CERN—TH.5377/89

CURIOSITIES AT $c=24$

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ABSTRACT

Two examples of $c = 24$ holomorphic conformal field theories are given that do not correspond to Niemeier lattices or $\mathbf{Z}_2$ twists of Niemeier lattices. One of these examples clarifies a recently discovered modular invariant of F_4 at level 6, which can be interpreted as the “complement” of $SU(3)$ level 2 within a new $c = 24$ theory.

^{*} Work supported in part by the US-Israel Binational Science Foundation, and the Israel Academy of Science.

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One of the most interesting problems in the classification of (rational) conformal field theories is that of enumerating all theories with just one primary field (with respect to some extended Virasoro algebra). Since the simplest examples of such theories are the even self-dual lattices, it is natural to call such theories *self-dual* conformal field theories. Such theories have trivial fusion rules and modular transformations ($S = 1$), and it is precisely for that reason that all approaches to classification that use the fusion rules [1-3] are blind to such theories. In other words, even if one has a classification of all fusion rules and the corresponding matrices S and T , one cannot know all conformal field theories without knowing all self-dual conformal field theories.

The matrix T representing the modular transformation $\tau \rightarrow \tau + 1$ of such a theory is simply $\exp(-2\pi i \frac{c}{24})$, and from the condition $(ST)^3 = 1$ one learns then that c must be a multiple of 8 [1]. Denoting the (only) character of a self-dual theory as $\mathcal{X}(\tau)$, one can easily classify the most general modular invariant combinations of such characters. Apart from the diagonal invariant, all cubic powers of the characters are modular invariant if $c = 8$ or $16 \bmod 24$, and the characters themselves are modular invariant if $c = 0 \bmod 24$. In the latter cases $\mathcal{X}$ can be written as a polynomial in the absolute modular invariant function j . This is all one can learn from the fusion rules and the matrices S and T .

Apart from the obvious importance for the classification of CFT's, there is a second, related reason why it would be interesting to have a complete list of the self-dual theories, namely to complete the classification of the ten-dimensional heterotic strings. While the closed bosonic and type-II strings have been classified completely in their maximal dimensions 26 and 10 respectively (and most likely also in 25 and 9 dimensions because of the results of [4-6] and [7]), the classification of heterotic strings remains unsatisfactory even in ten dimensions. It is not obvious that the exhaustive classifications of twisted supersymmetric strings [8], or of ten-dimensional free fermion constructions [9,10] yield all possibilities, even though nothing new has been found during the last three years.

A method that simplifies the classification problem of type-II and heterotic strings in d space-time dimensions is to replace their NSR-sectors by an $SO(d+6) \times E_8$ (or $SO(d+22)$) Kac-Moody algebra, in such a way that modular invariance is preserved. In this way one associates with every type-II or heterotic string a compactified bosonic string. A detailed explanation of this procedure has appeared in several papers [11]. When applied to ten-dimensional strings, one obtains a bosonic string with $(D_8 E_8)_L \times (D_8 E_8)_R$ Kac-Moody

algebra from a type-II string and a $(C_{16})_L \times (D_8 E_8)_R$ conformal field theory from a heterotic string, where C_{16} denotes some, not yet determined, $c = 16$ conformal field theory.

Finding all the modular invariants of ten-dimensional type-II strings amounts thus merely to finding all modular invariants of D_8 , and this problem is easily solved. Hence (see [12]) the complete set of such theories consists of the type-IIA and type-IIB strings, plus the two tachyonic theories found in [13]. To classify the heterotic strings, one can instead of $(C_{16})_L \times (D_8 E_8)_R$ consider self-dual theories $C_{16} D_8$, since there is a one-to-one correspondence between the modular invariants of these two theories. This trick was applied in [12] to obtain all known ten-dimensional strings with a rank 16 gauge group, by simply reading them off from the table of the 24 Niemeier lattices [14,15]. One simply selects all entries in the table that contain a level-1 $SO(16)$ Kac-Moody algebra. It is obvious that the identical procedure, applied to a list of all $c = 24$ self-dual conformal field theories, will yield *all* ten-dimensional heterotic string theories. For this reason it would be very interesting to know the complete list of such theories.

Although there is no proof that the ten-dimensional strings have all been found, this is certainly true for the *supersymmetric* ten-dimensional heterotic strings. For these theories anomaly cancellation is such a strong constraint that it provides a proof of completeness. It follows immediately that all self-dual conformal field theories are known for $c = 8$ and $c = 16$, and correspond simply to the E_8 , $E_8 \times E_8$ and D_{16} even self-dual lattices (the same conclusion has been obtained in [16], with not entirely different arguments). In principle one can use the same arguments for $c = 24$ theories, at least for those with an associated Kac-Moody algebra with $c = 24$. For every such theory one can define a character valued partition function, which has a known behaviour under modular transformations [17]. From the modular transformations one can derive trace-identities for the representations appearing at each level, which one can try to solve. We will not attempt to explore this possibility here.

In addition to the Niemeier lattices, a second class of known $c = 24$ theories are their $\mathbf{Z}_2$ orbifolds [16][18]. The latter are constructed by “modding out” the automorphism $X^i \rightarrow -X^i, i = 1, \dots, 24$. The number of spin-1 fields in such a theory is always half the number of roots of the Niemeier lattice, and in particular it is zero for the $\mathbf{Z}_2$ twisted Leech lattice. The purpose of this paper is to provide two new examples, each of which illustrates one of the motivations for enumerating the $c = 24$ theories completely.

Our first example can be obtained by inverting the procedure explained above. Since we know of the existence of one ten-dimensional heterotic string with a gauge group of rank 8 rather than 16, it follows that a $c = 24$ self-dual conformal field theory can be obtained from it by combining it with an $SO(16)$ factor. It was shown in [19] that the E_8 theory can be described as an E_8 level 2 Kac-Moody algebra, tensored with an Ising model. Both these CFT's have three primary fields: for E_8 they are identified as the singlet, the 3875 and the 248; for the Ising model one has the well-known possibilities $h = 0$, $h = \frac{1}{2}$ and $h = \frac{1}{16}$. We label these primary fields respectively by $l = 0, 1$ and 2 , for E_8 as well as the Ising model, and denote their partition functions as $\mathcal{P}_{E_8}^l$ or $\mathcal{P}_{\text{Ising}}^l$ respectively. To specify the theory, we simply have to indicate which combinations of these fields belong to the spin-structures of the right-moving NSR-model, or equivalently, to the conjugacy classes of the right-moving D_8 factor. The result is

$$\begin{aligned}\mathcal{P}_0 &= \mathcal{P}_{E_8}^0 \mathcal{P}_{\text{Ising}}^0 + \mathcal{P}_{E_8}^1 \mathcal{P}_{\text{Ising}}^1 \\ \mathcal{P}_v &= \mathcal{P}_{E_8}^0 \mathcal{P}_{\text{Ising}}^1 + \mathcal{P}_{E_8}^1 \mathcal{P}_{\text{Ising}}^0 \\ \mathcal{P}_s &= \mathcal{P}_c = \mathcal{P}_{E_8}^2 \mathcal{P}_{\text{Ising}}^2\end{aligned}\tag{1}$$

Here $\mathcal{P}_x$ is the left-moving internal partition function that belongs to the conjugacy class (x) of the right-moving D_8 factor.

Replacing the right-moving $SO(16)$ factor by a left-moving one, one obviously gets an $(E_8)_{\text{level } 2} \times SO(17)$ Kac-Moody algebra, where $SO(17)$ is obtained by extending $SO(16)$ with the Ising model. The character of the resulting theory is the sum of the three character combinations $\mathcal{X}_1 \mathcal{X}_1$, $\mathcal{X}_{256} \mathcal{X}_{248}$ and $\mathcal{X}_{17} \mathcal{X}_{3875}$, whose ground states have conformal spins 0, 2 and 2 respectively. A useful check of this partition function is the total count of spin 2 fields (including descendants), which must be equal to 196884, the coefficient of q in the function j . This addition is shown in table I.

In counting excited states of Kac-Moody (or other) algebras one has to be careful to remove null-states. Using the Kac determinant [20] one can show that in level-1 $SO(N)$ algebras there is a null vector among the states $K_{-1}^{(a)} K_{-1}^{(b)}$, which belongs to the representation with Dynkin labels $(02000\dots)$. Higher level ($k > 1$) Kac-Moody algebras do not have any null states at the second excited level, since if they did the same states would be negative norm states for $k = 1$. However, negative norm states can only appear as excitations of null-states, and there are no null states at the first excited level.

$J_{-1}^{(a)} J_{-1}^{(b)} 0\rangle$	30876
$K_{-1}^{(a)} K_{-1}^{(b)} 0\rangle$	9316
null vectors in (02000000)	–6783
$J_{-1}^a K_{-1}^b 0\rangle$	33728
$J_{-2}^a 0\rangle$	248
$K_{-2}^a 0\rangle$	136
$ (17, 3875)\rangle$	65875
$ (256, 248)\rangle$	63488
total	196884

Table I. *Spin 2 states in the $(E_8)_{\text{level } 2} \times SO(17)$ theory. The E_8 generators are denoted by J_n^a , the $SO(17)$ generators by K_n^a . Note that the state $K_{-1}^{(a)} K_{-1}^{(b)} |0\rangle$ contains null vectors, as indicated.*

This example is not one of the $\mathbf{Z}_2$ -twisted Niemeier lattices mentioned above, for the simple reason that it has 384 spin-1 fields, which is not equal to half the number of roots of any Niemeier lattice. There does however exist a different orbifold construction of it, which is analogous to the procedure used in [8] in their construction of non-supersymmetric string theories. One starts with the Niemeier lattice $(E_8)^3$, and considers an automorphism that interchanges two E_8 factors, combined with a shift by a D_8 vector weight in the third E_8 factor.

Our second example has a far more complicated structure. It is related to a rather mysterious modular invariant of the Kac-Moody algebra F_4 at level 6 [21]. Almost all integer spin modular invariants (*i.e.* modular invariants that can be written as sum of squares) of Kac-Moody algebras belong to one of two classes. The first class, found in [22], can be understood in terms of integer spin currents that belong to the center of the Kac-Moody algebra (here we use the definition of the center of a CFT given in [21]). These invariants can presumably be understood in terms of strings on group manifolds modulo a subgroup of the center of the group (indeed, the D -invariants of $SU(2)$ can be regarded as $SO(3)$ invariants), even though they were obtained in a rather different way. The second class, first described in [23], corresponds to modular invariants of a group H that are really diagonal invariants of a group G , into which H is conformally embedded. The F_4 invariant

is the only example we know that does not belong to either category. This is obvious since F_4 has a trivial center, and since no spin 1 primary field appears in the identity block. Furthermore F_4 level 6 does not appear in the tables of conformal embeddings [24], not even in combination with some other Kac-Moody algebra.

The following combination of characters was found to be modular invariant in [21].

$$|\mathcal{X}_{0000} + \mathcal{X}_{0004} + \mathcal{X}_{0030} + \mathcal{X}_{1100}|^2 + |\mathcal{X}_{0003} + \mathcal{X}_{0006} + \mathcal{X}_{0021} + \mathcal{X}_{2010}|^2 \\ + 2|\mathcal{X}_{0101} + \mathcal{X}_{1012}|^2 + 2|\mathcal{X}_{0102} + \mathcal{X}_{2000}|^2, \quad (2)$$

where the subscripts are the Dynkin-labels of the ground states.

A first step in an attempt to interpret this invariant is to determine the matrix S and the fusion rules in terms of the new primary field appearing within the squares of (2). To obtain the new fusion rules one can proceed as follows. First one projects the matrix S of the original theory on the invariant subspace spanned by the four character combinations appearing in (2). In the invariant subspace one uses a normalized basis of character combinations $\frac{1}{2}[\mathcal{X}_{0000} + \mathcal{X}_{0004} + \mathcal{X}_{0030} + \mathcal{X}_{1100}]$, $\frac{1}{\sqrt{2}}[\mathcal{X}_{0101} + \mathcal{X}_{1012}]$, etc. The resulting four-by-four matrix $\hat{S}$ is obviously unitary, but it is not the S -matrix of the new theory, and it does not produce integer fusion rule coefficients.

The origin of this problem is that the theory does not have four, but six primary fields. The fact that the last two terms appear with multiplicity 2 indicates that there are really two fields, which should be treated as different primary fields of the extended algebra (see [25,26] for examples, and [27] for a proof of this fact). Denote the six characters appearing in (2) as $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3^+, \mathcal{X}_3^-, \mathcal{X}_4^+, \mathcal{X}_4^-$, so that (2) is the sum of their squares. The new S matrix must reduce to $\hat{S}$ when restricted to a basis $\mathcal{X}_1, \mathcal{X}_2, \frac{1}{\sqrt{2}}(\mathcal{X}_3^+ + \mathcal{X}_3^-), \frac{1}{\sqrt{2}}(\mathcal{X}_4^+ + \mathcal{X}_4^-)$, from which we conclude that it can be expressed in terms of the matrix elements of s_{ij} of $\hat{S}$ plus three new parameters

$$S = \begin{pmatrix} s_{11} & s_{12} & \frac{1}{\sqrt{2}}s_{13} & \frac{1}{\sqrt{2}}s_{13} & \frac{1}{\sqrt{2}}s_{14} & \frac{1}{\sqrt{2}}s_{14} \\ s_{21} & s_{22} & \frac{1}{\sqrt{2}}s_{23} & \frac{1}{\sqrt{2}}s_{23} & \frac{1}{\sqrt{2}}s_{24} & \frac{1}{\sqrt{2}}s_{24} \\ \frac{1}{\sqrt{2}}s_{31} & \frac{1}{\sqrt{2}}s_{32} & \frac{1}{2}s_{33} + \frac{1}{2}\alpha & \frac{1}{2}s_{33} - \frac{1}{2}\alpha & \frac{1}{2}s_{34} + \frac{1}{2}\beta & \frac{1}{2}s_{34} - \frac{1}{2}\beta \\ \frac{1}{\sqrt{2}}s_{31} & \frac{1}{\sqrt{2}}s_{32} & \frac{1}{2}s_{33} - \frac{1}{2}\alpha & \frac{1}{2}s_{33} + \frac{1}{2}\alpha & \frac{1}{2}s_{34} - \frac{1}{2}\beta & \frac{1}{2}s_{34} + \frac{1}{2}\beta \\ \frac{1}{\sqrt{2}}s_{41} & \frac{1}{\sqrt{2}}s_{42} & \frac{1}{2}s_{43} + \frac{1}{2}\beta & \frac{1}{2}s_{43} - \frac{1}{2}\beta & \frac{1}{2}s_{44} + \frac{1}{2}\gamma & \frac{1}{2}s_{44} - \frac{1}{2}\gamma \\ \frac{1}{\sqrt{2}}s_{41} & \frac{1}{\sqrt{2}}s_{42} & \frac{1}{2}s_{43} - \frac{1}{2}\beta & \frac{1}{2}s_{43} + \frac{1}{2}\beta & \frac{1}{2}s_{44} - \frac{1}{2}\gamma & \frac{1}{2}s_{44} + \frac{1}{2}\gamma \end{pmatrix}$$

To determine the parameters α, β and γ we use first of all the identity $S^2 = C$. The matrix

C is *a priori* unknown, but there are only a few possible choices. If we assume that all six fields are self-conjugate ($C = 1$), we find that the matrix

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

must be orthogonal; if we assume that charge conjugation interchanges the fields with character $\mathcal{X}_i^+$ and $\mathcal{X}_i^-$, we find that $i\sigma$ must be orthogonal. In both cases the 3 parameters can thus be expressed in terms of one angle ξ . To determine ξ , we require that all fusion rules, obtained from S using the formula of [1], have integer coefficients. This problem has no solution if $C = 1$, but we do find a solution if C is off-diagonal as described above.

Denoting the six fields in the obvious way, we find the following fusion rules (omitting trivial ones)

$$\begin{aligned} \phi_2 \times \phi_2 &= \phi_1 + \phi_2 \\ \phi_2 \times \phi_3^+ &= \phi_3^+ + \phi_4^- \\ \phi_2 \times \phi_4^+ &= \phi_3^- \\ \phi_3^+ \times \phi_3^+ &= \phi_3^- + \phi_4^+ \\ \phi_3^+ \times \phi_3^- &= \phi_1 + \phi_2 \\ \phi_3^+ \times \phi_4^+ &= \phi_2 \\ \phi_3^+ \times \phi_4^- &= \phi_3^- \\ \phi_4^+ \times \phi_4^+ &= \phi_4^- \\ \phi_4^+ \times \phi_4^- &= \phi_1 \end{aligned}$$

All remaining rules can be obtained by flipping simultaneously all signs. Note that the set of fields includes two simple currents (by definition these are currents that have unique fusion rules [21]), namely ϕ_4^+ and ϕ_4^- . These currents generate a $\mathbf{Z}_3$ center of the algebra. These fusion rules turn out to be identical to those of the $SU(3)$, level 2 Kac-Moody algebra,^{*} if one makes the following identification of primary fields

$$\begin{aligned} (1) &\leftrightarrow \phi_1 & (8) &\leftrightarrow \phi_2 \\ (3) &\leftrightarrow \phi_3^+ & (\bar{3}) &\leftrightarrow \phi_3^- \\ (6) &\leftrightarrow \phi_4^+ & (\bar{6}) &\leftrightarrow \phi_4^- \end{aligned} \tag{3}$$

However, it is not true that the two algebras are identical. In fact, their spectra of L_0

* This observation is due to Nathan Seiberg

$J_{-1}^{(a)} J_{-1}^{(b)} 0\rangle$	1378
$K_{-1}^{(a)} K_{-1}^{(b)} 0\rangle$	36
$J_{-1}^a K_{-1}^b 0\rangle$	416
$J_{-2}^a 0\rangle$	52
$K_{-2}^a 0\rangle$	8
$ (1, 0004)\rangle$	16302
$ (1, 1100)\rangle$	29712
$ (8, 0003)\rangle$	$8 \times 2652 = 21216$
$ (3 + \bar{3}, 0101)\rangle$	$6 \times 19278 = 115668$
$ (6 + \bar{6}, 2000)\rangle$	$12 \times 1053 = 12636$
total	196884

Table II. *Count of spin 2 states in the $(A_2)_{\text{level } 2}(F_4)_{\text{level } 6}$ theory*

eigenvalues turn out to be opposite, modulo 1, and their central charges are

$$c(F_4, \text{level } 6) = \frac{104}{5}; \quad c(SU(3), \text{level } 2) = \frac{16}{4},$$

so that the central charges of the two algebras add up to 24!

It follows that the two theories can be tensored to construct a self-dual $c = 24$ theory. Denoting the six characters of the F_4 modular invariant as Ψ_i , and the six characters of $SU(3)$ at level 2 as Ξ_i (where the labels are chosen according to the identification (3)) we can write down the following modular invariant partition function

$$\mathcal{P}(\tau) = \Phi_i(\tau) \Xi_i(\tau). \quad (4)$$

This combination is modular invariant under T because the conformal dimensions of the primary fields of the two theories are opposite (modulo 1); it is invariant under S because the modular transformation matrices representing S for the two theories are each others' complex conjugates (note that the fusion rules determine S only up to conjugation. However, the condition $(ST)^3 = C$ can be used to remove the remaining ambiguity).

To convince oneself that this is indeed a modular invariant theory one can again count the spin-2 states. The contributions are listed in table II.

It is slightly more difficult to argue that this theory is not a $\mathbf{Z}_2$ twisted Niemeier lattice, since there exists a Niemeier lattice with the correct number (120) of roots. This Niemeier lattice has an $(A_4)^6$ root system. However, its Z_2 twisted version is a $SO(5)^6$ (level 2) Kac-Moody algebra, and not $F_4 \times SU(3)$. Of course we certainly cannot rule out the possibility that this theory can be obtained by an entirely different orbifold procedure.

This example illustrates the point made in the beginning of this paper, namely that the self-dual $c = 8k$ theories can be used to discover new modular invariants of other conformal field theories. Whenever one can split such theories into two factors $\mathcal{C}_1$ and $\mathcal{C}_2$, with central charges adding up to $c = 8k$, the single character of the self-dual theory becomes a sum of products of characters of the two factors, as in (4). If one of the factors is a good conformal field theory with a well-defined matrix S and correct fusion rules, the other is also well-defined, and has a modular transformation matrix S^* . In particular one can write down a diagonal invariant for each factor separately. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are Kac-Moody algebras, one will often just get the diagonal invariant of each Kac-Moody algebra. The example given above shows that less trivial results are also possible.

Using a terminology introduced in [28] (where this procedure of “splitting” a self-dual CFT with $c = 8k$ has been used for the special case of self-dual lattices), we call $\mathcal{C}_1$ and $\mathcal{C}_2$ each others’ *complement*. Examples of complementary CFT’s (for $c = 8$) are $SU(3)$ and E_6 , $SU(2)$ and E_7 , F_4 and G_2 , all at level 1. It is an interesting question whether, for sufficiently large k , every rational CFT has a complement. It is however not hard to see that every Kac-Moody-algebra at every level has a complement, since one can always construct an embedding of the required Dynkin index in a sufficiently large even self-dual lattice. In general, such a construction yields a complement that is a coset theory; it is far from obvious that one can always find a complement that is itself a Kac-Moody algebra.

The two $c = 24$ theories described in this paper can be understood in terms of the complementary CFT’s E_8 (level 2) and $SO(17)$ (level 1), and $SU(3)$ (level 2) with the new theory based on the F_4 (level 6) modular invariant of [21]. Most of these theories have other, simpler complements. For example, E_8 (level 2) is a complement of the Ising model, together with which it forms the $E_8 \times E_8$ self-dual lattice (note that $SO(16n - 1)$ (level 1), $n \geq 1$, are also complements of the Ising model); $SU(3)$ (level 2) has a complement $SU(3) \times G_2$ (both level 1).

In some cases, the self-dual conformal field theories can be used to solve the following

problem. Suppose one wants to construct a modular invariant theory with a given left-moving sector, (given by some rational CFT $\mathcal{C}_1$), and an unspecified right-moving sector X , with the same central charge as $\mathcal{C}_1$. Suppose $\mathcal{C}_1$ has a complement $\mathcal{C}_2$, and let $8k$ be the smallest value of the central charge of the self-dual theory for which $\mathcal{C}_2$ exists. Then, given a list of all $c = 8k$ self-dual conformal field theories, one can classify all possible conformal field theories X that can be combined with $\mathcal{C}_1$ to give a modular invariant theory. The solutions for X are simply all complements of $\mathcal{C}_2$ that appear in the list of $c = 8k$ theories, which includes of course $\mathcal{C}_1$. For example, there is a non-trivial modular invariant for $[E_8(\text{level } 2)]_L[SO(31)(\text{level } 1)]_R$, because both the left-moving and right-moving theories are complements of the Ising model. In [28] this method was used to classify all even self-dual Lorentzian lattices whose left-moving sector is a (simply-laced) Lie-algebra lattice of rank six or less.

Note that the complement $(\text{CFT})_c$ to a given CFT can always be viewed as a coset $(c = 8k \text{ theory})/\text{CFT}$. Thus in a loose sense the complement can be viewed as the (non-unique) “inverse” of a CFT, and with proper care one may replace (CFT) by $(\overline{\text{CFT}})_c$ (where the bar distinguishes left-movers from right-movers) or $1/(\text{CFT})$ (as in coset models) by $(\text{CFT})_c$ to map one modular invariant theory into another. This interplay between chiral coset models and left-right theories was recently discussed within the three-dimensional Chern-Simons approach to the construction of coset models [29]. Obviously, the range of application of such methods is severely limited by the absence of an enumeration of the self-dual CFT’s. This enumeration is already complete for $c = 8$ and $c = 16$ and can probably be completed for $c = 24$, but will never be completed (at least not in any explicit form) for larger values of c , since we know already that the number of self-dual lattices alone is simply too big.

Despite this somewhat sobering fact, the enumeration of the $c = 24$ theories remains a very intriguing mathematical problem, which is interesting in its own right, and which can probably be solved.

ACKNOWLEDGEMENTS

We thank Nathan Seiberg for asking the right questions about the F_4 invariant, for many stimulating and fruitful discussions, and for his comments on the manuscript.

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