FUSION AND BRAIDING IN EXTENDED CONFORMAL THEORIES

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ABSTRACT

We define the chiral conformal blocks of integer-spin extended (W-algebra) conformal theories by fusion of elementary ones. The braid group representation matrices realizing the exchange algebra are computed. They are shown to coincide with the Boltzmann weights - in a certain limit of the spectral parameter - of the face models of Jimbo et al. In this article we restrict ourselves to the case corresponding to symmetric tensor representations of A_n .

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1. Introduction

A lot of progress has been made recently in elucidating the relations between conformal field theory and braid groups¹⁻⁷⁾ on the one hand, and quantum groups⁶⁻¹²⁾ on the other. The braid properties of a conformal field theory are closely related⁷⁾ to its fusion algebra¹³⁾. Determining the braid group representations carried by the conformal blocks is an important step towards the complete classification of the conformal field theories, and hence of classical string vacua.

It has become more and more clear recently - at least for the standard minimal models¹⁴⁾ - that the braiding matrices of conformal field theories are intimately connected to the R matrices (Boltzmann weights) of exactly solvable (lattice) models. First, the braiding matrix of the fundamental $\phi_{(21)}$ field in conformal field theory was determined by Gervais and Neveu¹⁵⁾ in their study of the quantum Liouville theory^{15,16)}. This quantum Liouville theory is essentially a theory of one free bosonic field with an (imaginary) background charge at infinity (Feigin-Fuchs¹⁷⁾ representation). The same representation of conformal field theories (for real background charge) was also developed in parallel by Dotsenko and Fateev¹⁸⁾. This braiding matrix of the $\phi_{(21)}$ field is related⁵⁾ by the vertex-SOS transformation to the R matrices of the exactly solvable 8-vertex model (associated with the spin $\frac{1}{2}$ representation of $su(2)$)¹⁹⁾. It has also been shown²⁰⁾ that it coincides - modulo a non-trivial change of basis - with the R -matrix of the quantum group $sl(2)_q$.

In a recent article⁵⁾, Felder, Froehlich and Keller (FFK) have computed the braiding matrices R for arbitrary $\phi_{(nm)}$ fields of the minimal model series. They use the Feigin-Fuchs representation¹⁷⁾ and exploit the fact that $\phi_{(n+1,m)}$ can be obtained by fusion¹⁴⁾ from $\phi_{(nm)}$ and $\phi_{(21)}$. As a consequence, the braiding matrices for $\phi_{(n+1,m)}$ are also determined by "fusion" of the braiding matrices for $\phi_{(nm)}$ and those for $\phi_{(21)}$. This gives a recurrence formula to compute all braiding matrices in terms of the elementary ones already known. By construction, they all satisfy the quantum Yang-Baxter equation²¹⁾, since it only expresses the associativity of the operator product algebra.

Quite independently, exactly solvable two-dimensional lattice models have been developed by Date, Jimbo, Miwa, Kuniba and Okado²²⁻²⁷⁾. The Boltzmann weights of these models solve the star-triangle relation which is nothing but the Yang-Baxter equation. In their first article²²⁾, Jimbo et al. construct the Boltzmann weights $W_{n-1,n'-1}$ corresponding to an arbitrary pair of $su(2)$ representations of spins $\frac{1}{2}(n-1)$ and $\frac{1}{2}(n'-1)$ by a fusion procedure in terms of the elementary Boltzmann weights associated with a pair of spin $\frac{1}{2}$ representations. FFK have shown⁵⁾ that the braiding matrix of $\phi_{(nm)}$ with

$\phi_{(n',m')}$ in the conformal field theory coincides - up to some irrelevant factors - with the product $W_{n-1,n'-1} W_{m-1,m'-1}$ of these Boltzmann weights, in a certain limit of the spectral parameter.

Relating such basic properties of a conformal field theory as is the exchange algebra, realized by its braiding matrices, to the Boltzmann weights of exactly solvable models is of twofold importance: besides giving us a deeper insight into the structure of conformal field theories, it might ultimately provide a method to go beyond criticality.

It is challenging to generalize this correspondence to all conformal field theories. On the one hand, there are several ways to extend the Virasoro algebra: the conformal field theory might have additional Kac-Moody symmetry (Wess-Zumino-Witten models²⁸⁾) or contain symmetry generators of integer conformal spin larger than two. The latter case, often referred to as extended conformal theories or W-algebras, was introduced by Fateev and Zamolodchikov²⁹⁾ and studied in more detail by several authors^{30-37,20)}. These extended conformal theories are related to coset models $\hat{g}_k \oplus \hat{g}_1 / \hat{g}_{k+1}$ for a (simply-laced) affine algebra $\hat{g}$. The standard (unextended) conformal theories correspond to³⁸⁾ $\hat{g} = A_1^{(1)} = \hat{su}(2)$ (for a review see ref.³⁹⁾). On the other hand, Jimbo et al. have extended their work on the fusion of the eight vertex SOS models²²⁾ (corresponding to $A_1^{(1)}$) to exactly solvable lattice models, called "interaction round a face" (IRF) models (see also ref.⁴⁰⁾). In these models the local states taken on by the fluctuation variables on the (square) lattice *sites* are the dominant integral weights of $A_n^{(1)}$ (ref.²³⁻²⁵⁾) or of $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ (ref.²⁷⁾). Jimbo et al. also introduced the $A_n^{(1)}$ face models where, in addition, new fluctuation variables assuming weights of the representations of $sl(n+1, \mathbb{C})$ (specified by Young tableaux) are placed on the *edges* of the lattice²⁶⁾. Jimbo et al. already noticed in ref.²⁵⁾ that their local state probabilities are given by theta function identities realizing the decomposition of the characters in a GKO construction for $A_n^{(1)} \oplus A_n^{(1)} / A_n^{(1)}$ with levels k and 1, and that the branching coefficients represent irreducible Virasoro characters such that the corresponding central charge and highest weights are those of the W-algebras associated with A_n as studied in refs²⁹⁻³⁵⁾. As already mentioned, it would be important to understand better this correspondence by directly relating the braiding matrices of the extended conformal theories to the Boltzmann weights. It is the aim of the present article to contribute to establishing this relation for the A_n case. We will however restrict ourselves to the case where the edge variables of ref.²⁶⁾ take only values among the symmetric tensor representations.

In the study of the Toda field theories based on any simple Lie algebra g of rank n ,

Gervais and the present author³²⁻³⁵⁾ have shown that these theories yield realizations of extended conformal symmetry (W-algebras) in terms of n free bosonic fields with a background charge at infinity. This Feigin-Fuchs¹⁷⁾ -type construction was also carried out by Fateev and Lykhanov³⁰⁾. In our opinion, the Toda field theoretic approach has the advantage that its quantization yields very naturally the W-generators³²⁻³⁵⁾, central charges and highest weights^{34,35)} for all simple Lie algebras g . For $g = A_n$, we have obtained³⁴⁾ the braiding matrix for certain fundamental fields, which generalize the $\phi_{(21)}$ of the non-extended theories. These fundamental fields correspond to the (fundamental) vector representation of A_n . As we will show in the present paper, this braiding matrix coincides up to irrelevant factors with the Boltzmann weight W_{11} - in a certain limit of the spectral parameter - constructed by Jimbo et al.²⁵⁾ for their $A_n^{(1)}$ face models.

Here we will define arbitrary rank N symmetric tensor fields by fusion of the fundamental fields. This allows us to obtain the braiding matrices R_{NM} between rank N and rank M fields by fusion of the R_{11} . This directly generalizes the results of FFK⁵⁾ to extended conformal algebras. Again, we will show that the R_{NM} coincide with the Boltzmann weights W_{NM} (in the appropriate limit, and up to irrelevant factors) of the $A_n^{(1)}$ face models of Jimbo et al.²⁶⁾ ^{*}. The case of arbitrary Young tableaux (including the antisymmetric tensor representations) is left for the future.

The plan of the paper goes as follow: In section 2, we review the description of extended conformal theories in terms of n free bosonic fields with a background charge and define the chiral screened vertex operators. We recall from ref.³⁴⁾ the form of the braiding matrix R_{11} for the fundamental conformal blocks and discuss the vertex-IRF correspondence in the present context. In section 3, we give the fusion procedure for the conformal blocks and the braiding matrices. We establish the precise relation between the braiding matrices R_{NM} and the Boltzmann weights W_{NM} . Section 4 contains conclusions and outlooks. In order that this paper be self-contained, we have added an appendix: it gives a brief summary of the work on Toda field theories of refs³²⁻³⁵⁾ and contains the dictionary between the present notation and the one of refs.³²⁻³⁵⁾

^{*} More precisely, R_{NM} factorizes into a pair $R_{N^+M^+}^{(+)}$, $R_{N^-M^-}^{(-)}$, each of the $R^{(+)}$ and $R^{(-)}$ coinciding with the limit of a $W_{N^+M^+}$ (up to irrelevant factors).

2. Extended conformal theories, their chiral screened vertex operators, and the fundamental braiding matrix

2.1 FEIGIN-FUCHS-TYPE REPRESENTATION OF EXTENDED CONFORMAL THEORIES

In extended conformal theories²⁹⁻³⁷⁾, the algebraic structure is enlarged beyond the Virasoro algebra. In addition to the energy-momentum tensor $T(z) \equiv W^{(2)}(z)$ which is a conformal (primary) field of weight 2 (modulo the conformal anomaly), there are conformal fields $W^{(k)}(z)$ of weight $k \in \mathbb{Z}$. In general the values of k are in one-to-one correspondence with the orders of the Casimir operators of a given Lie algebra g , e.g. $k = 2, 3, \dots, n+1$ for A_n . The $W^{(k)}$ form a closed operator product algebra (W-algebra) in the enveloping algebra:[†]

$$W^{(k)}(z)W^{(k')}(z') = \frac{c_{k+k'}^{kk'}}{(z-z')^{k+k'}} + \sum_{l=2}^{k+k'-1} \frac{c_{k+k'-l}^{kk'}}{(z-z')^{k+k'-l}} P_l(W^{(j)}(z)) \quad (2.1)$$

where $P_l(W^{(j)})$ is some polynomial of the $\partial^r W^{(j)}$ such that $\sum j + \sum r = l$. The best known example is Zamolodchikov's spin-3 algebra²⁹⁾ ($W \equiv W^{(3)}, T \equiv W^{(2)}$):

$$\begin{aligned} W(z)W(z') &= \frac{c/3}{(z-z')^6} + \frac{2T(z')}{(z-z')^4} + \frac{\partial T(z')}{(z-z')^3} \\ &+ \frac{3}{10} \frac{\partial^2 T(z')}{(z-z')^2} + \frac{1}{15} \frac{\partial^3 T(z')}{z-z'} + \frac{2\beta \mathcal{L}(z')}{(z-z')^2} + \frac{\beta \partial \mathcal{L}(z')}{z-z'} \\ \mathcal{L}(z) &=: T(z)T(z) : - \frac{3}{10} \partial^2 T(z) \quad , \quad \beta = \frac{16}{22+5c} \end{aligned} \quad (2.2)$$

Although the $W^{(k)}$ do not form a *Lie* algebra, the algebraic structure (2.1) is all one needs to define generalized Verma modules³⁴⁾. The representation theory of the W-algebras is much less well known than the one of the standard Virasoro algebra. However, using a Feigin-Fuchs-type realization, it was possible to obtain the spectrum of highest weights^{29-30,34)} in general. For the unitary $c < n = \text{rank } g$ series (and for simply-laced g) these results are confirmed by the GKO construction³¹⁾ based on $\hat{g}_k \oplus \hat{g}_1/\hat{g}_{k+1}$. There exist also some results⁴¹⁾ concerning the zeros of the generalized Kac determinant.

[†] Although at present there exists no proof that this is true for any n and any k, k' , there is a lot of evidence from sample calculations^{34,35)} that it is.

We will use the Feigin-Fuchs-type representation of these extended algebras^{29-30,32-35)} as it comes out naturally in the study of the quantum Toda field theory.³²⁻³⁵⁾ * For simplicity, here we will restrict ourselves to the case of simply-laced g (see refs³³⁻³⁵⁾ for the non-simply-laced case). This Feigin-Fuchs-type construction uses n free bosonic fields ϕ_i with a background charge placed at infinity:

$$\langle \phi_i(z) \phi_j(z') \rangle = -\delta_{ij} \ln(z - z') \quad (2.3)$$

The ϕ_i are standard Fubini-Veneziano fields. Normal-ordered chiral vertex operators are defined by

$$V_\alpha(z) = : e^{i\alpha \cdot \phi(z)} : \equiv T_\alpha z^{\alpha \cdot a_0} \exp \left(\alpha \cdot \sum_{n=1}^{\infty} a_{-n} \frac{z^n}{n} \right) \exp \left(-\alpha \cdot \sum_{n=1}^{\infty} a_n \frac{z^{-n}}{n} \right) \quad (2.4)$$

where $[a_n^i, a_m^j] = \delta^{ij} n \delta_{n,-m}$ and $[a_n^i, T_\alpha] = \delta_{n,0} \alpha^i T_\alpha$.

One introduces the simple roots $e_i, i = 1, \dots, n$, of g (normalized as $e_i \cdot e_i = 2$), the n fundamental weights λ_i satisfying

$$\lambda_i \cdot e_j = \delta_{ij} \quad (2.5)$$

and the weights h_μ of the fundamental vector representation ($h_1 = \lambda_1$). For $g = A_n$ e.g. one has $\mu = 1, \dots, n+1$, $\sum_\mu h_\mu = 0$, $h_\mu \cdot h_\nu = \delta_{\mu\nu} - \frac{1}{n+1}$, $e_i = h_i - h_{i+1}$, $\lambda_i = \sum_{\mu=1}^i h_\mu$ and $h_\mu = \lambda_\mu - \lambda_{\mu-1}$ ($\lambda_0 = \lambda_{n+1} = 0$). The Weyl vector ρ is defined as half the sum of the positive roots and equals

$$\rho = \sum_{i=1}^n \lambda_i \quad (2.6)$$

The energy-momentum tensor then reads

$$T(z) = -\frac{1}{2} \partial \phi(z) \cdot \partial \phi(z) + 2i\alpha_0 \rho \cdot \partial^2 \phi(z) \quad (2.7)$$

where α_0 is the background charge which is related to the Virasoro central charge by

$$c = n - 48\alpha_0^2 \rho^2 = n(1 - h_c(h_c + 1)4\alpha_0^2) \quad (2.8)$$

h_c being the (dual) Coxeter number of g ($h_c = n+1$ for A_n). In a similar way, the other fields $W^{(k)}$ are of the form $W^{(k)} \simeq \sum_{r=1}^k \partial^r \phi \prod_{l=1}^{k-r} \partial \phi$. Their exact expression^{30,33-34)} is not relevant to us here.

* A brief summary of the content of refs³²⁻³⁵⁾ relevant to the present article is given in the appendix.

The vertex operators (2.4) are (chiral) primary fields of conformal dimension

$$\Delta(\alpha) = \frac{1}{2}\alpha \cdot (\alpha - 4\alpha_0\rho) = \frac{1}{2}(\alpha - 2\alpha_0\rho)^2 + \frac{c-n}{24} \quad (2.9)$$

Screened vertex operators (see next subsection) are built up from the screening operators which are vertex operators with unit conformal dimensions. Define

$$V_{\alpha_{\pm}}^i(z) = V_{\alpha_{\pm}e_i}(z) \equiv : e^{i\alpha_{\pm}e_i \cdot \phi(z)} : \quad (2.10)$$

which have unit conformal dimension provided

$$\alpha_{\pm}^2 - 2\alpha_0\alpha_{\pm} - 1 = 0 \quad (2.11)$$

As usual in Feigin-Fuchs-type constructions¹⁸⁾, the spectrum of highest weights of the theory is determined by the screening charges $\alpha_{\pm}$. It is found to be given by^{30,34)} ($m = (m_1, \dots, m_n)$)

$$\Delta_{mm'} = \Delta(\alpha_{mm'}), \quad \alpha_{mm'} = - \sum_{i=1}^n ((m_i - 1)\alpha_- + (m'_i - 1)\alpha_+) \lambda_i \quad (2.12)$$

Note that, since $\alpha_+ + \alpha_- = 2\alpha_0$, we have $\alpha_{mm'} - 2\alpha_0\rho = - \sum_{i=1}^n (m_i\alpha_- + m'_i\alpha_+)\lambda_i$. Substituting this equation into (2.9) and expressing α_0 in terms of c via (2.8) one obtains

$$\Delta_{mm'} = \frac{n-c}{96\rho^2} \left(\sum_{i=1}^n \left(m_i + m'_i + (m_i - m'_i) \sqrt{\frac{n(2h_c + 1)^2 - c}{n-c}} \right) \lambda_i \right)^2 + \frac{c-n}{24} \quad (2.13)$$

In ref.^{34,35)} it was argued that the rational theories are precisely such that the square-root appearing in this equation (and hence the highest weights themselves) are rational. This requirement, which is equivalent to having rational α_-^2 , then gives the generalized BPZ series of minimal models:

$$\alpha_-^2 = \frac{r}{s} \Leftrightarrow c = n \left(1 - \frac{(r-s)^2 h_c(h_c + 1)}{rs} \right) \quad (2.14)$$

The Hilbert space of the extended conformal theories is a direct sum of irreducible representation spaces for a pair of (left and right) W -algebras:

$$\mathcal{H} = \bigoplus_L \mathcal{H}_L \otimes \mathcal{H}_L \quad (2.15)$$

where L labels the irreducible highest weight representations. Generically, these correspond to a fixed eigenvalue of the zero-mode a_0 . The highest weight states are obtained by

applying the chiral primary vertex operators (2.4) to the $sl(2, \mathbb{C})$ invariant vacuum state $|0\rangle$ (satisfying $a_n^i|0\rangle = 0, \forall n \geq 0, \forall i$):

$$|(l')\rangle = \lim_{z \rightarrow 0} V_{\alpha_{l'}}(z)|0\rangle \quad (2.16)$$

which is a Fock space vacuum state with a_0 eigenvalue $\alpha_{l'}$. Hence the index L in (2.15) corresponds to (l') . For rational theories, L only takes on a finite number of values.

The primary fields of the extended conformal theory decompose into holomorphic and antiholomorphic parts:

$$\Phi_{\Delta_{mm'}} = \sum_{L,K} d_{(mm')L}^K V_{(mm')L}^K(z) \otimes V_{(mm')L}^K(\bar{z}) \quad (2.17)$$

$V_{(mm')L}^K$ has conformal dimension $\Delta_{mm'}$ and maps $\mathcal{H}_L$ to $\mathcal{H}_K$.

2.2 THE SCREENED VERTEX OPERATORS

The chiral primary fields are screened vertex operators $V_{(mm')L}^K$ built up from (un-screened) vertex operators (2.4) with $\alpha = \alpha_{mm'}$ and the screening operators which are integrals over closed contours of the vertex operators $V_{\alpha_{\pm}}^i$ with unit conformal dimensions (see (2.10,11)). Hence, the screening operators have zero conformal dimension, and the screened vertex operators have the same dimensions as the unscreened ones, $\Delta = \Delta(\alpha_{mm'})$. However, the screening operators $\oint V_{\alpha_{\pm}}^i$ shift the eigenvalues of the zero-mode a_0 by $\alpha_{\pm} e_i$.

The integration contours for the screened vertex operators

$$V_{mm'}^{i_1 \dots i_r j_1 \dots j_{r'}}(z) = \int V_{\alpha_{mm'}}(z) V_{\alpha_-}^{i_1}(u_1) \dots V_{\alpha_-}^{i_r}(u_r) V_{\alpha_+}^{j_1}(v_1) \dots V_{\alpha_+}^{j_{r'}}(v_{r'}) \prod_{a=1}^r \prod_{b=1}^{r'} du_a dv_b \quad (2.18)$$

have to be chosen carefully. For the unextended conformal theories ($\simeq A_1$) this is discussed in detail by FFK⁵⁾, and also by Dotsenko and Fateev¹⁸⁾. Since the integrand is many-valued, it must be defined by analytic continuation from some reference configuration of the integration variables. The latter is chosen to have all u_a, v_b ordered on the positive real axis (cf ref.⁵⁾ for A_1) as $0 < v_{r'} < \dots < v_1 < u_r < \dots < u_1 < z$, see fig. 1.

The screened vertex operators (2.18) should be thought of as inserted in correlation functions $\langle 0|V(z_N)V(z_{N-1})\dots V(z_2)V(z_1)|0\rangle$. This of course shows that the integration contours in fig. 1 *cannot* be deformed arbitrarily. In particular, if one considers a screened

vertex operator acting on $\mathcal{H}_L$ one should imagine a $V_{\alpha_{\mu'}}(z_1)$ inserted at $z_1 = 0$ in fig. 1. Since

$$V_{\alpha_{\pm}}^i(w)V_{\alpha_{\pm}}^j(w') = (w - w')^{\alpha_{\pm}^2 e_i \cdot e_j} : V_{\alpha_{\pm}}^i(w)V_{\alpha_{\pm}}^j(w') : \quad (2.19a)$$

$$V_{\alpha_{\pm}}^i(w)V_{\alpha_{\mp}}^j(w') = (w - w')^{-e_i \cdot e_j} : V_{\alpha_{\pm}}^i(w)V_{\alpha_{\mp}}^j(w') : \quad (2.19b)$$

(with $e_i \cdot e_j = 0, 2, -1$ for any simply-laced g) the contours over which vertex operators V^i and V^j with $e_i \cdot e_j = 0$ are integrated can be interchanged without altering the value of (2.18). This is true, since the difference between the two orderings then is given by $\oint_w \oint_{|w'-w|=\epsilon} : V^i(w)V^j(w') :$ which vanishes due to the regularity of the normal-ordered product as $w' \rightarrow w$. The same is true for $V_{\alpha_+}^i V_{\alpha_-}^i$ (see ref.⁵⁾) and also for $V_{\alpha_-}^i V_{\alpha_+}^j$ if $e_i \cdot e_j = -1$. Hence we have

$$\int_{|u| > |v|} V_{\alpha_-}^i(u)V_{\alpha_+}^j(v)dudv = \int_{|v| > |u|} V_{\alpha_+}^j(v)V_{\alpha_-}^i(u)dudv, \quad \forall i, j \quad (2.20)$$

In the other cases, interchanging the contours is non-trivial and requires their decomposition into pieces with appropriate phase factors for each piece. For the unextended conformal theories this has been discussed in detail by Dotsenko and Fateev¹⁸⁾ and FFK.⁵⁾

In the following we restrict ourselves to the A_n theories. We define the "fundamental" conformal blocks to be the following screened vertex operators associated with the fundamental vector representation of A_n :

$$\begin{aligned} \hat{V}_{\mu}^{-}(z) &= \int V_{\alpha_{(21\dots 1)(1\dots 1)}}(z)V_{\alpha_-}^1(u_1)V_{\alpha_-}^2(u_2)\dots V_{\alpha_-}^{\mu-1}(u_{\mu-1})du_{\mu-1}\dots du_1 \\ \hat{V}_{\mu}^{+}(z) &= \int V_{\alpha_{(1\dots 1)(21\dots 1)}}(z)V_{\alpha_+}^1(v_1)V_{\alpha_+}^2(v_2)\dots V_{\alpha_+}^{\mu-1}(v_{\mu-1})dv_{\mu-1}\dots dv_1 \end{aligned} \quad (2.21)$$

These operators shift the a_0 eigenvalue by $-\alpha_{\pm}(\lambda_1 - e_1 - \dots e_{\mu-1}) = -\alpha_{\pm}h_{\mu}$. Recall that h_{μ} is the μ^{th} weight of the vector representation of A_n .

In order to indicate clearly between which Hilbert spaces the screened vertex operators are acting, it is convenient to write

$$V_{ML}^K(z) \equiv V_{(mm')(ll')}^{(kk')}(z) = e^{i\theta_{(ll')}}^{i_1\dots i_r j_1\dots j_{r'}} V_{mm'}^{i_1\dots i_r j_1\dots j_{r'}}(z) \quad : \quad \mathcal{H}_{(ll')} \rightarrow \mathcal{H}_{(kk')} \quad (2.22)$$

with some phase factor θ (see below). Note that the data $(mm'), (ll')$ and (kk') do not unambiguously specify the screened vertex operator on the r.h.s. . They tell us which and

how many $V_{\alpha_{\pm}}^i$ do occur on the r.h.s., but they do not fix their order. The problem of ordering is quite subtle in general (except for the cases discussed above: $e_i \cdot e_j = 0$ or $V_{\alpha_+}^i V_{\alpha_-}^i$), but has a simple solution for the cases of interest in the present paper: when $\alpha_{ll'}$ and $\alpha_{kk'}$ differ by $-\alpha_{\pm} h_{\mu}$, $\mu = 1, \dots, n+1$, we define

$$V_{(\lambda_1, 0)L}^{L-\alpha_{\pm} h_{\mu}}(z) = e^{i\theta_L^{12\dots\mu-1}} \hat{V}_{\mu}^{\pm}(z) \equiv \mathcal{V}_{\mu}^{\pm}(z) \quad (2.23a)$$

Here, $M = (\lambda_1, 0)$ is meant to be $m = (2, 1, \dots, 1)$, $m' = (1, \dots, 1)$ such that $\alpha_{mm'} = -\lambda_1 \alpha_-$, and $L - \alpha_{\pm} h_{\mu}$ is meant to represent those indices (kk') such that $\alpha_{kk'} = \alpha_{ll'} - \alpha_{\pm} h_{\mu}$.^{*}

The phase factors are chosen to satisfy the following condition

$$\begin{aligned} \theta_{\alpha_L}^{i_1 \dots i_r \tilde{i}_1 \dots \tilde{i}_r j_1 \dots j_r \tilde{j}_1 \dots \tilde{j}_r} &= \theta_{\alpha_L - \alpha_- (m_s \lambda_s - e_{i_1} - \dots - e_{i_r}) - \alpha_+ (m'_s \lambda_s - e_{j_1} - \dots - e_{j_r})}^{i_1 \dots i_r \tilde{i}_1 \dots \tilde{i}_r j_1 \dots j_r \tilde{j}_1 \dots \tilde{j}_r} \\ &\quad + \theta_{\alpha_L}^{i_1 \dots i_r j_1 \dots j_r \tilde{j}_1 \dots \tilde{j}_r} \\ &\quad - \pi \left(\alpha_- (e_{i_1} + \dots + e_{i_r}) + \alpha_+ (e_{j_1} + \dots + e_{j_r}) \right) \cdot (\alpha_- m_s \lambda_s + \alpha_+ m'_s \lambda_s) \end{aligned} \quad (2.24)$$

which will ensure that the fusion of screened vertex operators (2.22) will look as simple as possible (see section 3). Superscripts i and $\tilde{i}$ refer to screening operators $V_{\alpha_-}^i$, $V_{\alpha_-}^{\tilde{i}}$ while j and $\tilde{j}$ refer to $V_{\alpha_+}^j$, $V_{\alpha_+}^{\tilde{j}}$. The simplest solution to (2.24) is

$$\begin{aligned} \theta_{\alpha_L}^{i_1 \dots i_r j_1 \dots j_r \tilde{j}_1 \dots \tilde{j}_r} &= -\pi \left(\alpha_- (e_{i_1} + \dots + e_{i_r}) + \alpha_+ (e_{j_1} + \dots + e_{j_r}) \right) \\ &\quad \cdot \left(\frac{1}{2} \alpha_- (e_{i_1} + \dots + e_{i_r}) + \frac{1}{2} \alpha_+ (e_{j_1} + \dots + e_{j_r}) + \alpha_L \right) \end{aligned} \quad (2.25)$$

However, one can always add a term $\Delta \theta_{\alpha_L}^{i_1 \dots i_r j_1 \dots j_r \tilde{j}_1 \dots \tilde{j}_r} = (e_{i_1} + \dots + e_{i_r}) \cdot \beta + (e_{j_1} + \dots + e_{j_r}) \cdot \gamma$ with constant vectors β, γ without affecting the validity of (2.24). For the operators $\mathcal{V}_{\mu}^-$ one has simply

$$\mathcal{V}_{\mu}^- \Big|_{\mathcal{H}_L} = (-)^{\sum_{i=1}^{\mu-1} (l'_i - 1)} \exp i\pi \alpha_-^2 \left(-1 + \sum_{i=1}^{\mu-1} (l_i - 1) \right) \hat{V}_{\mu}^- \Big|_{\mathcal{H}_L} \quad (2.23b)$$

and similar for $\mathcal{V}_{\mu}^+$.

^{*} Explicitly one has $k_i = l_i + \delta_{i,\mu} - \delta_{i,\mu-1}$, $k'_i = l'_i$ for α_- and $k_i = l_i$, $k'_i = l'_i + \delta_{i,\mu} - \delta_{i,\mu-1}$ for α_+

2.3 THE BRAIDING MATRIX, ITS VERTEX AND IRF REPRESENTATION, AND THE YANG-BAXTER EQUATION

In general, the braiding matrix R is defined by[†]

$$V_{MD}^C(z)V_{NA}^D(w) = \sum_B R(C, M, N, A)_{DB} V_{NB}^C(w)V_{MA}^B(z) \quad (2.26)$$

C and A label the final and initial Hilbert spaces, D and B the intermediate ones on the right-hand and left-hand sides, respectively. M and N label the A_n representation with which the V 's are associated. The operator product on the right-hand side is well-defined for $|w| > |z|$. The left-hand side must be understood as the analytic continuation from the region $|w| < |z|$ along a path which we choose such that z circumvents w counterclockwise (see fig.2).

It is sometimes convenient to represent the R -matrix graphically as

$$R(C, M, N, A)_{DB} \equiv \begin{array}{ccc} & A & B \\ & \boxed{(M, N)} & \\ D & & C \end{array} \quad (2.27)$$

This notation makes it clear that one goes from $\mathcal{H}_A$ (with a_0 eigenvalue $\alpha_{aa'}$) to $\mathcal{H}_C$ (with a_0 eigenvalue $\alpha_{cc'}$) via $\mathcal{H}_B$ ($a_0 = \alpha_{bb'}$) on the r.h.s. or via $\mathcal{H}_D$ ($a_0 = \alpha_{dd'}$) on the l.h.s.. With the correspondence to the Boltzmann weights $W_{MN} \begin{pmatrix} A & B \\ D & C \end{pmatrix}$ of Jimbo et al²²⁻²⁶ in mind (see next section), we call (2.26,27) the IRF (interaction round a face) or SOS or spin representation of the R -matrix.

There exists also a dual representation of the R -matrix, called the vertex representation. Here the word "vertex" does not refer to the chiral vertices of the conformal theory, but to the geometric object which is dual to the plaquette (face) of (2.27), see fig. 3. Instead of labelling R by the Hilbert space indices A, B, C, D , in the dual vertex representation R is labelled by the *differences* of the a_0 eigenvalues between these Hilbert spaces, i.e. by the shifts of a_0 realized by the chiral vertex operators V_{NA}^D , etc. Call $\alpha = \alpha_{bb'} - \alpha_{aa'}$, $\beta = \alpha_{cc'} - \alpha_{bb'}$, $\gamma = \alpha_{cc'} - \alpha_{dd'}$, $\delta = \alpha_{dd'} - \alpha_{aa'}$ ($\alpha + \beta = \gamma + \delta$). Then, replacing A, B, C, D

[†] We use the same notation as FFK⁵⁾.

by $\alpha, \beta, \gamma, \delta$ and A we write^{*}

$$V_{M\gamma}(z)V_{N\delta}(w)\Big|_{\mathcal{H}_A} = \sum_{\alpha, \beta: \alpha+\beta=\gamma+\delta} R_{(MN)\gamma\delta}^{\beta\alpha}(A) V_{N\beta}(w) V_{M\alpha}(z)\Big|_{\mathcal{H}_A} \quad (2.28)$$

This is the vertex representation of the R -matrix (and of the chiral vertex operators). It can be denoted graphically as

$$R_{(MN)\gamma\delta}^{\beta\alpha} \equiv \begin{array}{c} A \\ \delta \quad \boxed{\begin{array}{c} \alpha \\ (MN) \\ \gamma \end{array}} \quad \beta \end{array} \equiv \begin{array}{c} (M, \alpha) \\ \text{---} \text{---} \text{---} \\ (N, \delta) \quad \text{---} \text{---} \text{---} \\ (M, \gamma) \end{array} \quad (N, \beta) \quad (2.29)$$

The last representation is familiar from the theory of exactly solvable models and describes the two-particle scattering.

The R -matrix obeys a cubic relation, the Yang-Baxter equation²¹⁾ (YBE) or Star-Triangle relation (STR) which simply translates the associativity of the operator product of three V 's: In the product $V_{KE}^D V_{MF}^E V_{NA}^F$ one can exchange the three V 's in different orders with the same result, see fig. 4. This is written as

$$\begin{aligned} & \sum_G R(G, K, N, A)_{FB} R(D, K, M, F)_{EG} R(D, M, N, B)_{GC} \\ &= \sum_G R(E, M, N, A)_{FG} R(C, K, M, A)_{GB} R(D, K, N, G)_{EC} \end{aligned} \quad (2.30)$$

which is nothing but the STR or YBE in the IRF representation. This is depicted in fig. 5a. Alternatively, we can write the YBE in the vertex representation:

$$\begin{aligned} & R_{(MN)\nu\mu}^{\sigma\kappa}(A) R_{(KN)\rho\sigma}^{\alpha\lambda}(A+\kappa) R_{(KM)\lambda\kappa}^{\beta\gamma}(A) \\ &= R_{(KM)\rho\nu}^{\sigma\kappa}(A+\mu) R_{(KN)\kappa\mu}^{\lambda\gamma}(A) R_{(MN)\sigma\lambda}^{\alpha\beta}(A+\gamma) \end{aligned} \quad (2.31)$$

This is depicted in fig. 5b. It is often abbreviated as $R_{(12)} R_{(13)} R_{(23)} = R_{(23)} R_{(13)} R_{(12)}$. We warn the reader that the arguments $A, A+\kappa$ etc. of the R -matrices are *not* the spectral parameter (cf. next section).

* $R(A)$ means $R(\alpha_{aa'})$.

2.4 THE BRAIDING MATRIX FOR THE FUNDAMENTAL CONFORMAL BLOCKS

In this subsection we will give the braiding matrix for the chiral screened vertex operators $\mathcal{V}_\mu^\pm$. The basic identity needed for determining the braiding matrices is (with the same convention for the analytic continuation as for (2.26), see fig. 2):

$$V_\alpha(z)V_\beta(w) = e^{i\pi\alpha\cdot\beta}V_\beta(w)V_\alpha(z) \quad (2.32)$$

for (unscreened) vertex operators (2.4).

First of all, the braiding of $\mathcal{V}_\mu^+$ with $\mathcal{V}_\nu^-$ (see fig. 6) is trivial: According to (2.20) all integration contours of the $V_{\alpha_+}^i$ can be interchanged with those of the $V_{\alpha_-}^j$. In the same way, one can pass a V_{α_+} (V_{α_-}) contour across $V_{\alpha_{(21\dots 1)(1\dots 1)}}(w)$ ($V_{\alpha_{(1\dots 1)(21\dots 1)}}(z)$) since

$$\oint_{|u-w|=\epsilon} V_{\alpha_{(21\dots 1)(1\dots 1)}}(w)V_{\alpha_+}^i(u)du = \oint_{|u-z|=\epsilon} V_{\alpha_-}^j(u)V_{\alpha_{(1\dots 1)(21\dots 1)}}(z)du = 0 \quad (2.33)$$

Hence the braiding of $\mathcal{V}_\mu^+$ with $\mathcal{V}_\nu^-$ is only determined by the braiding of $V_{\alpha_{(1\dots 1)(21\dots 1)}}(z)$ with $V_{\alpha_{(21\dots 1)(1\dots 1)}}(w)$ which gives a simple phase factor $e^{i\pi\lambda_1^2\alpha_-\alpha_+}$:

$$\begin{aligned} & R(L - \alpha_+h_\mu - \alpha_-h_\nu, (0, \lambda_1), (\lambda_1, 0), L)_{L-\alpha_-h_\nu, L-\alpha_+h_\mu} \\ &= R(L - \alpha_+h_\mu - \alpha_-h_\nu, (\lambda_1, 0), (0, \lambda_1), L)_{L-\alpha_+h_\nu, L-\alpha_-h_\mu} = e^{-i\pi\lambda_1^2} \end{aligned} \quad (2.34)$$

where (for A_n)

$$\lambda_1^2 = \frac{n}{n+1} \quad (2.35)$$

Next we give the braiding matrix of $\mathcal{V}_\mu^-$ ($\mathcal{V}_\mu^+$) with $\mathcal{V}_\nu^-$ ($\mathcal{V}_\nu^+$). In this case, the value of l' (l) remains fixed and we will not write it explicitly. The R -matrix will then be denoted simply by $r(c, m, n, a)_{db}$ (resp. $r'(c', m', n', a')_{d'b'}$).^{*} In general we write

$$V_{(m, (1\dots 1))(da')}^{(ca')}(z)V_{(n, (1\dots 1))(aa')}(w) = \sum_b r(c, m, n, a)_{db} V_{(n, (1\dots 1))(ba')}^{(ca')}(w)V_{(m, (1\dots 1))(aa')}(z) \quad (2.36)$$

or more explicitly (using the vertex representation of the r -matrix)

$$\mathcal{V}_\mu^-(z)\mathcal{V}_\nu^-(z) \Big|_{\mathcal{H}_{(aa')}} = \sum_{\kappa, \rho: h_\kappa + h_\rho = h_\mu + h_\nu} r_{\mu\nu}^{\kappa\rho}(a) \mathcal{V}_\kappa^-(w)\mathcal{V}_\rho^-(z) \Big|_{\mathcal{H}_{(aa')}} \quad (2.37a)$$

^{*} A priori, r could depend also on a' . However, it will turn out that this is not the case.

$$r_{\mu\nu}^{\kappa\rho}(a) \equiv r_{(\lambda_1, \lambda_1)\mu\nu}^{\kappa\rho}(a) = r(a - \alpha_- h_\mu - \alpha_- h_\nu, \lambda_1, \lambda_1, a)_{a-\alpha_- h_\nu, a-\alpha_- h_\rho}$$

$$= \begin{array}{c} a \quad \rho \\ \nu \quad \square \quad \kappa \\ \mu \end{array} \quad (2.37b)$$

We will concentrate on the $\mathcal{V}^-$, the equations for the $\mathcal{V}^+$ being completely analogous. The braiding matrix of $\mathcal{V}_1^-(z)$ with $\mathcal{V}_1^-(w)$ is trivial since there are no screening charges and one has (see (2.32)):

$$r_{11}^{11}(l) \equiv r(l - 2\alpha_- h_1, \lambda_1, \lambda_1, l)_{l-\alpha_- h_1, l-\alpha_- h_1} = e^{i\pi\lambda_1^2\alpha_-^2} = x^{\frac{1}{2}\lambda_1^2} \quad (2.38)$$

where, following ref⁵⁾, we introduced

$$x = e^{2i\pi\alpha_-^2}, \quad x' = e^{2i\pi\alpha_+^2} \quad (2.39)$$

The braiding of $\mathcal{V}_1^-$ with $\mathcal{V}_2^-$ is more complicated, since the latter now involves one screening operator (see fig. 7). The computation of this braiding matrix involves the decomposition of the integration contour in fig. 7 into two pieces, one from w to z and one from z to w as indicated in fig. 7. For each of the two pieces, we move the reference points for the analytic continuations to a given standard position, which results in multiplying by certain phase factors, see e.g. fig. 8. Doing the same manipulations for $\mathcal{V}_1^-(z)\mathcal{V}_2^-(w)$, $\mathcal{V}_2^-(z)\mathcal{V}_1^-(w)$, $\mathcal{V}_1^-(w)\mathcal{V}_2^-(z)$ and $\mathcal{V}_2^-(w)\mathcal{V}_1^-(z)$ allows us to express all four products in terms of two integrals multiplied by certain phase factors. Solving for $\mathcal{V}_1^-(z)\mathcal{V}_2^-(w)$, $\mathcal{V}_2^-(z)\mathcal{V}_1^-(w)$, in terms of $\mathcal{V}_1^-(w)\mathcal{V}_2^-(z)$, $\mathcal{V}_2^-(w)\mathcal{V}_1^-(z)$ gives the braiding matrix. The details of this calculation are a step-by-step repetition of those for the non-extended theories⁵⁾, only the phase factors are modified. We refer to the paper by FFK⁵⁾ and only give the result:

$$\begin{aligned} r_{12}^{12}(l) &\equiv r(l - \alpha_- h_1 - \alpha_- h_2, \lambda_1, \lambda_1, l)_{l-\alpha_- h_2, l-\alpha_- h_2} = x^{-\frac{1}{2}(1-\lambda_1^2) + \frac{1}{2}l_1} \frac{[1]}{[l_1]} \\ r_{21}^{21}(l) &\equiv r(l - \alpha_- h_1 - \alpha_- h_2, \lambda_1, \lambda_1, l)_{l-\alpha_- h_1, l-\alpha_- h_1} = -x^{-\frac{1}{2}(1-\lambda_1^2) - \frac{1}{2}l_1} \frac{[1]}{[l_1]} \\ r_{12}^{21}(l) &\equiv r(l - \alpha_- h_1 - \alpha_- h_2, \lambda_1, \lambda_1, l)_{l-\alpha_- h_2, l-\alpha_- h_1} = x^{-\frac{1}{2}(1-\lambda_1^2)} \frac{[l_1 - 1]}{[l_1]} \\ r_{21}^{12}(l) &\equiv r(l - \alpha_- h_1 - \alpha_- h_2, \lambda_1, \lambda_1, l)_{l-\alpha_- h_1, l-\alpha_- h_2} = x^{-\frac{1}{2}(1-\lambda_1^2)} \frac{[l_1 + 1]}{[l_1]} \end{aligned} \quad (2.40)$$

where we introduced the following notation, borrowed from quantum groups:

$$[a] = \frac{x^{\frac{1}{2}a} - x^{-\frac{1}{2}a}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \quad (2.41)$$

The corresponding braiding matrices r' for the $\mathcal{V}_{l,2}^+$ are obtained by replacing $x \rightarrow x'$ and $l \rightarrow l'$.

The same braiding matrices were obtained in our study of the Toda field theories³⁴⁾, although in a different notation (see the appendix). Also the $\mathcal{V}_\mu$ were "normalized" differently. In general, a change of "normalization"

$$V_{ML}^K(z) \longrightarrow f_{ML}^K V_{ML}^K(z) \quad (2.42)$$

with some c -number function f_{ML}^K modifies the braiding matrix (2.26) according to

$$R(C, M, N, A)_{DB} \longrightarrow \frac{f_{MD}^C f_{NA}^D}{f_{NB}^C f_{MA}^B} R(C, M, N, A)_{DB} \quad (2.43)$$

Of course, the new R 's still satisfy the YBE (2.30). In ref.³⁴⁾ the braiding matrices were obtained using the fact that the four-point functions $\langle 0 | V_{\alpha_{kk'}} \mathcal{V}_1^\pm(z) \mathcal{V}_2^\pm(w) V_{\alpha_{ll'}} | 0 \rangle$, etc. are hypergeometric functions.

Moreover, it was shown in ref.³⁴⁾ that there exists a permutation symmetry among the $\mathcal{V}_\mu^-$. Indeed, there exists a canonical Fock space transformation $\mathcal{P}_{\mu\nu}$ mapping all oscillators a_n^i to a different set of oscillators $(a^{(\mu\nu)})_n^i$ such that

1. the new set still satisfies $[(a^{(\mu\nu)})_n^i, (a^{(\mu\nu)})_m^j] = \delta^{ij} n \delta_{n,-m}$ and
2. in terms of the $(a^{(\mu\nu)})_m^j$ the $\mathcal{V}_\mu^-$ is written as an *unscreened* vertex operator and $\hat{\mathcal{V}}_\nu^-$ contains only one screening operator.

From a Lie algebra representation theoretic point of view, $\mathcal{P}_{\mu\nu}$ permutes the weights h_κ of the vector representation of A_n such that $h_1 \rightarrow h_\mu, h_2 \rightarrow h_\nu$. This allowed us to deduce³⁴⁾ all $r_{\mu\nu}^{\kappa\rho}$ from $r_{11}^{11}, r_{12}^{12}, r_{21}^{21}, r_{12}^{21}, r_{21}^{12}$ (see the appendix). In particular, $r_{\mu\mu}^{\mu\mu} = r_{11}^{11}$ for all μ , which is readily verified for $\mu = 2$ by a direct computation in the same way as

for eqs (2.40). With the present notation we have:

$$\begin{aligned}
r_{\mu\mu}^{\mu\mu}(l) &\equiv r(l - 2\alpha_- h_\mu, \lambda_1, \lambda_1, l)_{l-\alpha_- h_\mu, l-\alpha_- h_\mu} = x^{\frac{1}{2}\lambda_1^2} \\
r_{\mu\nu}^{\mu\nu}(l) &\equiv r(l - \alpha_- h_\mu - \alpha_- h_\nu, \lambda_1, \lambda_1, l)_{l-\alpha_- h_\nu, l-\alpha_- h_\nu} = -x^{-\frac{1}{2}(1-\lambda_1^2) - \frac{1}{2}l_{\mu\nu}} \frac{[1]}{[l_{\mu\nu}]} \quad (\mu \neq \nu) \\
r_{\mu\nu}^{\nu\mu}(l) &\equiv r(l - \alpha_- h_\mu - \alpha_- h_\nu, \lambda_1, \lambda_1, l)_{l-\alpha_- h_\nu, l-\alpha_- h_\mu} = x^{-\frac{1}{2}(1-\lambda_1^2)} \frac{[l_{\mu\nu} + 1]}{[l_{\mu\nu}]} \quad (\mu \neq \nu)
\end{aligned} \tag{2.44}$$

where

$$l_{\mu\nu} = \sum_{i=\nu}^{\mu-1} l_i \quad \text{if } \mu > \nu, \quad l_{\mu\nu} = -l_{\nu\mu} \quad \text{if } \nu > \mu \tag{2.45}$$

It is obvious that all equations (2.37-45) are equally valid for the $\mathcal{V}^+$, provided one replaces $x \rightarrow x'$, $l_i \rightarrow l'_i$.

Note that the matrix $r_{\mu\nu}^{\rho\sigma}$ decomposes into 1×1 blocks $r_{\mu\mu}^{\mu\mu}$ and 2×2 blocks $r_{(\mu\nu)}$:

$$r_{(\mu\nu)} = \begin{pmatrix} r_{\mu\nu}^{\mu\nu} & r_{\mu\nu}^{\nu\mu} \\ r_{\nu\mu}^{\mu\nu} & r_{\nu\mu}^{\nu\mu} \end{pmatrix} \tag{2.46}$$

The determinant of the latter matrix equals $\det r_{(\mu\nu)} = -x^{\lambda_1^2-1}$. It follows that the inverse of $r_{(\mu\nu)}$ is obtained by simply replacing x by x^{-1} . The same is obviously true for $r_{\mu\mu}^{\mu\mu}$, and hence also for the complete matrix $r_{\mu\nu}^{\rho\sigma}$:

$$r(x^{-1}) = r(x)^{-1} \tag{2.47}$$

It also follows that the determinant of the complete matrix $r_{\mu\nu}^{\rho\sigma}$ is ± 1 :

$$\det r(x) = (-1)^{\frac{1}{2}n(n+1)} \tag{2.48}$$

Finally, let us remark that in defining the braiding matrix in (2.26,28) one could have chosen the opposite sense for the analytic continuation (see fig. 2). Then all the phase factors would have been reversed, i.e. x (x') replaced by x^{-1} (x'^{-1}) in eqs (2.40,44). According to the preceding remark, this amounts to replacing $r(x)$ by its inverse. This is nothing but a consistency check, since braiding two operators in one sense and then in the opposite one should give back the initial configuration (see fig. 9).

3. Fusion of the conformal blocks and of the braiding matrices

3.1 FUSION OF THE CHIRAL SCREENED VERTEX OPERATORS

We will define the fusion of the chiral (screened) vertex operators $\mathcal{V}_\mu$ associated with the vector representation of A_n (highest weight: λ_1). This will yield chiral screened vertex operators $\mathcal{V}_{\mu_1\mu_2\ldots\mu_N}$ associated with the rank N symmetric tensor representation of A_n (highest weight: $N\lambda_1$). For the A_1 case, this has been done in ref.⁵⁾.

First consider the fusion of an operator $\hat{\mathcal{V}}_\mu^\pm(\zeta)$ as given in (2.21) (without the phase factor (2.25)) with a general screened vertex operator (2.18) $V_{mm'}^{i_1\ldots i_r j_1\ldots j_r}(z)$. The short distance singularity of the OPE is $(\zeta - z)^{-\alpha_\pm \lambda_1 \cdot \alpha_{mm'}}$ and one might be tempted to define the operator resulting from the fusion as

$$\lim_{\zeta \rightarrow z} (\zeta - z)^{\alpha_\pm \lambda_1 \cdot \alpha_{mm'}} \hat{\mathcal{V}}_\mu^\pm(\zeta) V_{mm'}^{i_1\ldots i_r j_1\ldots j_r}(z) \quad (3.1)$$

However, this only works for $\mu = 1$, when $\hat{\mathcal{V}}_\mu$ is an unscreened vertex operator. For $\mu \neq 1$, we have to change the reference point for the analytic continuations from

$$v_{r'} < \ldots v_1 < u_r < \ldots u_1 < z < w_{\mu-1} < \ldots w_1 < \zeta \quad (3.2a)$$

(the w 's denote the integration variables of $\hat{\mathcal{V}}_\mu^\pm$) to

$$v_{r'} < \ldots v_1 < u_r < \ldots u_1 < w_{\mu-1} < \ldots w_1 < z < \zeta \quad (3.2b)$$

which results in a phase factor

$$\exp\left(i\pi\alpha_\pm \sum_{i=1}^{\mu-1} e_i \cdot \alpha_{mm'}\right) = \exp\left(-i\pi \sum_{i=1}^{\mu-1} ((m_i - 1)\alpha_\pm \alpha_- + (m'_i - 1)\alpha_\pm \alpha_+)\right) \quad (3.3)$$

One obtains (fixing $\hat{\mathcal{V}}^-$ to simplify the notation):

$$\begin{aligned} V_{m+(10\ldots 0),m'}^{12\ldots\mu-1 i_1\ldots i_r j_1\ldots j_r}(z) &= e^{+i\pi \sum_{i=1}^{\mu-1} (m_i - 1)\alpha_-^2} (-)^{\sum_{i=1}^{\mu-1} (m'_i - 1)} \\ &\times \lim_{\zeta \rightarrow z} (\zeta - z)^{\alpha_- \lambda_1 \cdot \alpha_{mm'}} \hat{\mathcal{V}}_\mu^-(\zeta) V_{mm'}^{i_1\ldots i_r j_1\ldots j_r}(z) \end{aligned} \quad (3.4)$$

If we rewrite this equation in terms of the V_{ML}^K and $\mathcal{V}_\mu^-$, involving the phase factors (2.25), we find that *all* phase factors precisely cancel, since the phases θ were chosen in exactly

such a way as to cancel the factor (3.3) , see eq. (2.24) :

$$V_{(m+(10\dots0),m')L}^{K-\alpha-h_\mu}(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{\alpha-\lambda_1 \cdot \alpha_{mm'}} \mathcal{V}_\mu^-(\zeta) V_{(mm')L}^K(z) \quad (3.5)$$

and similar for $\mathcal{V}_\mu^+$. In fact this argument and the choice (2.25) of the phase factors θ are general enough to show that the fusion of two arbitrary screened vertex operators

$$V_{(n+m,n'+m')A}^C(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{-\alpha_{nn'} \cdot \alpha_{mm'}} V_{(nn')B}^C(\zeta) V_{(mm')A}^B(z) \quad (3.6)$$

does not contain any left-over phase factor.*

The simplest case is the fusion of $\mathcal{V}_\mu^-$ with $\mathcal{V}_\nu^-$:

$$\mathcal{V}_{\mu\nu}^-(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{-\lambda_1^2 \alpha_-^2} \mathcal{V}_\mu^-(\zeta) \mathcal{V}_\nu^-(z) \quad (3.7)$$

We now proceed to show that

$$\mathcal{V}_{\mu\nu}^- = \mathcal{V}_{\nu\mu}^- \quad (3.8)$$

Indeed, one has for $\mu \neq \nu$:

$$\begin{aligned} \mathcal{V}_{\mu\nu}^-(z) &= \lim_{\zeta \rightarrow z} (z - \zeta)^{-\lambda_1^2 \alpha_-^2} e^{-i\pi \alpha_-^2 \lambda_1^2} (r_{\mu\nu}^{\mu\nu} \mathcal{V}_\mu^-(z) \mathcal{V}_\nu^-(\zeta) + r_{\mu\nu}^{\nu\mu} \mathcal{V}_\nu^-(z) \mathcal{V}_\mu^-(\zeta)) \\ &= e^{-i\pi \alpha_-^2 \lambda_1^2} (r_{\mu\nu}^{\mu\nu} \mathcal{V}_{\mu\nu}^-(z) + r_{\mu\nu}^{\nu\mu} \mathcal{V}_{\nu\mu}^-(z)) \end{aligned} \quad (3.9)$$

It is easy to verify that the r -matrices (2.44) satisfy

$$r_{\mu\nu}^{\mu\nu} + r_{\mu\nu}^{\nu\mu} = x^{\frac{1}{2} \lambda_1^2} = e^{i\pi \alpha_-^2 \lambda_1^2} \quad (3.10)$$

which proves (3.8).

* In the general case (3.6) however one has to make some convention for the ordering of the screening operators, as was done for the $\mathcal{V}_\mu$ and as results for the $\mathcal{V}_{\mu_1 \dots \mu_N}$ in the following.

It is now straightforward to give the general fusion procedure. We define

$$\mathcal{V}_{\mu\mu_1\ldots\mu_N}^-(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{-N\lambda_1^2\alpha_-^2} \mathcal{V}_{\mu}^-(\zeta) \mathcal{V}_{\mu_1\ldots\mu_N}^-(z) \quad (3.11)$$

We will show that this is completely symmetric in all indices:

$$\mathcal{V}_{\mu_1\ldots\mu_N}^- = \mathcal{V}_{(\mu_1\ldots\mu_N)}^- \quad (3.12)$$

For $N = 2$, this was already shown above. For arbitrary N we have

$$\begin{aligned} \mathcal{V}_{\mu\nu\mu_1\ldots\mu_{N-1}}^-(z) &= \lim_{\zeta \rightarrow z} \lim_{\xi \rightarrow z} (\zeta - z)^{-(N-1)\lambda_1^2\alpha_-^2} (\zeta - \xi)^{-\lambda_1^2\alpha_-^2} (\xi - z)^{-(N-1)\lambda_1^2\alpha_-^2} \\ &\quad \times \mathcal{V}_{\mu}^-(\zeta) \mathcal{V}_{\nu}^-(\xi) \mathcal{V}_{\mu_1\ldots\mu_{N-1}}^-(z) \end{aligned} \quad (3.13)$$

Using the braiding matrices (2.44) to exchange $\mathcal{V}(\zeta)$ and $\mathcal{V}(\xi)$, and the relation (3.10), we obtain as before

$$\lim_{\zeta \rightarrow z} (\zeta - z)^{-N\lambda_1^2\alpha_-^2} \mathcal{V}_{\mu}^-(\zeta) \mathcal{V}_{\nu\mu_1\ldots\mu_{N-1}}^-(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{-N\lambda_1^2\alpha_-^2} \mathcal{V}_{\nu}^-(\zeta) \mathcal{V}_{\mu\mu_1\ldots\mu_{N-1}}^-(z) \quad (3.14)$$

Assuming that (3.12) holds for N , this shows that (3.12) also holds for $N + 1$; q.e.d.. The symmetry (3.12) guarantees that it does not matter whether one first fuses $\mathcal{V}_{\mu}$ with $\mathcal{V}_{\nu}$ and then the resulting operator with $\mathcal{V}_{\rho}$ or whether one does it in a different order. The result only depends on *which* operators have been fused.

Thus we have obtained chiral screened vertex operators associated with symmetric tensor representations of A_n of arbitrary rank N by fusion of the fundamental ($N = 1$) operators $\mathcal{V}_{\mu}$.

The preceding discussion for the $\mathcal{V}^-$ is equally valid for the fusion of operators $\mathcal{V}^+$. If we fuse a $\mathcal{V}_{\mu}^-$ with a $\mathcal{V}_{\nu}^+$, again due to (2.24) all phase factors cancel and one has

$$\mathcal{V}_{\mu-\nu+}(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{\lambda_1^2} \mathcal{V}_{\mu}^-(\zeta) \mathcal{V}_{\nu}^+(z) \quad (3.15)$$

Using the R -matrix (2.34) a simple calculation analogous to (3.9) shows that

$$\mathcal{V}_{\mu-\nu+} = \mathcal{V}_{\nu+\mu-} \quad (3.16)$$

Hence a general operator obtained by fusion in an arbitrary order of N operators $\mathcal{V}^-$ and

N' operators $\mathcal{V}^+$ is

$$\mathcal{V}_{\mu_1^- \dots \mu_N^- \nu_1^+ \dots \nu_{N'}^+}(z) = \lim_{\zeta \rightarrow z} (\zeta - z)^{NN' \lambda_1^2} \mathcal{V}_{\mu_1^- \dots \mu_N^-}(\zeta) \mathcal{V}_{\nu_1^+ \dots \nu_{N'}^+}(z) \quad (3.17)$$

and is associated with a pair of rank N and rank N' symmetric tensor representations of A_n . Recall from section 2.2 that this operator is also denoted

$$V_{(N\lambda_1, N'\lambda_1)L}^{L-\alpha-\lambda-\alpha+\lambda'} = \mathcal{V}_{\mu_1^- \dots \mu_N^- \nu_1^+ \dots \nu_{N'}^+} \Big|_{\mathcal{H}_L}$$

where $\lambda = \sum_{i=1}^N h_{\mu_i}$ and $\lambda' = \sum_{i=1}^{N'} h_{\nu_i}$ are weights of the rank N and rank N' symmetric tensor representations.

Finally, the fusion of any two operators of the type (3.17) is very simple:

$$\begin{aligned} & \mathcal{V}_{\mu_1^- \dots \mu_N^- \rho_1^- \dots \rho_M^- \nu_1^+ \dots \nu_{N'}^+ \sigma_1^+ \dots \sigma_{M'}^+}(z) \\ &= \lim_{\zeta \rightarrow z} (\zeta - z)^{(NM' + N'M - NM\alpha_-^2 - N'M'\alpha_+^2)\lambda_1^2} \mathcal{V}_{\mu_1^- \dots \mu_N^- \nu_1^+ \dots \nu_{N'}^+}(\zeta) \mathcal{V}_{\rho_1^- \dots \rho_M^- \sigma_1^+ \dots \sigma_{M'}^+}(z) \end{aligned} \quad (3.18)$$

3.2 FUSION OF THE BRAIDING MATRICES

The fusion procedure of the previous subsection for the chiral vertex operators implies a fusion procedure for the corresponding R -matrices.

Consider first the braiding of $\mathcal{V}_\mu^-(z)$ with $\mathcal{V}_{\mu_1 \dots \mu_M}^-(w)$. Writing the product of the two operators as $(\lambda = \sum_{i=1}^M h_{\mu_i})$

$$\begin{aligned} & V_{(\lambda_1, 0)L-\alpha-\lambda}^{L-\alpha-\lambda-\alpha-h_\mu}(z) V_{(M\lambda_1, 0)L}^{L-\alpha-\lambda}(w) \\ &= \lim_{\zeta \rightarrow w} (\zeta - w)^{-(M-1)\lambda_1^2 \alpha_-^2} V_{(\lambda_1, 0)L-\alpha-\lambda}^{L-\alpha-\lambda-\alpha-h_\mu}(z) V_{(\lambda_1, 0)L-\alpha-\lambda+\alpha-h_{\mu_i}}^{L-\alpha-\lambda}(\zeta) V_{((M-1)\lambda_1, 0)L}^{L-\alpha-\lambda+\alpha-h_{\mu_i}}(w) \end{aligned} \quad (3.19)$$

and braiding $V(z)$ to the right we get the following fusion rule for the r -matrices

$$r(c, \lambda_1, M\lambda_1, a)_{db} = \sum_{\tilde{b}} r(c, \lambda_1, \lambda_1, \tilde{a})_{d\tilde{b}} r(\tilde{b}, \lambda_1, (M-1)\lambda_1, a)_{\tilde{a}b} \quad (3.20)$$

In the case (3.19) one has e.g. $a = l$, $b = l - \alpha_- h_\rho$, $c = l - \alpha_-(h_\mu + \lambda)$, $d = l - \alpha_- \lambda$, $\tilde{b} = l - \alpha_-(h_\mu + \lambda - h_{\mu_i})$, $\tilde{a} = l - \alpha_-(\lambda - h_{\mu_i})$. Instead of choosing $\tilde{a} = l - \alpha_-(\lambda - h_{\mu_i})$ one could have chosen $\tilde{a} = l - \alpha_-(\lambda - h_{\mu_j})$ with the same result: any $\tilde{a}$ compatible with the fusion (3.11) can be chosen. $\tilde{b}$ can take at most two values (cf. (2.44)). An analogous relation holds for the r^l -matrices.

It is useful to rewrite the recursion relation (3.20) using the diagrammatic notation (2.27):

$$\begin{array}{c} a \\ \square \\ d \end{array} \begin{array}{c} b \\ (1, M) \\ c \end{array} = \sum_{\tilde{b}} \begin{array}{c} a \\ \square \\ \tilde{a} \end{array} \begin{array}{c} b \\ (1, M-1) \\ \tilde{b} \end{array} \times \begin{array}{c} \tilde{a} \\ \square \\ d \end{array} \begin{array}{c} \tilde{b} \\ (1, 1) \\ c \end{array} \quad (3.21)$$

In the same way one finds

$$\begin{array}{c} a \\ \square \\ d \end{array} \begin{array}{c} b \\ (N, 1) \\ c \end{array} = \sum_{\tilde{a}} \begin{array}{c} a \\ \square \\ d \end{array} \begin{array}{c} \tilde{a} \\ (N-1, 1) \\ \tilde{c} \end{array} \times \begin{array}{c} \tilde{a} \\ \square \\ \tilde{c} \end{array} \begin{array}{c} b \\ (1, 1) \\ c \end{array} \quad (3.22)$$

One sees that $r(c, \lambda_1, M\lambda_1, a)_{db}$ is obtained by multiplying M elementary braiding matrices $r(\cdot, \lambda_1, \lambda_1, \cdot)$.. "vertically", whereas $r(c, N\lambda_1, \lambda_1, a)_{db}$ is obtained by multiplying N elementary $r(\cdot, \lambda_1, \lambda_1, \cdot)$.. "horizontally".

Next, one obtains the general recursion formula for $r(c, N\lambda_1, M\lambda_1, a)_{db}$ using the fusion relations as before:

$$r(c, (N+1)\lambda_1, M\lambda_1, a)_{db} = \sum_{\tilde{a}} r(c, \lambda_1, M\lambda_1, \tilde{a})_{\tilde{c}b} r(\tilde{c}, N\lambda_1, M\lambda_1, a)_{d\tilde{a}} \quad (3.23a)$$

$$r(c, N\lambda_1, (M+1)\lambda_1, a)_{db} = \sum_{\tilde{b}} r(c, N\lambda_1, \lambda_1, \tilde{a})_{d\tilde{b}} r(\tilde{b}, N\lambda_1, M\lambda_1, a)_{\tilde{a}b} \quad (3.23b)$$

The diagrammatic representation of these equations is analogous to (3.21,22) (which are just the special cases $M=1$ and $N=1$). Again, the same relations are true for the r' .

Finally we must consider the braiding of general operators $\mathcal{V}_{\mu_1^- \dots \mu_N^- \nu_1^+ \dots \nu_{N'}^+}$ (cf. (3.17)). The same argument as the one that led to eq. (2.34) shows that the braiding of $\mathcal{V}_{\mu_1^- \dots \mu_N^-}$ with $\mathcal{V}_{\rho_1^+ \dots \rho_M^+}$ simply gives a phase factor:

$$R(C, (N\lambda_1, 0), (0, M\lambda_1), A)_{DB} = R(C, (0, M\lambda_1), (N\lambda_1, 0), A)_{DB} = e^{-i\pi N M \lambda_1^2} \quad (3.24)$$

Then using (3.17) we have

$$\begin{aligned} & R(C, (N\lambda_1, N'\lambda_1), (M\lambda_1, M'\lambda_1), A)_{DB} \\ &= e^{-i\pi(NM'+N'M)\lambda_1^2} r(c, N\lambda_1, M\lambda_1, a)_{db} r'(c', N'\lambda_1, M'\lambda_1, a')_{d'b'} \end{aligned} \quad (3.25)$$

Equations (3.20-25) together with (2.44) give the braiding matrices for any two chiral screened vertex operators (conformal blocks), each of them being labelled by a pair of

symmetric tensor representations (rank N and N') of A_n . The complete R -matrix (3.25) factorizes - up to a trivial phase factor independent of A, B, C, D - into the product of two r (r')-matrices. In the next subsection we will show that these r (r')-matrices coincide with an appropriate limit of the Boltzmann weights W_{NM} for the A_n face models of Jimbo et al.²⁶⁾.

3.3 RELATION WITH THE BOLTZMANN WEIGHTS OF THE A_n FACE MODELS

We will establish this relation in two steps: first we relate the fundamental braiding matrices $r_{\mu\nu}^{\rho\sigma}$ (eq.(2.44)) to the Boltzmann weights of the A_n face models associated with the vector representation, as constructed by Jimbo et al. in ref.²⁵⁾ and ref.²⁷⁾. It is the formulation given in the latter article which is closest to the one of the present paper. Then we will point out that the fusion procedure of ref.²⁶⁾ for the Boltzmann weights is the same as the one for the braiding matrices r (r') given in the preceding subsection.

Jimbo et al. consider two-dimensional solvable lattice models with fluctuation variables a, b, c, d interacting by four around a plaquette (face). The corresponding Boltzmann weight is denoted by

$$W_{11} \begin{pmatrix} a & b \\ d & c \end{pmatrix} \equiv \begin{array}{c} a \quad b \\ \square \\ d \quad c \end{array} \quad (3.26)$$

The fluctuation variables a, b, c, d are taken to be dominant integral weights of $A_n^{(1)}$ of a fixed level²⁷⁾. They are constrained by

$$b - a, c - b, d - a, c - d \in \{h_1, \dots, h_{n+1}\} \quad (3.27)$$

i.e. the difference of adjacent site variables has to be a weight of the vector representation of A_n . Thus, if $b - a = h_\rho$, $c - b = h_\kappa$, $d - a = h_\nu$, $c - d = h_\mu$ ($h_\mu + h_\nu = h_\rho + h_\kappa$), the Boltzmann weights can equally well be written (cf. subsections 2.3 and 2.4)

$$W_{11} \begin{pmatrix} a & a + h_\rho \\ a + h_\nu & a + h_\mu + h_\nu \end{pmatrix} = \begin{array}{c} a \quad \rho \\ \nu \square \kappa \\ \mu \end{array} \quad (3.28)$$

These Boltzmann weights of Jimbo et al.²⁷⁾ are given in terms of elliptic theta

functions, depending on a spectral parameter u :^{*}

$$[u]_J \equiv \theta_1 \left(\frac{\pi u}{\mathcal{L}}, p \right) = 2p^{\frac{1}{2}} \sin \frac{\pi u}{\mathcal{L}} \prod_{k=1}^{\infty} \left(1 - 2p^k \cos \frac{2\pi u}{\mathcal{L}} + p^{2k} \right) (1 - p^k) \quad (3.29)$$

where $\mathcal{L}$ is some arbitrary fixed complex parameter. Jimbo et al. introduce

$$a_{\mu\nu} = (\bar{a} + \rho) \cdot (h_{\mu} - h_{\nu}) \quad (3.30)$$

where $\bar{a}$ denotes the "classical" part of a . In fact, due to the constraint (3.27) all local states a, b, c, d are confined to the set $a^0 + \{\sum_{\mu} m_{\mu} h_{\mu}, m_{\mu} \in \mathbb{Z}\}$ with some fixed a^0 . Choosing $a^0 = 0$, $a = \sum_{\mu=1}^{n+1} m_{\mu} h_{\mu} = \sum_{i=1}^n (m_i - m_{i+1}) \lambda_i$ and identifying $m_i - m_{i+1} \equiv l_i - 1$ one has $a + \rho = \sum_{i=1}^n l_i \lambda_i$. Hence

$$a_{\mu\nu} = -l_{\mu\nu} \quad (3.31)$$

where $l_{\mu\nu}$ was introduced in eq. (2.45). The Boltzmann weights are then²⁷⁾

$$\begin{aligned} \begin{array}{c} \mu \\ \square \\ \mu \end{array} &= \frac{[1+u]_J}{[1]_J} \\ \begin{array}{c} \nu \\ \square \\ \mu \end{array} &= \frac{[l_{\mu\nu} - u]_J}{[l_{\mu\nu}]_J} \quad (\mu \neq \nu) \\ \begin{array}{c} \mu \\ \square \\ \nu \end{array} &= \frac{[u]_J [l_{\mu\nu} + 1]_J}{[1]_J [l_{\mu\nu}]_J} \quad (\mu \neq \nu) \end{aligned} \quad (3.32)$$

To relate these Boltzmann weights to the braiding matrices r , one has first to take the limit[†] $p \rightarrow 0$:

$$\lim_{p \rightarrow 0} \frac{[u]_J}{[v]_J} = \frac{[u]}{[v]} \quad (3.33)$$

where $[u]$ is defined in (2.41), provided we identify

$$x = e^{\frac{2i\pi}{\mathcal{L}}} \quad (3.34)$$

Then one has to take the spectral parameter u in (3.32) to $-i\infty$ ($+i\infty$) if $\text{Re} \frac{1}{\mathcal{L}} > 0$ (< 0).

* The subscript J of the bracket serves to distinguish the bracket used by Jimbo et al. from the one defined above in eq.(2.41).

† For the A_1 case, this was discussed in ref.⁵⁾.

One obtains

$$x^{-\frac{1}{2}(1-\lambda_1^2)}(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \lim_{u \rightarrow -i\infty} \lim_{p \rightarrow 0} x^{-\frac{1}{2}u} \left(\begin{array}{c} \mu \\ \mu \square \mu \\ \mu \\ \nu \square \mu \\ \mu \\ \nu \square \nu \\ \mu \end{array} \right)_{\text{Jimbo et al}} = \begin{pmatrix} r_{\mu\mu}^{\mu\mu}(l) \\ r_{\mu\nu}^{\mu\nu}(l) \\ r_{\mu\nu}^{\nu\mu}(l) \end{pmatrix} \quad (3.35)$$

Using the IRF representation of the r -matrix, this reads

$$x^{-\frac{1}{2}(1-\lambda_1^2)}(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \lim_{u \rightarrow -i\infty} \lim_{p \rightarrow 0} x^{-\frac{1}{2}u} W_{11} \left(\begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) = r(c, \lambda_1, \lambda_1, a)_{db} \quad (3.36)$$

Hence the elementary braiding matrices r (r') coincide with the Boltzmann weights W_{11} in an appropriate limit and up to an irrelevant global factor.

Comparing the fusion relations (3.21-23) for the braiding matrices r with the way Jimbo et al.²⁶⁾ define the Boltzmann weights W_{NM} for the A_n face models in terms of the fundamental W_{11} , one finds a one-to-one correspondence between the formulac.* We conclude

$$r(c, N\lambda_1, M\lambda_1, a)_{db} = x^{-\frac{1}{2}NM(1-\lambda_1^2)}(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^{NM} \times \lim_{u \rightarrow -i\infty} \lim_{p \rightarrow 0} x^{-\frac{1}{2}NMu} W_{NM} \left(\begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) \quad (3.37)$$

* Our eqs (3.21,22) correspond exactly to their formula in lemma 2.1 and 2.2 defining $h_N = W_{N1}$, $v_M = W_{1M}$ and our eqs (3.23a,b) to their eqs (2.6,7) for W_{NM} .

4. Conclusion and outlook

We have given a precise definition of chiral screened vertex operators in extended conformal field theories (W-algebras) associated with A_n . A certain subset of them, corresponding to the vector representation, plays the role of fundamental operators. We have shown how to obtain chiral screened vertex operators, corresponding to rank N symmetric tensor representations of A_n , by fusion of the fundamental ones. The braiding matrices describing the exchange of "rank N " with "rank M " operators have been obtained by a corresponding fusion procedure out of the elementary braiding matrix of ref.³⁴⁾. The fused braiding matrices have been shown to factorize - up to trivial factors - into a pair of Boltzmann weights W_{NM} - in a certain limit of the spectral parameter - of the A_n face models of Jimbo et al.²⁶⁾

It remains to generalize these results to arbitrary A_n representations, including anti-symmetric tensors. This will be done elsewhere.

Jimbo et al. also constructed Boltzmann weights W_{11} corresponding to the vector representations of B_n , C_n and D_n .²⁷⁾ It is still an open problem to compute the braiding matrices of the associated extended conformal theories, although the latter have already been discussed to a certain extent in the frame of the quantum Toda field theories.³⁴⁾ Of course it is tempting to conjecture that the relation (3.36) between the Boltzmann weights and the braiding matrices still holds in these cases. (At least for those states a, b, c, d which differ by the vector weights of the A_{n-1} subalgebra of B_n, C_n, D_n this is certainly true, since the corresponding subset of screened vertex operators is the same as in the A_{n-1} case.) To be precise, with the normalizations of ref.²⁷⁾, the generalization of (3.36) to B_n, C_n, D_n should have an extra global factor $(1 - x^{-\frac{1}{2}th_c})$ on the l.h.s., and $x^{-\frac{1}{2}u}$ should be replaced by x^{-u} (h_c is the dual Coxeter number and $t = 2$ for C_n and $t = 1$ for B_n, D_n):

$$(1 - x^{-\frac{1}{2}th_c})x^{-\frac{1}{2}(1-\lambda_1^2)}(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \lim_{u \rightarrow -i\infty} \lim_{p \rightarrow 0} x^{-u} W_{11} \left(\begin{array}{c|c} a & b \\ d & c \end{array} \middle| u \right) = r(c, \lambda_1, \lambda_1, a)_{db} \quad (4.1)$$

APPENDIX

In this appendix we will very briefly sketch some of the results of Bilal and Gervais on the Toda field theories³²⁻³⁵⁾ relevant to the present article, and relate their notations to the one used in the present paper. For a more detailed review, we refer the reader to ref.³⁵⁾. Here, for simplicity, we will restrict ourselves to the A_n case.

The Toda field theories are completely integrable two-dimensional field theories of n interacting bosonic fields $\varphi_i(\sigma, \tau)$. The equations of motion are

$$(\partial_\sigma^2 - \partial_\tau^2)\varphi_i = e^{\sum_{j=1}^n K_{ij}\varphi_j} \quad (A.1)$$

where $K_{ij} = 2e_i \cdot e_j / e_j^2$ is the Cartan matrix of a simple Lie algebra g of rank n . For A_n : $K_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$. These equations of motion can be derived by varying an action functional $\frac{1}{\hbar}S[\varphi_i]$, $\hbar$ being the coupling constant of the system. By suitably adjusting a total derivative term in the action, the canonical energy-momentum tensor is traceless: the Toda field theories are conformally invariant theories.

The classical solutions of the field equations (A.1) are of the form³²⁾

$$e^{-\varphi_i} = \frac{\pm 1}{i!} \sum_{\mu_1, \dots, \mu_i=1}^{n+1} \chi_{\mu_1 \dots \mu_i}(\sigma - \tau) \psi_{\mu_1 \dots \mu_i}(\sigma + \tau) \quad (A.2)$$

where

$$\psi_{\mu_1 \dots \mu_i} = i! \psi_{[\mu_1} \partial \psi_{\mu_2} \dots \partial^{(i-1)} \psi_{\mu_i]} \quad (A.3)$$

$[\mu_1 \dots \mu_i]$ denotes complete antisymmetrization of the indices. $\chi_{\mu_1 \dots \mu_i}$ is defined similarly. In the following we will concentrate on the left-moving fields ψ . The χ 's are treated similarly. One sees that the solutions $e^{-\varphi_i}$ are built up from rank i antisymmetric tensors formed out of $n+1$ fundamental fields ψ_μ . Hence, $e^{-\varphi_i}$ is associated with the A_n representation having the fundamental weight λ_i as highest weight. The fundamental fields ψ_μ are constructed out of n free (Fubini-Veneziano) fields $q_i(x^\pm)$ ($x^\pm = \sigma \pm \tau$), $p_i = \partial q_i$:

$$\begin{aligned} \psi_1(x^+) &= e^{\sum_j G_{1j} q_j(x^+)}, \\ \psi_\mu(x^+) &= \psi_1(x^+) \int_{x_1}^{x^+} dx_1 e^{-q_1(x_1)} \int_{x_1}^{x_1} \dots \int_{x_{\mu-2}}^{x_{\mu-1}} dx_{\mu-1} e^{-q_{\mu-1}(x_{\mu-1})} \end{aligned} \quad (A.4)$$

where $G_{ij} = \lambda_i \cdot \lambda_j$ denotes the inverse of the Cartan matrix K_{ij} .

Starting from the explicit classical solution (A.4) it is straightforward to derive^{32,33)} an (ordinary) differential equation for the ψ_μ :

$$(\partial - h_{n+1} \cdot \Lambda) \dots (\partial - h_2 \cdot \Lambda)(\partial - h_1 \cdot \Lambda) \psi_\mu = 0 \quad (A.5)$$

where

$$\Lambda(\sigma + \tau) = \sum_{i=1}^n \lambda_i p_i(\sigma + \tau) \quad (A.6)$$

Rewriting (A.5) as

$$(-\partial^{n+1} + U_{(2)}\partial^{n-1} + U_{(3)}\partial^{n-2} + \dots + U_{(n+1)}) \psi_\mu = 0 \quad (A.7)$$

allows us to express the "potentials" $U_{(k)}$ as combinations of Λ 's and their derivatives. $U_{(2)}$ turns out to be proportional to the energy-momentum tensor and hence will generate a Virasoro algebra. The other $U_{(k)}$ are related to the other W-algebra generators $W^{(k)}$, e.g. $W^{(3)} \sim U_{(3)} - \frac{n-1}{2}\partial U_{(2)}$.

Starting from the classical action $\frac{1}{\hbar}S[\varphi_i]$, Bilal and Gervais^{32,33)} introduce canonical Poisson brackets for the φ_i and their conjugate fields $\pi_i \sim \frac{1}{\hbar}K_{ij}\partial_\tau\varphi_j$. They have shown³³⁾ that these brackets are equivalent to free field (harmonic oscillator) Poisson brackets for the p_i -fields (and hence also for the Λ). This allowed them to determine the Poisson bracket algebra of the $U_{(k)}$ and to show that the W-algebra closes at the level of Poisson brackets³³⁾.

The Toda field theories are quantized^{32,34)} by replacing the canonical p_j Poisson brackets by i times the commutator. In terms of the Fourier modes p_n^j of $p_j(\sigma + \tau)$ one has:

$$[p_n^i, p_m^j] = 2\hbar e_i \cdot e_j n \delta_{n,-m} \quad (A.8)$$

All fields have to be normal-ordered with respect to the p_n^i harmonic oscillators. Requiring that $:e^{-\eta\varphi_i}:$ remains a conformal dimension-one primary field in the quantum theory (classically $\eta = 1$), such that the quantum analogue of (A.4) makes sense, determines η as one of the two solutions $\eta_\pm$ of

$$2\hbar\eta^2 - \eta + 1 = 0 \quad (A.9)$$

The quantum analogue of (A.4) then gives the fundamental fields ψ_μ as screened vertex operators. It was also shown in ref.³⁴⁾ (at least for A_2) that the same equation (A.9)

ensures that the differential equation (A.7) carries over to the quantum theory as a suitably normal-ordered differential equation for the quantum operators ψ_μ .

Let us now establish the relation between the notation of refs^{32–35)} used up to now in this appendix and the one used in the main part of the present article.

The basic oscillators are related by (cf. (2.4) and (A.8))

$$p_n^j = (2\hbar)^{\frac{1}{2}} e_j \cdot a_n - i\delta_{n,0} \quad (A.10)$$

in such a way that the $sl(2, \mathbb{C})$ invariant vacuum $a_0 = 0$ corresponds to $p_0^j = -i$, $\forall j$.

The coupling constant $\hbar$ is related to the background charge α_0 by

$$2\alpha_0 = \frac{i}{(2\hbar)^{\frac{1}{2}}} \quad (A.11)$$

and the screening charges $\eta_\pm$ and $\alpha_\pm$ are related by

$$\alpha_\pm = i(2\hbar)^{\frac{1}{2}} \eta_\pm \quad (A.12)$$

such that $: e^{i\alpha_\pm \phi} : \simeq : e^{-\eta_\pm \phi} :.$ Hence, upon quantization and rotating to a Euclidean formulation, the fundamental fields $\psi_\mu^\pm$ ($\pm$ refer to using $\eta_\pm$) are to be identified with the fundamental screened vertex operators $\mathcal{V}_\mu$:

$$\psi_\mu^\pm \simeq \mathcal{V}_\mu^\pm \quad (A.13)$$

However, one should be careful about the relative normalizations (see below).

As already described in section 2.4, the braiding matrix $\hat{r}$ for the operators ψ_μ was obtained in ref.³⁴⁾ by exploiting the $\mathcal{P}_{\mu\nu}$ -symmetry and the hypergeometric nature of the four-point function. It reads:

$$\begin{aligned} \hat{r}_{\mu\mu}^{\mu\mu}(\Lambda_0) &= e^{-2\pi i \lambda_1^2 \hbar \eta^2 \epsilon} \\ \hat{r}_{\mu\nu}^{\mu\nu}(\Lambda_0) &= e^{-i\pi(y_{\mu\nu} - 2\hbar\eta^2(1-\lambda_1^2))\epsilon} \frac{\sin 2\pi\hbar\eta^2}{\sin \pi y_{\mu\nu}} \quad (\mu \neq \nu) \\ \hat{r}_{\mu\nu}^{\nu\mu}(\Lambda_0) &= e^{2\pi i \hbar \eta^2 (1-\lambda_1^2)\epsilon} \frac{\Gamma(1+y_{\mu\nu})\Gamma(y_{\mu\nu})}{\Gamma(1+y_{\mu\nu}+2\hbar\eta^2)\Gamma(y_{\mu\nu}-2\hbar\eta^2)} \quad (\mu \neq \nu) \end{aligned} \quad (A.14)$$

where

$$y_{\mu\nu} = i\eta(h_\mu - h_\nu) \cdot \Lambda_0 \quad (A.15)$$

η_- (η_+) has to be used for $\hat{r}$ ($\hat{r}'$). $\epsilon = \pm$ according to which sense of analytic continuation is chosen ($\epsilon = +$ for fig. 2).

As was remarked by Gervais⁴²⁾, one can get rid of the Γ -functions in the last equation (A.14) and obtain a completely trigonometric r -matrix, by multiplying the ψ_μ by an appropriately chosen function of the zero-mode Λ_0 (cf. eqs (2.42,43)): ^{*}

$$\psi_\mu \longrightarrow \hat{\psi}_\mu = \left(\prod_{\nu \neq \mu} \Gamma(1 + y_{\mu\nu}) \right)^{-1} \psi_\mu \quad (\text{A.16})$$

The braiding matrix r for the new $\hat{\psi}_\mu$ operators is the same as before, except that the last equation (A.14) for $\hat{r}_{\mu\nu}^{\nu\mu}$ is changed: [†]

$$r_{\mu\mu}^{\mu\mu} = \hat{r}_{\mu\mu}^{\mu\mu}, \quad r_{\mu\nu}^{\mu\nu} = \hat{r}_{\mu\nu}^{\mu\nu}, \quad r_{\mu\nu}^{\nu\mu} = e^{2\pi i \hbar \eta^2 (1 - \lambda_1^2) \epsilon} \frac{\sin \pi (y_{\mu\nu} - 2\hbar \eta^2)}{\sin \pi y_{\mu\nu}} \quad (\text{A.17})$$

Next, as was shown in ref.³⁴⁾, the spectrum of the zero-modes Λ_0 is given by the application of $(l_i - 1)$ operators : $e^{\eta_- G_{ij} q_j}$: and of $(l'_i - 1)$ operators : $e^{\eta_+ G_{ij} q_j}$: to the $sl(2, \mathbb{C})$ invariant vacuum state $p_0^j = -i, \forall j$. This results in a Λ_0 eigenvalue

$$\Lambda_0(l, l') = -2i\hbar \sum_j (\eta_- l_j + \eta_+ l'_j) \lambda_j \quad (\text{A.18})$$

Of course, by (A.10), this is equivalent to eq. (2.12) for α_W . Then we have

$$y_{\mu\nu}^{(-)}(l, l') = i\eta_- (h_\mu - h_\nu) \cdot \Lambda_0(l, l') = -2\hbar \eta_-^2 l_{\mu\nu} - l'_{\mu\nu} \quad (\text{A.19})$$

$l_{\mu\nu}$ is defined in eq. (2.45). It is now straightforward to substitute (A.19) into the first two equations of (A.14) and into (A.17), replace $-2\hbar \eta_\pm^2$ by $\alpha_\pm^2$, and obtain the equations (2.44) for the r -matrix given in section 2.4.

* Equation (A.16) is not the only possible choice. Multiplying ψ_μ by $\left(\prod_{\nu \neq \mu} \Gamma(1 + 2\hbar \eta^2 - y_{\mu\nu}) \right)^{-1}$ leads to the same result.

† If one had chosen instead to multiply the ψ_μ by $\prod_{\nu \neq \mu} \Gamma(-y_{\mu\nu})$ or by $\prod_{\nu \neq \mu} \Gamma(y_{\mu\nu} + 2\hbar \eta^2)$, the result would be:

$$r_{\mu\nu}^{\nu\mu} = e^{2\pi i \hbar \eta^2 (1 - \lambda_1^2) \epsilon} \frac{\sin \pi (y_{\mu\nu} + 2\hbar \eta^2)}{\sin \pi y_{\mu\nu}}$$

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FIGURE CAPTIONS

1. The integration contours for the screened vertex operators (2.18) are depicted. The dashed lines give the paths of analytic continuation for the integration variables from the reference configuration on the real axis.
2. The path of analytic continuation in the definition of the braiding matrix
3. Dualizing the plaquette (face) yields a vertex.
4. The basic relation between the generators of the braid group
5. The STR(YBE) in the IRF representation (a) and in the vertex representation (b)
6. The integration contours for $\mathcal{V}_\mu(z)\mathcal{V}_\nu(w)$
7. The integration contour for $\mathcal{V}_2(z)\mathcal{V}_1(w)$ is given. For the computation of the braiding matrix, this contour has to be decomposed into two pieces, one from z to w , and one from w to z , as indicated.
8. Changing a reference point for the analytic continuation of the integrand results in multiplying by certain phase factors. Here $V_{\alpha_-}^1(u)$ is moved around the origin, where a $V_{\alpha_{ll'}}(0)$ is inserted, resulting in a phase factor $e^{2i\pi\alpha_{ll'}\cdot e_1\alpha_-}$.
9. The braiding of two operators in one sense and then in the opposite one gives back the initial configuration.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5a

FIGURE 5b

FIGURE 6

FIGURE 7

FIGURE 9

=

$$x \cdot e^{2i\pi \alpha_1 \cdot e_1 \cdot \alpha_1}$$

FIGURE 8