained from the phase of the wave function (10, 12). Neglecting terms of order k_γ^{-1} , it reads

$$\begin{aligned}m_\perp &= ie \frac{4N\alpha}{R} \frac{1}{1+3\tilde{z}} \left(3x(1-x) \frac{\tilde{k}_\perp}{\Sigma} + \tilde{k}_\perp \right) ; \tilde{z} < -1 \\ m_\perp &= -ie \frac{2N\alpha}{R} \left(\frac{12x(1-x)}{(1-\tilde{z})^2} \frac{\tilde{k}_\perp}{\Sigma} + \tilde{k}_\perp \right) ; |\tilde{z}| < 1\end{aligned}\quad (26)$$

Since the coherent production length is ℓ_c (which sets the pace for the variation of the phase as a function of z), and since $\ell_c \ll L_b$, one can consider separately

the production process occurring inside the bunch ($|z| < L_b/2$) and the rest. This will be our first goal. We are in a regime where m is negligible in front of the bending momenta. One notices that, as in the case of beamstrahlung, the modulus square of the matrix elements is proportional to the derivative of the phase of the amplitude [(23) and (26)]. We shall again exploit this relation.

The points of the stationary phase are complex. They coincide on the real axis only in the coplanar case, when $K_\perp$ and $k_\perp$ are proportional. In that case, the stationary point $\tilde{z}_0$ corresponds to

$$(1 - \tilde{z}_0)^2 = - \frac{12x(1-x)}{\tilde{k}_\perp} \frac{\tilde{k}_\perp}{\varepsilon} \quad (27)$$

expanding $\partial\phi/\partial\tilde{z}$ around $\tilde{z}_0$, one finds

$$\frac{\partial\phi}{\partial\tilde{z}} = - \frac{\Sigma N\alpha}{12(x(1-x))^2} \frac{\tilde{k}_\perp^3 \varepsilon}{\tilde{K}_\perp} (\tilde{z}_0 - \tilde{z})^2 + \dots \quad (28)$$

and a phase variation given by

$$\phi(\tilde{z}) = \frac{\Lambda^3}{36(x(1-x))^2} \frac{\tilde{k}_\perp^3 \varepsilon}{\tilde{K}_\perp} (\tilde{z}_0 - \tilde{z})^3 + \dots \quad (29)$$

One again finds the cubic variation met in the case of beamstrahlung and the special rôle of λ_c as a coherent length. This is of course only meant as a clarifying remark, since we actually do not need this coplanar approximation.

3. - THE PRODUCTION RATE PROPORTIONAL TO THE BUNCH LENGTH

This corresponds to $|\tilde{z}| < 1$, which we can consider separately since $\lambda_c \ll L_b$. The production amplitude, written as a two-component vector, to be contracted with the photon polarization vector, reads (26)

$$M_\perp = -ie \int dz \int dx dy e^{i\phi(\vec{x})} \frac{2N\alpha}{R} \left(\frac{12x(1-x)}{(1-\tilde{z})^2} \frac{\tilde{k}_\perp}{\varepsilon} + \tilde{k}_\perp \right) \quad (30)$$

The integral over x and y is easily done using the stationary phase conditions (21). The coefficient of r^2 in (17) is

$$- \frac{2}{3} \frac{(N\alpha)^2}{L_\ell^2 R^4} \left(\frac{k}{2} - \tilde{z}\right)^3 \frac{1}{k_y x(1-x)} = - \Sigma \frac{N\alpha}{R^2} \frac{(1-\tilde{z})^3}{12x(1-x)} \quad (31)$$

Integrating over x and y , one obtains the phase at r (21), as already used in (23), and an overall factor

$$- i \frac{\pi R^2}{\Sigma N\alpha} \frac{12x(1-x)}{(1-\tilde{z})^3}$$

The modulus square of the production amplitude averaged over initial polarization is therefore

$$\begin{aligned} \overline{|M|^2} = & \frac{4\pi\alpha}{2} L_\ell^2 \frac{\pi^2 R^2}{\Sigma^2} (12x(1-x))^2 \int d\tilde{z} d\tilde{z}' \frac{1}{(1-\tilde{z})^3 (1-\tilde{z}')^3} \\ & e^{i(\phi(\tilde{z}) - \phi(\tilde{z}'))} \left(\frac{12x(1-x)}{(1-\tilde{z})^2} \frac{\tilde{k}_\perp}{\Sigma} + \tilde{h}_\perp \right) \cdot \left(\frac{12x(1-x)}{(1-\tilde{z}')^2} \frac{\tilde{k}_\perp}{\Sigma} + \tilde{h}_\perp \right) \end{aligned} \quad (32)$$

The integrals over $\tilde{z}$ and $\tilde{z}'$ now extend over the -1 to $+1$ interval. We have averaged over initial polarization. The production rate is related to $|\bar{M}|^2$ through a phase-space integral, namely

$$\mathcal{I} = \frac{1}{(2\pi)^5} \frac{1}{8 k_y^2} \int \overline{|M|^2} \frac{dx}{x(1-x)} d^2\vec{k}_\perp d^2\vec{k}'_\perp \quad (33)$$

Combining (32) and (33), we get

$$\frac{\mathcal{I}(x)}{\pi R^2} = \frac{\alpha}{\pi} (N\alpha)^2 \frac{18}{\pi^2} x(1-x) \int d\tilde{z} d\tilde{z}' F(\tilde{z}, \tilde{z}') \quad (34)$$

with

$$F(\tilde{z}, \tilde{z}') = \int d^2 \tilde{k}_\perp d^2 \tilde{k}'_\perp \frac{1}{(1-\tilde{z})^3 (1-\tilde{z}')^3} e^{i(\phi(\tilde{z}) - \phi(\tilde{z}'))}$$

$$\left(\frac{12 \times (1-x)}{(1-\tilde{z})^2} \frac{\tilde{k}_\perp}{\Sigma} + \tilde{k}'_\perp \right) \cdot \left(\frac{12 \times (1-x)}{(1-\tilde{z}')^2} \frac{\tilde{k}'_\perp}{\Sigma} + \tilde{k}_\perp \right)$$

(35)

The phase $\phi(\tilde{z})$, which we need up to a constant term, is obtained by integrating (23), namely

$$\phi(\tilde{z}, \tilde{z}') = \frac{\Sigma N \alpha}{4 \times (1-x)} \left\{ (12 \times (1-x))^2 \frac{\tilde{k}_\perp^2}{\Sigma^2} T(\tilde{z}, \tilde{z}') \right.$$

$$\left. + 24 \times (1-x) \frac{\tilde{k}_\perp}{\Sigma} \cdot \tilde{k}'_\perp U(\tilde{z}, \tilde{z}') + (\tilde{k}_\perp^2 + 4 \tilde{m}^2)(\tilde{z} - \tilde{z}') \right\}$$

(36)

where

$$T(\tilde{z}, \tilde{z}') = \frac{1}{3} \left(\frac{1}{(1-\tilde{z})^3} - \frac{1}{(1-\tilde{z}')^3} \right)$$

and

$$U(\tilde{z}, \tilde{z}') = \frac{1}{1-\tilde{z}} - \frac{1}{1-\tilde{z}'}$$

(37)

Here $\phi(\tilde{z}, \tilde{z}') = \phi(\tilde{z}) - \phi(\tilde{z}')$. This global phase factor is written as

$$\phi(\tilde{z}, \tilde{z}') = A \frac{\tilde{k}_\perp^2}{\Sigma^2} + B \frac{\tilde{k}_\perp}{\Sigma} \cdot \tilde{k}'_\perp + C \tilde{k}_\perp^2 + D \tilde{m}^2$$

(38)

where (36)

$$\begin{aligned}
 A &= \frac{\Sigma N \alpha}{4x(1-x)} \left((12x(1-x))^2 T(\tilde{z}, \tilde{z}') \right) \\
 B &= \frac{\Sigma N \alpha}{4x(1-x)} 24x(1-x) U(\tilde{z}, \tilde{z}') \\
 C &= \frac{\Sigma N \alpha}{4x(1-x)} (\tilde{z} - \tilde{z}') \\
 D &= 4C
 \end{aligned} \tag{39}$$

Our method follows that developed in the case of beamstrahlung when discussing fluctuations in bunch density [5]. We define a quantity

$$F_0(\tilde{z}, \tilde{z}') = \int d^2 \tilde{h}_\perp d^2 \tilde{h}'_\perp e^{i\phi(\tilde{z}, \tilde{z}')} \tag{40}$$

and exploit the close relationship between the argument of the phase factor and the matrix element squared to generate the different terms in (35) through partial differentiation of (40). Acting on F_0 ,

$$\frac{\partial}{\partial A} \rightarrow i \frac{\tilde{k}_\perp^2}{\Sigma^2} \quad \frac{\partial}{\partial B} \rightarrow i \frac{\tilde{k}_\perp}{\Sigma} \cdot \tilde{h}_\perp \quad \frac{\partial}{\partial C} \rightarrow i \tilde{h}_\perp^2 \tag{41}$$

and, neglecting $\tilde{m}^2$ in front of $\tilde{k}_\perp^2$, we can write (35) as

$$\begin{aligned}
 F(\tilde{z}, \tilde{z}') &= -i \frac{1}{(1-\tilde{z})^2 (1-\tilde{z}')^2} \left\{ \frac{(12x(1-x))^2}{(1-\tilde{z})^2 (1-\tilde{z}')^2} \frac{\partial}{\partial A} \right. \\
 &\quad \left. + 12x(1-x) \left(\frac{1}{(1-\tilde{z})^2} + \frac{1}{(1-\tilde{z}')^2} \right) \frac{\partial}{\partial B} + \frac{\partial}{\partial C} \right\} F_0(\tilde{z}, \tilde{z}')
 \end{aligned} \tag{42}$$

As previously said, it is clear that the particular z, z' dependence brought about by the new coherence conditions bears some formal analogy to the effect of a varying density along the bunch axis. We can therefore use the same simplifying trick [5].

From (37) and (39) we conclude that, acting on F_0 ,

$$\begin{aligned}\frac{\partial}{\partial \tilde{z}} &\equiv \frac{\Sigma N \alpha}{4x(1-x)} \left\{ \frac{(12x(1-x))^2}{(1-\tilde{z})^4} \frac{\partial}{\partial A} + \frac{24x(1-x)}{(1-\tilde{z})^2} \frac{\partial}{\partial B} + \frac{\partial}{\partial C} \right\} \\ \frac{\partial}{\partial \tilde{z}'} &\equiv - \frac{\Sigma N \alpha}{4x(1-x)} \left\{ \frac{(12x(1-x))^2}{(1-\tilde{z}')^4} \frac{\partial}{\partial A} + \frac{24x(1-x)}{(1-\tilde{z}')^2} \frac{\partial}{\partial B} + \frac{\partial}{\partial C} \right\}\end{aligned}\quad (43)$$

and that (42) can thus be rewritten as

$$\begin{aligned}F(\tilde{z}, \tilde{z}') &= i \frac{1}{(1-\tilde{z})^3(1-\tilde{z}')^3} \left\{ \frac{(12x(1-x))^2}{2} \left(\frac{1}{(1-\tilde{z})^2} - \frac{1}{(1-\tilde{z}')^2} \right)^2 \frac{\partial}{\partial A} \right. \\ &\quad \left. - \frac{2x(1-x)}{\Sigma N \alpha} \left(\frac{\partial}{\partial \tilde{z}} - \frac{\partial}{\partial \tilde{z}'} \right) \right\} F_0(\tilde{z}, \tilde{z}')\end{aligned}\quad (44)$$

The rapid oscillation of $\phi(\tilde{z}, \tilde{z}')$ warrants our extending the integral to infinity, and integration by parts gives

$$\begin{aligned}F(\tilde{z}, \tilde{z}') &= i \left\{ \frac{(12x(1-x))^2}{2} \left(\frac{1}{(1-\tilde{z})^2} - \frac{1}{(1-\tilde{z}')^2} \right)^2 + \right. \\ &\quad \left. + \frac{6x(1-x)}{\Sigma N \alpha} \left(\frac{1}{1-\tilde{z}} - \frac{1}{1-\tilde{z}'} \right) \frac{1}{(1-\tilde{z})^3(1-\tilde{z}')^3} \right\} \frac{\partial}{\partial A} F_0(\tilde{z}, \tilde{z}')\end{aligned}\quad (45)$$

On this form we can use the fact that the variation of the phase occurs over a scale of Λ^{-1} and that the integration over $\mathcal{Q} = (\tilde{z} - \tilde{z}')$ will be practically cut off beyond Λ^{-1} . The first term in the bracket is of order Λ^{-2} , while the second one is of order Λ^{-4} . Since Λ is a large parameter, we can neglect the second term in front of the first one. If we were to keep it also, this would correspond to contributions of order Λ^{-1} in the rate, like those calculated in the case of varying density for beamstrahlung. This would go beyond the reasonable limit of validity of our cylindrical bunch approach, but corresponds to the next step in our programme.

We now move to the phase-space integral in (40). Using the form (38) for the phase factor, we change variables to

$$\frac{\bar{K}_\perp}{\Sigma} = \frac{\tilde{K}_\perp}{\Sigma} + \frac{B}{2A} \tilde{h}_\perp \quad (46)$$

to write it as

$$\phi(\tilde{z}, \tilde{z}') = A \frac{\bar{K}_\perp^2}{\Sigma^2} + \frac{4AC - B^2}{4A} \tilde{h}_\perp^2 \quad (47)$$

$$F_0(\tilde{z}, \tilde{z}') = \int d^2 \tilde{h}_\perp d^2 \tilde{h}'_\perp e^{i \left(A \frac{\bar{K}_\perp^2}{\Sigma^2} + \frac{4AC - B^2}{4A} \tilde{h}_\perp^2 \right)} \quad (48)$$

We again use the fact that the integral is in practice limited to a $\mathcal{C} = \tilde{z} - \tilde{z}'$ interval of order Λ^{-1} . It is an easy matter to check that A is of order Λ^2 ($\Lambda \mathcal{C}$) while $(4AC - B^2)/4A$ is of order $(\Lambda \mathcal{C})^3$. Since the coherence conditions maintain $\Lambda \mathcal{C}$ to a value of order 1, the first coefficient is much larger than the second one. This means that the integrand in (48) will oscillate very quickly with increasing $(\bar{K}_\perp/\Sigma)^2$ and much more slowly with increasing $\tilde{h}_\perp^2$. We may then perform the integral over $\tilde{K}_\perp$ irrespective of its actual (and complicated) boundaries, and write

$$F_0(\tilde{z}, \tilde{z}') = i \frac{\pi \Sigma^2}{A} \int d^2 \tilde{h}_\perp e^{i \frac{4AC - B^2}{4A} \tilde{h}_\perp^2} \quad (49)$$

We now go back to (45). The derivative with respect to A will give two terms, respectively proportional to A^{-2} and $B^2 A^{-3}$. We already know that the first term will be of order Λ^{-4} when the second one is found to be of order Λ^{-2} . We can therefore again neglect the first one as compared to the second one, and are left with a single leading term after exploiting all the simplifications associated with the oscillating phase within the bunch. It reads:

$$F(\tilde{z}, \tilde{z}') = -i \frac{(12 \times (1-x))^2}{2} \left(\frac{1}{(1-\tilde{z})^2} - \frac{1}{(1-\tilde{z}')^2} \right)^2 \frac{1}{(1-\tilde{z})^3 (1-\tilde{z}')^3} \frac{\pi \Sigma^2}{A} \frac{B^2}{4A^2} \int \tilde{h}_\perp^2 d^2 \tilde{h}_\perp e^{i \frac{4AC - B^2}{4A} \tilde{h}_\perp^2} \quad (50)$$

It is convenient to change variables to $\xi = 1 - \tilde{z}$. In practice, the oscillation of the phase will cluster the integration to similar values of ξ and ξ' , differing by at most Λ^{-1} . While we should keep proper track of the different powers of $\mathcal{C} = \xi' - \xi$, we can therefore equate ξ' to ξ in all the terms which do not vanish at $\mathcal{C} = 0$. This is particularly the case when calculating the limit of the phase-space integral. Bending occurs only between $\tilde{z}$ and 1, and the maximum normalized transverse momentum is therefore $\frac{1}{2}(1 - \tilde{z}) = \xi/2$. The integral over $\tilde{k}_\perp^2$ extends therefore to $\xi^2/4$. We obtain for the different relevant terms in (50):

$$\left(\frac{1}{(1-\tilde{z})^2} - \frac{1}{(1-\tilde{z}')^2} \right)^2 \frac{1}{(1-\tilde{z})^3 (1-\tilde{z}')^3} \approx \frac{4\mathcal{C}^2}{\xi^{12}} \quad (51)$$

$$\mathcal{C}^2 \frac{B^2}{A^3} \approx \frac{1}{(36)^2} \frac{1}{(x(1-x))^3} \frac{1}{\Lambda^3} \xi^8 \mathcal{C}$$

$$\frac{4AC - B^2}{4A} \approx \frac{\Lambda^3 \mathcal{C}^3}{12x(1-x)\xi^2}$$

The phase-space integral is then transformed into

$$\pi \int_0^{\xi^2/4} u du e^{i \frac{\Lambda^3 \mathcal{C}^3}{12x(1-x)\xi^2} u} = \frac{\pi \xi^4}{16} \int_0^1 v dv e^{i \frac{\Lambda^3 \mathcal{C}^3}{48x(1-x)} v} \quad (52)$$

We now notice the remarkable matching previously mentioned. The ξ^4 factor resulting from the phase-space integral just cancels the ξ^{-4} factor resulting from the product of the first two terms in (51), which appears in (50). There is no longer any dependence on ξ ! The contribution to the production rate is accordingly linear in L_b .

We now combine (34) and (50) with (51) and (52):

$$\frac{I(x)}{\pi R^2} = \frac{\kappa}{\pi} \frac{\Lambda^3}{8} (-i) \int_0^1 v dv \int_{-\infty}^{+\infty} e^{i \frac{\Lambda^3 \mathcal{C}^3}{48x(1-x)} v} \mathcal{C} d\mathcal{C} \quad (53)$$

and perform the integral over $\mathcal{C}$ which can be freely extended to infinity because of the rapid damping at large $\mathcal{C}$. The integral over $\frac{1}{2}(z+z')$ already performed in (53) gives merely a factor 2. One finds

$$\int_{-\infty}^{+\infty} e^{i \frac{\Lambda^3 \eta^3}{48 x(1-x)} v} \eta d\eta = i \frac{1}{\sqrt{3}} \Gamma(2/3) \frac{(48 x(1-x))^{2/3}}{\Lambda^3 v^{2/3}} \quad (54)$$

The integral over v now simply gives a factor $3/4$:

$$\frac{I(x)}{\pi R^2} = \frac{\alpha}{\pi} \Lambda \sqrt{3} \frac{6^{2/3}}{8} \Gamma(2/3) (x(1-x))^{2/3} \quad (55)$$

and hence (2), using (9). As in the case of beamstrahlung, the small factor α/π is partly compensated by a large factor Λ . The probability of pair production is therefore considerable. With $\Lambda = 50$ [3], it is of the order of 15%. There is here an extra numerical factor of $(16)^{2/3}$ as compared to the value given in Ref. [7], a factor 16 having there been omitted in the denominator of the exponent in (52). The main final result, as stated in Eq. (3) of Ref. [7] and Eq. (2) here, remains the same.

4. - FRONT-EDGE EFFECTS

In the case of beamstrahlung, we found a term of order $\alpha/\pi \ln L_e/\lambda_c$ which originated from the radiation amplitude being extended outside the bunch. We now turn to similar effects for pair production, which are also of pure quantum origin.

The contribution to the rate proportional to Λ originates from an integral where both $\tilde{z}$ and $\tilde{z}'$ run from -1 to $+1$. We now include those contributions where at least one of these variables is less than -1 , thus considering the front-edge effect.

We first look at the case in which both $\tilde{z}$ and $\tilde{z}'$ are less than -1 . The production amplitude is now given by the first term in (26). The contribution to the production rate is accordingly

$$\frac{I_0(x)}{\pi R^2} = \frac{\alpha}{\pi} \frac{(N\alpha)^2}{\pi^2} \frac{9}{8} x(1-x) \int_{-\infty}^{-1} d\tilde{z} \int_{-\infty}^{-1} d\tilde{z}' F(\tilde{z}, \tilde{z}') \quad (56)$$

with

$$F(\tilde{z}, \tilde{z}') = \int \frac{|3x(1-x) \frac{\tilde{k}_\perp}{\tilde{\Sigma}} + \tilde{h}_\perp|^2}{(1+3\tilde{z})^2 (1+3\tilde{z}')^2} e^{i\phi(\tilde{z}, \tilde{z}')} d^2 \tilde{h}_\perp d^2 \tilde{h}'_\perp \quad (57)$$

We write the new phase factor $\phi(\tilde{z}, \tilde{z}')$ as

$$\phi(\tilde{z}, \tilde{z}') = A \frac{\tilde{k}_\perp^2}{\tilde{\Sigma}^2} + B \frac{\tilde{k}_\perp}{\tilde{\Sigma}} \cdot \tilde{h}_\perp + C \tilde{h}_\perp^2 + D \tilde{m}^2 \quad (58)$$

and find, in that case, that

$$\begin{aligned} A &= 3\Lambda^3 x(1-x) \left(\frac{1}{1+3\tilde{z}} - \frac{1}{1+3\tilde{z}'} \right) \\ B &= \frac{2A}{3x(1-x)} \\ C &= \frac{A}{9(x(1-x))^2} \\ D &= \frac{\Lambda^3}{x(1-x)} (\tilde{z} - \tilde{z}') \end{aligned} \quad (59)$$

We introduce F_0 (40), writing now

$$F(\tilde{z}, \tilde{z}') = \frac{-i}{(1+3\tilde{z})^2 (1+3\tilde{z}')^2} \left((3x(1-x))^2 \frac{\partial}{\partial A} + 6x(1-x) \frac{\partial}{\partial B} + \frac{\partial}{\partial C} \right) \quad (60)$$

$$F_0(\tilde{z}, \tilde{z}')$$

Acting on F_0 ,

$$\frac{\partial}{\partial \tilde{z}'} \equiv \frac{\Lambda^3}{x(1-x)} \left(\frac{1}{(1+3\tilde{z}')^2} \left((3x(1-x))^2 \frac{\partial}{\partial A} + 6x(1-x) \frac{\partial}{\partial B} + \frac{\partial}{\partial C} \right) + \frac{\partial}{\partial D} \right) \quad (61)$$

and we can therefore rewrite (60) as

$$F(\tilde{z}, \tilde{z}') = \frac{x(1-x)}{\Lambda^3} \frac{-i}{(1+3\tilde{z})^2} \frac{\partial}{\partial \tilde{z}'} F_0(\tilde{z}, \tilde{z}') \quad (62)$$

neglecting the D derivative which gives a term proportional to $\tilde{m}^2$.

One readily integrates over $\tilde{z}'$, writing

$$\frac{I_0(x)}{\pi R^2} = \frac{-i\alpha}{\pi} \frac{(N\alpha)^2}{\pi^2} \frac{q}{8} \frac{(x(1-x))^2}{\Lambda^3} \int_{-\infty}^{-1} d\tilde{z} \frac{1}{(1+3\tilde{z})^2} F_0(\tilde{z}, -1) \quad (63)$$

Since we are dealing with an edge effect, it is convenient to shift the position to the origin by introducing the new variables

$$\begin{aligned} z &= \tilde{z} + 1 \\ z' &= \tilde{z}' + 1 \end{aligned} \quad (64)$$

and writing (63) as

$$\frac{I_0(x)}{\pi R^2} = \frac{-i\alpha}{\pi} \frac{(N\alpha)^2}{\pi^2} \frac{q}{8} \frac{(x(1-x))^2}{\Lambda^3} \int_{-\infty}^0 dz \frac{1}{(3z-z)^2} F_0(z, 0) \quad (65)$$

It is of order Λ^0 , as seen using (49).

We turn now to the contribution originating from $Z < 0$ and $Z' > 0$, or vice versa. We obtain a new contribution to the production rate which we write, in analogy with (56), as

$$\frac{I_1(x)}{\pi R^2} = \frac{\alpha}{\pi} \frac{(N\alpha)^2}{\pi^2} \frac{q}{2} x(1-x) \int_{-\infty}^{-1} d\tilde{z} \int_{-1}^{\infty} d\tilde{z}' \left(F(\tilde{z}, \tilde{z}') + F(\tilde{z}', \tilde{z}) \right) \quad (66)$$

with now

$$F(\tilde{z}, \tilde{z}') = - \int \frac{1}{(1+3\tilde{z})^2(1-\tilde{z}')^3} \left(\frac{12x(1-x)}{(1-\tilde{z}')^2} \frac{\tilde{k}_\perp}{\Sigma} + \tilde{h}_\perp \right) \cdot \left(3x(1-x) \frac{\tilde{k}_\perp}{\Sigma} + \tilde{h}_\perp \right) e^{i\phi(\tilde{z}, \tilde{z}')} d^2\tilde{h}_\perp d^2\tilde{h}'_\perp \quad (67)$$

The extra minus sign as compared to (57) corresponds to the two different global signs as they appear in (26).

We switch again to the variables (64) and use the Z, Z' symmetry, which should eventually appear in the rate, to write:

$$\begin{aligned} \frac{1}{2} \frac{I_1(x)}{\pi R^2} &= i \frac{x}{\pi} \frac{(N\alpha)^2}{\pi^2} x(1-x) \frac{g}{2} \int_{-\infty}^0 dz \int_0^\infty dz' \\ &\frac{1}{(2-3z)^2(2-z')^3} \left\{ \frac{36(x(1-x))^2}{(2-z')^2} \frac{\partial}{\partial A} + 3x(1-x) \left(1 + \frac{4}{(2-z')^2} \right) \frac{\partial}{\partial B} \right. \\ &\quad \left. + \frac{\partial}{\partial C} \right\} F_0(z, z') \end{aligned} \quad (68)$$

with, as before in Eq. (40),

$$F_0(z, z') = \int d^2\tilde{h}_\perp d^2\tilde{h}'_\perp e^{i\phi(z, z')}$$

The new phase factor has been written in the same formal way as before:

$$\phi(z, z') = A \frac{\tilde{k}_\perp^2}{\Sigma^2} + B \frac{\tilde{k}_\perp}{\Sigma} \cdot \tilde{h}_\perp + C \tilde{h}_\perp^2 + D \tilde{m}^2 \quad (69)$$

The phases calculated separately inside and outside the bunch have been matched at $Z = Z' = 0$, and taken to be zero there. The functions in (69) now take different forms, namely

$$\begin{aligned}
 A &= \frac{3}{2} \Lambda^3 x(1-x) \left\{ \frac{3z}{2-3z} - \left(\left(1 - \frac{z'}{2}\right)^{-3} - 1 \right) \right\} \\
 B &= 3\Lambda^3 \left\{ \frac{z}{2-3z} - \left(\left(1 - \frac{z'}{2}\right)^{-1} - 1 \right) \right\} \\
 C &= \frac{\Lambda^3}{2x(1-x)} \left\{ \frac{z}{2-3z} - \frac{z'}{2} \right\} \\
 D &= \frac{\Lambda^3}{x(1-x)} z
 \end{aligned} \tag{70}$$

We again remark that, acting on F_0 (40), we can make the substitution

$$\begin{aligned}
 \frac{\partial}{\partial z} &\equiv \frac{\Lambda^3}{x(1-x)(2-3z)^2} \left\{ 9(x(1-x))^2 \frac{\partial}{\partial A} + 6x(1-x) \frac{\partial}{\partial B} + \frac{\partial}{\partial C} + (2-3z)^2 \frac{\partial}{\partial D} \right\} \\
 \frac{\partial}{\partial z'} &\equiv -\frac{\Lambda^3}{4x(1-x)} \left\{ 9(x(1-x))^2 \left(1 - \frac{z'}{2}\right)^{-4} \frac{\partial}{\partial A} \right. \\
 &\quad \left. + 6x(1-x) \left(1 - \frac{z'}{2}\right)^{-2} \frac{\partial}{\partial B} + \frac{\partial}{\partial C} \right\}
 \end{aligned} \tag{71}$$

which we use to rewrite (68) in the form

$$\begin{aligned}
 \frac{I_1(x)}{\pi R^2} &= i \frac{\alpha}{\pi} \frac{(Nd)^2}{\pi^2} x(1-x) \frac{q}{2} \int_{-\infty}^0 dz \int_0^{\infty} dz' \frac{1}{(2-3z)^2 (2-z')^3} \\
 &\quad \left\{ -\frac{q}{2} (x(1-x))^2 \left(1 - \left(1 - \frac{z'}{2}\right)^{-2}\right)^2 \frac{\partial}{\partial A} + \frac{2x(1-x)}{\Lambda^3} \left(\frac{(2-3z)^2}{4} \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right) \right. \\
 &\quad \left. - \frac{(2-3z)^2}{2} \frac{\partial}{\partial D} \right\} F_0(z, z')
 \end{aligned} \tag{72}$$

The change of variable (46) in F_0 separates two terms, A and $C - (B^2/4A)$, which we must now consider for values of Z' which are practically limited by the coherence condition to the range $0 < Z' < \Lambda^{-1}$. We thus expand them in Z' , keeping

$$A = \frac{9\Lambda^3 x(1-x)}{2} \left(\frac{z}{2-3z} - \frac{z'}{2} \right)$$

$$C - \frac{B^2}{4A} = - \frac{\Lambda^3 z'^3}{12x(1-x)} \quad (73)$$

The first term is therefore once more much larger than the second one for values of Z' of order Λ^{-1} . One can again integrate over $\tilde{k}_\perp^2$, extending the integral to infinity:

$$F_0(z, z') = \frac{\pi \Sigma^2}{A} \int_0^1 dy e^{i\left(C - \frac{B^2}{4A}\right)y} e^{i \frac{\Lambda^3 z}{x(1-x)} \tilde{m}^2} \quad (74)$$

The integral over $\tilde{k}_\perp^2$ now extends to 1, since the relevant production zone is limited to the front of the bunch. This is used in writing (74).

We next analyze the different terms in (72). The derivative with respect to A , acting on A^{-1} in (74), gives a term proportional to A^{-2} in (72) whereas, when acting on the exponent, it gives a term proportional to $B^2/4A^2$ and an overall factor proportional to $B^2/4A^3$ in (72):

$$\frac{B^2}{4A^3} = \frac{2}{81} \frac{1}{\Lambda^3} \frac{1}{(x(1-x))^3} \left(\frac{z}{2-3z} - \frac{z'}{2} \right)^{-1} \quad (75)$$

of order Λ^2 with respect to the former one. We can hence neglect the former contribution as compared to the latter.

The derivative with respect to Z is cancelled by the derivative with respect to D , both terms being proportional to $\tilde{m}^2$, and hence small. The derivative with respect to Z' brings in an extra factor Z'^2 when acting on F_0 , as seen using (73), which is in practice a small term. However, the derivative with respect to A is also multiplied by a factor Z'^2 present in (72). The two contributions are therefore of the same order, both giving contributions of order Λ^0 to the production rate.

However, we can integrate by parts with respect to Z' . The derivative of the $(2-Z')^{-3}$ term now gives a contribution of order Λ^{-1} as compared to the derivative of F_0 . The leading contribution comes then from the integrated term, which is easily calculated:

$$i \frac{\kappa}{\pi} \frac{(N\alpha)^2}{\pi^2} \frac{(x(1-x))^2}{\Lambda^3} \frac{9}{16} \int_{-\infty}^0 dz \frac{1}{(3z-2)^2} F_0(z,0) \quad (76)$$

The full contribution to I_1 is twice that. It just cancels the contribution of I_0 which we obtained previously (65).

From Eqs. (72)-(74) and (76) we can now write the full contribution from the front-edge effect as

$$\begin{aligned} \frac{I_E(x)}{\pi R^2} = & -i \frac{\kappa}{\pi} \Lambda^3 \int_{-\infty}^0 dz \int_0^{\infty} dz' \frac{1}{(2-3z)^2} \frac{1}{(2-z')^3} \left(\frac{z}{2-3z} - \frac{z'}{z} \right)^{-1} \\ & \int_0^1 \eta d\eta e^{-i \frac{\Lambda^3 z'^3}{12x(1-x)} \eta} z'^2 e^{i \frac{L_4 z}{x(1-x)L_e}} \end{aligned} \quad (77)$$

It is globally of order Λ^0 , which is typical of edge effects. We have neglected terms of order Λ^{-1} .

The limit $Z' = 0$ should be of no significance since the integrand includes a factor Z'^2 . The minimum relevant value of Z' is actually of order Λ^{-1} . Since we have only a logarithmic behaviour in the integral over Z , as the limit of integration approaches the edge, and since we wish to select the larger $\ln \Lambda$ term, we can replace Z' by its mean practical value Λ^{-1} in the denominators in (77), which we write as

$$\begin{aligned} \frac{I_E(x)}{\pi R^2} = & -i \frac{\kappa}{\pi} \frac{\Lambda^3}{4} \int_{-\infty}^0 \frac{e^{i \frac{L_4 z}{x(1-x)L_e}}}{(2-3z)^2 \left(\frac{2z}{2-3z} - \Lambda^{-1} \right)} \\ & \int_0^{\infty} dz' \int_0^1 \eta d\eta e^{-i \frac{\Lambda^3 z'^3}{12x(1-x)} \eta} z'^2 \end{aligned} \quad (78)$$

We now perform the integral over Z' :

$$-i \int_0^{\infty} z'^2 dz' e^{-i \frac{\Lambda^3 z'^3}{12x(1-x)}} = \frac{4x(1-x)}{\Lambda^3} \quad (79)$$

The integral over η simply gives a factor 1, and the leading contribution from the edge effect is accordingly

$$\frac{I_E(x)}{\pi R^2} = -\frac{\alpha}{\pi} x(1-x) \int_{-\infty}^0 \frac{e^{i \frac{L_e z}{x(1-x)L_e}}}{(2-3z)^2 \left(\frac{2z}{2-3z} - \Lambda^{-1} \right)} \quad (80)$$

Using (9) we find a spectrum in the Dirac case of the type

$$I_E(x) \sim (1-x)^2 + x^2 \quad (81)$$

The integral behaves logarithmically as Z approaches the edge, namely $1/4 \ln L_b/L_c$. There is a cut-off at large $|Z|$ provided by the rapidly-varying phase and the range of integration extends only to a distance of order L_e/L_b . We have to compare this to the bunch length corresponding to $\Delta Z = 2$.

We can first take the long bunch limit $L_b \gg L_e$. In this case we can neglect the variation of $(2-3Z)$ over the practical range of integration, as is clearly the case when considering the behaviour at small $|Z|$ (Λ^{-1}), as previously done. We have a leading term $1/4 \ln L_e/L_b$. In that case the leading (logarithmic) term in (80) is in the spinless case [multiplying by (9) for the Dirac case]

$$\frac{I_E(x)}{\pi R^2} \approx \frac{\alpha}{\pi} \frac{1}{4} x(1-x) \ln \frac{L_e}{L_c} \quad (82)$$

i.e., a term similar to the one met in the case of beamstrahlung [3].

However, we may also look at the case of a "short" bunch $L_e \gg L_b$. Under such conditions the variation of $(2-3Z)$ can no longer be neglected over L_e , and the lower limit of the integral gives in practice a negligible value. In this "short" bunch approximation we are left with a leading contribution coming from the low $|Z|$ limit only, namely

$$\frac{I_E(x)}{\pi R^2} \approx \frac{\alpha}{\pi} \frac{1}{4} x(1-x) \ln \frac{L_b}{L_e} \quad (83)$$

This difference of behaviour, depending on how the quantum coherence length extends with respect to the bunch length, can of course be traced back to the special conditions which we met in the case of pair production (21). They depend very much on how close one is with respect to the bunch, as opposed to what one finds in the beamstrahlung case. Considering (21) in the case $\tilde{z} \sim -L_e/L_b$, the location of the point of the stationary phase is determined by the momentum kick given to the electron if the bunch length is relatively small (high $\tilde{z}$ limit). It is, however, determined by the differential bending of the electron and positron if the bunch length is relatively long (low $\tilde{z}$ limit). It is therefore natural to reach different values for the corresponding contributions to the rate in the two cases. There is always a logarithmic term. However, it is the shortest of the two (long) lengths L_e and L_b which sets the scale.

5. - CONCLUSIONS

We have calculated the leading contributions to the production rate of an electron-positron pair by a photon impinging on a dense bunch. This rate is considerable insofar as a small factor α/π is partly compensated by a large factor. Pair production in the deep quantum regime is therefore a serious hazard. There is a leading term proportional to the bunch length or, more precisely, to Λ where $\Lambda = L_b/\lambda_c$. The relevant coherent production length λ_c is the same as the one obtained for beamstrahlung in the deep quantum regime.

There is also a non-negligible term originating from the front-edge effect. It is, however, proportional to $\ln L_e/\lambda_c$ or $\ln L_b/\lambda_c$ depending on which of the two lengths, L_b or L_e , is the longest. The shortest of the two limits the value of that term through a logarithmic dependence.

The expression at which we arrived in the long bunch limit (82), multiplied by the spin factor (9), is identical to that which was more easily arrived at in the "black box" approach followed in our first paper on beamstrahlung [3], extending the calculation to the production of a pair, limited by a volume closed by the edge of the bunch.

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FIGURE CAPTIONS

Fig. 1a : Feynman graph for pair production by a photon in a bunch. The crosses correspond to the interaction of the electron and positron with the bunch.

Fig. 1b : Feynman graph for pair production from an electron in the strong field of a bunch.

Fig. 2 : Pair production by a photon entering a bunch. A schematic view. In practice $L_b \gg R$.

Fig. 1a

Fig. 1b

Fig. 2