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**QUANTUM SUPERSYMMETRY AND THE
SUPERGEOMETRY OF FOUR-DIMENSIONAL
SUPERSTRINGS**

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Abstract

The origin of world-sheet supersymmetry in the free-fermion models used in many constructions of four-dimensional heterotic strings and superstrings is shown to be due to the quantum-mechanical effect of an anomaly. A component action which may be used as the basis for studying supermodular invariance on an arbitrary genus Riemann surface and a superpace formulation are given.

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Superstring theories can be formulated in terms of two-dimensional field theories that are invariant under world-sheet supersymmetry. We shall here be concerned mainly with heterotic strings with $(1,0)$ supersymmetry. In the simplest such theory [1], the supersymmetry acts linearly, transforming free bosons and fermions into one another. We shall refer to this as the NSR model. The supercharge Q squares to give the right-moving momentum P_R , $\{Q, Q\} = P_R$, which is the standard $(1,0)$ superalgebra [2]. It is straightforward to couple the theory to $(1,0)$ supergravity [3], allowing the theory to be quantized covariantly using the path integral approach (see [4] for a review and references).

An interesting alternative is to use a free-fermion theory. These models have a non-linear supersymmetry [5,6] and they have been used extensively in the construction of four-dimensional superstrings [7]. However, the supersymmetry is of a rather unusual kind [8]. Although there is a conserved spin- $\frac{1}{2}$ fermionic charge Q , the global symmetry algebra of the classical action is $\{Q, Q\} = 0$ instead of the $(1,0)$ superalgebra. Upon quantization, the corresponding quantum operators are found to satisfy the $(1,0)$ algebra $\{Q, Q\} = aP_R$, where a is a loop-counting parameter (proportional to $h\alpha'$). Thus world-sheet supersymmetry only arises quantum mechanically. In particular, it is not possible to couple the system to $(1,0)$ supergravity at the classical level [8] (or to any other ordinary $(1,0)$ multiplets), so that it would appear that one could not quantize using the path integral approach. One would like to formulate the theory on arbitrary genus super-Riemann surfaces so as to be able to construct covariant loop amplitudes. If such a formulation is not possible, one cannot even begin to properly address issues such as super-modular invariance. These problems seem particularly acute in the $d = 4$ string theories [7] which have both an NSR sector, which can be quantized super-covariantly in the conventional way, and a free-fermion sector, which cannot. The purpose of this paper is to show how to resolve these issues so that one can study the covariant quantization of such $d = 4$ superstring theories.

The (1,0) NSR model has the component action¹

$$S = \frac{1}{a} \int d^2\sigma \left(\frac{1}{2} \partial_+ X^i \partial_- X^i + i \frac{1}{2} \psi_+^i \partial_- \psi_+^i \right) , \quad (1)$$

where $i = 1, \dots, d$ and a is the inverse string tension. If X is taken to be dimensionless and $[\psi] = \frac{1}{2}$, then a is dimensionless, while if $[X] = 1$ and $[\psi] = \frac{3}{2}$, a is a length squared, $[a] = 2$. The supercurrent $G_{++}^{NSR}(\sigma^\pm)$ and energy-momentum tensor $T_{++}^{NSR}(\sigma^\pm)$ are

$$G_{++}^{NSR} = \frac{1}{2} \psi_+^i \partial_+ X^i , \quad T_{++}^{NSR} = \frac{1}{2} \partial_+ X^i \partial_+ X^i + i \frac{1}{2} \psi_+^i \partial_+ \psi_+^i , \quad (2)$$

(with implicit normal ordering). The supercurrent has the OPE

$$G^{NSR}(\sigma_1^\pm) G^{NSR}(\sigma_2^\pm) \sim \frac{1}{2} \frac{T^{NSR}(\sigma_2^\pm)}{\sigma_1^\pm - \sigma_2^\pm} - \frac{a}{6} \frac{d}{(\sigma_1^\pm - \sigma_2^\pm)^3} . \quad (3)$$

The free-fermion model has the action

$$S = \frac{i}{2a} \int d^2\sigma \lambda_+^A \partial_- \lambda_+^A , \quad (4)$$

where $A = 1, \dots, D$ and has “supercurrent” G and stress energy tensor T

$$G_{++} = \frac{1}{3} f_{ABC} \lambda_+^A \lambda_+^B \lambda_+^C , \quad T_{++} = i \frac{1}{2} \lambda_+^A \partial_+ \lambda_+^A , \quad (5)$$

where normal ordering is again implicit and f_{ABC} are the structure constants of some D -dimensional Lie group G satisfying

$$f_{[AB}{}^C f_{C]D}{}^E = 0 , \quad f_{ACD} f_{BCD} = \frac{1}{2} \delta_{AB} . \quad (6)$$

The OPE of the supercurrents is then [5,6]

$$G(\sigma_1^\pm) G(\sigma_2^\pm) \sim \frac{a T(\sigma_2^\pm)}{\sigma_1^\pm - \sigma_2^\pm} - \frac{a^2}{3} \frac{D}{(\sigma_1^\pm - \sigma_2^\pm)^3} . \quad (7)$$

The OPE (3) leads to the usual super-Virasoro algebra corresponding to classical superconformal symmetry violated at one loop by an anomaly corresponding to the

¹Our notation is as follows: μ, ν are curved world-sheet indices, $(a, b, \dots) = (\pm, =)$ are tangent space indices w.r.t. some zweibein $e_\mu{}^a$, such that $=$ denotes the self-dual component of a two-vector and $-$ the anti-self-dual one. Our Minkowski metric is $\eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and for flat world-sheets we do not distinguish between curved and tangent indices, so that in the flat case world-sheet coordinates are $\sigma^\mu = (\sigma^\pm, \sigma^\mp)$. $(\alpha, \beta, \dots) = +, -$ are spinor indices and the gamma matrices $\gamma_{\alpha\beta}^a$, $\gamma_{\alpha\beta}^3$ satisfy $\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_{++}^3 = \gamma_{--}^3 = 1$, all other components vanishing.

Schwinger term in (3). For (7), however, one finds zero on the right-hand side at the tree-level (the $a = 0$ limit), the stress-energy term arising at one loop while the Schwinger term arises only at two loops.

The action (1) is invariant under the fermionic symmetry

$$\delta_\epsilon \lambda_+^A = i\epsilon_- f_{BC}^A \lambda_+^B \lambda_+^C, \quad (8)$$

corresponding to the conserved current G , where $\epsilon_-(\sigma)$ is a fermionic parameter satisfying $\partial_- \epsilon_- = 0$. The transformations (8) have the algebra

$$[\delta_\epsilon, \delta_{\epsilon'}] = 0, \quad (9)$$

corresponding to the classical ($a = 0$) limit of (7). As (9) is not the $(1, 0)$ superalgebra, it is straightforward to see that the action (4) cannot be coupled to supergravity [8]. The best one can do is to gauge the symmetries of the classical theory, *i.e.* to minimally couple to world-sheet gravity through a zweibein e_μ^a and covariant derivative ∇_μ and to gauge the fermionic symmetry (8) by introducing a corresponding gauge field $\chi_{\mu-}$. The resulting action is

$$S = \frac{1}{a} \int d^2\sigma e \left[\frac{i}{2} \lambda_+^A \nabla_- \lambda_+^A - \frac{1}{3} f_{ABC} \chi_{--} \lambda_+^A \lambda_+^B \lambda_+^C \right], \quad (10)$$

which is invariant under the local fermionic and Weyl symmetries with arbitrary parameters $\epsilon_-(\sigma^\pm, \sigma^-)$, $\Omega(\sigma^\pm, \sigma^-)$ given by

$$\begin{aligned} \delta \lambda_+^A &= i\epsilon_- f_{BC}^A \lambda_+^B \lambda_+^C, \\ \delta \chi_{--} &= \nabla_- \epsilon_- - \frac{1}{2} \Omega \chi_{--}, \\ \delta e_\mu^a &= \Omega e_\mu^a, \end{aligned} \quad (11)$$

and also under world-sheet diffeomorphisms and tangent space rotations. It is also invariant under the “super-Weyl” transformation $\delta \chi_{+-} = \eta_+$ as χ_{+-} doesn’t appear in (10). The gauge fields $e_\mu^a, \chi_{\mu-}$ can be viewed as Lagrange multipliers whose field equations yield the Virasoro constraints and the constraint $G_{++} = 0$, respectively.

The action (10) can be used in a path integral quantization in which one functionally integrates over $\lambda_+^A, e_\mu^a, \chi_{\mu-}$. We shall now analyse the anomalies that arise. At one loop, there is an anomaly in the fermionic symmetry. Indeed, the diagram in fig. 1 gives the following momentum-space contribution to the effective action

$$\Gamma_1 = \left(\frac{1}{4\pi} \right) \int d^2p \chi_{--}(p) \lambda_+^A(p) \frac{p_+^2}{(p^2)} \chi_{--}(-p) \lambda_+^A(-p), \quad (12)$$

The variation of Γ_1 , under the fermionic symmetry gives a local term

$$\Delta\Gamma_1 = \left(\frac{1}{2\pi}\right) \int d^2\sigma \epsilon_- \chi_- = -\lambda_+^A \partial_+ \lambda_+^A \quad , \quad (13)$$

which can be shown to be the linearized form of the full one-loop anomaly in the fermionic symmetry (to linear order in χ , quadratic order in λ and zero'th order in $e_\mu{}^a$). (The non-local terms in the variation of Γ , cancel against terms from other graphs.) The anomaly is non-trivial in that it cannot be cancelled by adding any local counterterms. However, it can be cancelled by modifying the transformation rules (11) by a -dependent terms. Noting that the integrand in (13) is $\epsilon_- \chi_- = -T_{\mp\mp}$, one sees that the anomaly can be cancelled to linear order if the zweibein is no longer inert under the fermionic symmetry, but transforms as

$$\delta e_{\mu-} = -\left(\frac{a}{2\pi}\right) \epsilon_- \chi_{\mu-} \quad . \quad (14)$$

This is analogous to the Green-Schwarz mechanism in which gauge and gravitational anomalies are cancelled by attributing an $\hbar$ -dependent transformation rule to the anti-symmetric tensor gauge field which was classically inert under those symmetries [9,2].

A non-linear analysis using Wess-Zumino consistency conditions [10] shows that the one-loop anomaly can be cancelled completely if the ϵ -transformation rules (11) become

$$\delta e_{\mu\mp} = 0 \quad , \quad \delta e_{\mu-} = -\frac{a}{2\pi} \epsilon_- \chi_{\mu-} \quad , \quad (15)$$

$$\delta \chi_{\mu-} = \nabla_\mu \epsilon_- - \frac{a}{2\pi} e_{\mu\mp} \chi_{\mp-} \chi_- = -\epsilon_- \quad , \quad (16)$$

$$\delta \lambda_+^A = i \epsilon_- f_{BC}{}^A \lambda_+^B \lambda_+^C - \frac{a}{2\pi} \epsilon_- \chi_{\mp-} \lambda_+^A \quad . \quad (17)$$

Remarkably, if $a = -2\pi$, (15) and (16) become precisely the transformation rules of (1,0) supergravity [4] whereas for general a this is a scaled form of the transformations. The commutator of two local supersymmetry transformations now gives a general co-ordinate transformation

$$[\delta_\epsilon, \delta_{\epsilon'}] = -\frac{a}{2\pi} \delta_{GCT}(i\epsilon\epsilon') + \dots \quad , \quad (18)$$

albeit scaled by a factor of a .

In this way supergravity emerges at the quantum level. A similar analysis can be repeated in (1,0) superspace with co-ordinates $z^M = (\sigma^\mu, \theta_-)$. The gauge fields $e_\mu{}^a$ and $\chi_{\mu-}$ are assembled into a super-vielbein $E_M{}^A(z)$ and the supergeometry is

restricted by imposing constraints on the torsion $T_{MN}{}^P$ [4]. As well as the conventional constraints [3]

$$T_{+,+}{}^+ = T_{+,+}{}^= = T_{+,=}{}^A = T_{\neq,=}{}^a = 0 \quad (19)$$

one imposes

$$T_{++}{}^\neq = i \frac{a}{2\pi} \quad (20)$$

This should be compared to the usual (1,0) supergeometry in which (19) holds but (20) is replaced by $T_{++}{}^\neq = i$ [4]. These standard constraints are easily derived from the $N = 1$ constraints [11] by truncation. The classical free-fermion model (10) can be formulated in superspace provided $T_{++}{}^\neq = 0$, using a superfield $\Lambda_+^A(z)$ satisfying the non-linear constraint

$$\nabla_+ \Lambda_+^A + \frac{1}{2} f^A{}_{BC} \Lambda_+^B \Lambda_+^C = 0 \quad , \quad (21)$$

where $\nabla_+ = E_+{}^M D_M + \text{connections terms}$ and $D_+ = \partial/\partial\theta_-$. Acting on (21) with ∇_+ and using $\{\nabla_+, \nabla_+\} = 0$, one sees that a constraint of the form of (21) can only be consistent if the coefficients $f^A{}_{BC}$ satisfy the Jacobi identity, i.e. they are the structure constants of a Lie algebra. The solution to the constraint (21) is

$$\Lambda_+ = g^{-1} \nabla_+ g \quad (22)$$

where $\Lambda_+ = \Lambda_+{}^A T_A$ and T_A are the group generators while $g(\sigma^\mu, \theta_-)$ is a superfield taking values in the Lie group G . The superspace field equation $\nabla_- \Lambda_+ = 0$ can be derived from an action similar to the supersymmetric Wess-Zumino-Novikov-Witten action [12]

$$S = - \int d^2\sigma d\theta E \text{Tr} \{ \nabla_+ g^{-1} \nabla_- g + \int_0^1 dt \hat{g}^{-1} \partial_t \hat{g} (\nabla_- \hat{g}^{-1} \nabla_+ \hat{g} - \nabla_+ \hat{g}^{-1} \nabla_- \hat{g}) \} \quad (23)$$

where $\hat{g}(\sigma^\mu, \theta, t)$ depends on an additional bosonic co-ordinate t such that

$$\hat{g}(\sigma, \theta, 0) = 0 \quad , \quad \hat{g}(\sigma, \theta, 1) = g(\sigma, \theta) \quad .$$

However, on quantization, the cancellation of anomalies requires that $T_{++}{}^\neq$ becomes a -dependent as in (20), so the super-derivative $\nabla_+^{(a)}$ now satisfies $\{\nabla_+^{(a)}, \nabla_+^{(a)}\} = \frac{ia}{2\pi} \nabla_+^{(a)}$. Thus, the supergeometry becomes modified at the quantum level — we shall refer to this as a “quantum supergeometry”. The constraint (21) is modified by non-local terms so that the quantum action is not equivalent to that of the (1,0) supersymmetric WZNW model. Further details will be given elsewhere [10].

So far we have considered the “supersymmetry” anomaly at one loop. The integration over the fields λ_+^A also leads to gravitational and Weyl anomalies at one loop and supersymmetry and super-Weyl anomalies at two loops, all of which are non-trivial and proportional to the number of fermion fields, D . These anomalies can only be cancelled by contributions from other matter fields and ghost contributions. To this end, we add n left-moving free fermions $\hat{\rho}_-^I$, $I = 1, \dots, n$, which furnish a representation of the quantum super-algebra $\{Q, Q\} = iaP_R$ on-shell if $\{Q, \hat{\rho}\} = 0$ as $[p_R, \hat{\rho}_-^I] = 0$ on-shell as a result of the field equation $\nabla_- \hat{\rho}_-^I = 0$. This can be extended to an off-shell representation by introducing auxiliary fields $F_{\mp}^I$ and superfields $\rho_{\mp}^I = \hat{\rho}_{\mp}^I + \theta_- F_{\mp}^I$ and writing the quantum superspace action

$$S = \frac{i}{2a} \int d^2\sigma^- d\theta E \rho_-^I \nabla_+^{(a)} \rho_-^I. \quad (24)$$

Next we turn to the ghost sector. We suppose that, as in the usual NSR superstring, one is to integrate over e_{μ}^a and $\chi_{\mu-}$. We choose the superconformal gauge $e_{\mu}^a = S_{\mu}^a$, $\chi_{\mu} = 0$ and quantize using the BRST formalism. At one loop, there is a violation of the BRST Ward identities resulting from the supersymmetry anomaly discussed above. Cancellation of this requires modifying the BRST transformations by a -degrees terms corresponding to those in (15,16,17) and BRST invariance of the quantum action requires adding a -dependent terms to the ghost Lagrangian. The resulting ghost action can be written economically in quantum superspace as

$$S_{GH} = \frac{i}{a} \int d^2\sigma d\theta E^{(a)} [B_{==} \nabla_+^{(a)} C_{\mp} + B_{\mp+} \nabla_{=}^{(a)} C_{=}] , \quad (25)$$

in terms of ghost superfields $C_{\mp}, C_{=}$ and anti-ghost superfields $B_{==}, B_{\mp+}$. The ghost structure is then very similar to that of the usual (1,0) supergravity [13].

With this ghost structure and the matter sector (10,24) the cancellation of the one-loop gravitational and Weyl anomalies requires

$$n/2 = 26, \quad D/2 = 10 , \quad (26)$$

as one might expect. Remarkably these conditions are also sufficient for the cancellation of the “supersymmetry” anomalies. The integration over the λ -fields gives a “supersymmetry” anomaly proportional to $a^2 D$ from a 2-loop graph, while the ghosts contribute to the anomaly through a one-loop graph. There is one factor of a because it is a one-loop graph, but there is a second factor of a coming from an a -dependent vertex arising in the component expansion of (25), giving a result proportional to $-10a^2$. Thus, with matter fields that are representations of quantum

“supersymmetry”, the anomalies cancel for any value of a , provided (26) holds. In this way, anomaly-free superstring theories can be constructed in a completely fermionic formulation. Suppose, however, that one wished to include a sector consisting of d NSR multiplets (1). The fields X^i could then be interpreted as the co-ordinates of a d -dimensional space-time and four such multiplets are required by the $4 - d$ superstring models [7]. The NSR action is given by minimally coupling (1) to $e_\mu{}^a$ and to an NSR gravitino $\hat{\chi}_{\mu-}$ [3]. In the classical theory, χ_μ the gravitino for the supersymmetry (8) and $\hat{\chi}_\mu$ should be regarded as distinct as they are gauge fields for different supersymmetries with parameters ϵ_- and $\hat{\epsilon}_-$ respectively.

In the quantum theory there seem to be two possibilities: one can either keep the two supersymmetries distinct or choose to identify the two in some way.

First, suppose one regards them as distinct supersymmetries and suppose $d > 0$ and $D > 0$. Then there are two gravitini, χ_μ and $\hat{\chi}_\mu$, and two sets of superghosts, β_{--}, γ_+ and $\hat{\beta}_{--}, \hat{\gamma}_+$. The cancellation of gravitational and Weyl anomalies requires ($4 = 26-11-11$)

$$d + n/2 = 26, \quad d + D/2 = 4, \quad (27)$$

while the two supersymmetry anomalies can only cancel separately if $d = D/2 = 10$, which is inconsistent with (27). Thus this possibility doesn't lead to any new models.

To obtain anything new, one must then identify the two supersymmetries. Comparing (15,16) with the corresponding transformations for (1,0) supergravity [3], we see that if $a = 2\pi$, it is consistent to make the identification $\epsilon = \hat{\epsilon}$, $\chi_\mu = \hat{\chi}_\mu$. In this regime of strong string tension, quantum corrections are as important as classical effects and perturbation theory breaks down. Alternatively, if one is prepared to allow non-analytic behavior in the loop counting parameter a , one can make the identification

$$\epsilon_- = a^{\frac{1}{2}} \hat{\epsilon}_-, \quad \chi_{\mu-} = a^{-\frac{1}{2}} \hat{\chi}_{\mu-}, \quad (28)$$

so that the transformations (15,16) become identical to those of ordinary (1,0) supergravity. (It is also convenient to rescale λ , $\lambda_+^A = a^{-\frac{1}{2}} \hat{\lambda}_+^A$.) In either approach, we find that the parameter a enters the theory in a way which is very different from the way it enters the usual NSR string; it would be of considerable interest to investigate the physical implications of this. With either method of identifying the supersymmetries, there is only one set of superghosts and we find that cancellations of local gravitational, Weyl and supersymmetry anomalies requires

$$d + \frac{n}{2} = 26, \quad d + D/2 = 10, \quad (29)$$

in agreement with the results given in [8].

In conclusion, we have shown that the classical fermionic “supersymmetry” of the free-fermion model has a quantum anomaly whose cancellation requires modifying the transformation rules in such a way that the usual (1,0) supersymmetry emerges only at the quantum level. The free-fermion model might therefore be described as furnishing a non-linear representation of *quantum* supersymmetry. Presumably, a similar phenomenon occurs in the purely bosonic lattice models in which, from analysing the current algebra, it appears that supersymmetry also emerges at the quantum level [14].

We have seen that the four-dimensional superstrings constructed using a free-fermion sector are necessarily theories without a (naive) $a = 0$ limit and for which there appears to be no perturbation theory in a . The same seems likely to also be true for the $d = 4$ superstrings constructed using the covariant lattice approach.

Here we have only considered the linearized anomalies. By studying Wess-Zumino consistency conditions, we have shown that the conditions obtained are sufficient for the cancellation of the full non-linear anomalies [10]. These theories are now amenable to a path integral approach and the critical issue of supermodular invariance can be studied in these unusual realizations of local world-sheet supersymmetry.

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FIG. 1