QUATERNIONIC AND OCTONIONIC ORBIFOLDS

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ABSTRACT

We identify the 4- and 6-dimensional orbifolds embeddable in E_8 , the root system of which is constructed with octonions and icosians. The root system of E_6 expressed as integral octonions can be interpreted as a collection of the binary tetrahedral groups and the dicyclic groups of order 12.

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A new set of conformal field theories with $c=1$ are obtained from the Gaussian model with $SU(2) \times SU(2)$ symmetry by performing an orbifold projection on the current algebra via the action of finite subgroups of $SU(2)$ [1]. The quaternion groups are in one-to-one correspondence with the extended simply-laced Coxeter-Dynkin diagrams according to an $A_{n-1} \leftrightarrow$ cyclic group of order n , a $D_n \leftrightarrow$ dicyclic group of order $4(n-2)$, an $E_6 \leftrightarrow$ binary tetrahedral group, $\langle 3,3,2 \rangle$, an $E_7 \leftrightarrow$ binary octahedral group, $\langle 4,3,2 \rangle$, and an $E_8 \leftrightarrow$ binary icosahedral group, $\langle 5,3,2 \rangle$ [2]. Another correspondence between this classification of the quaternion groups and the $n=2$ superconformal field theories has been recently obtained [3]. These developments invite further investigations along the same line. Recently it has been shown that there exist relations between the root systems of $SO(8)$, F_4 , and E_8 and the finite subgroups of $SU(2)$, $\langle 3,3,2 \rangle$, $\langle 4,3,2 \rangle$, and $\langle 5,3,2 \rangle$ respectively [4]. On top of these facts, one of the most important feature of the E_8 is perhaps that its root system can be represented by the integral octonions corresponding to the non-associative discrete algebra of order 240 [5].

In what follows we describe the construction of the E_8 root system with octonions and icosians (quaternionic elements of the binary icosahedral group) and show the relevant discrete subgroups of $SU(2)$ associated with the 4-and 6-dimensional orbifolds. The attractive part of these orbifolds is that they cannot be obtained by the Weyl group automorphisms and can be used to reduce the rank of the gauge symmetries obtained by the compactifications [6] of the heterotic string [7].

The root systems of $SO(8)$ and F_4 are described by the following sets of quaternions provided they are multiplied by $\sqrt{2}$:

$$SO(8) : A_0 \quad ; \quad F_4 : A_0 + A_1 + A_2 + A_3 \quad (1)$$

where $A_a (a=0,1,2,3)$ are given by

A_0	A_1	A_2	A_3
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$\pm 1, \pm e_1, \pm e_2, \pm e_3$	$1/2(\pm 1 \pm e_1)$	$1/2(\pm 1 \pm e_2)$	$1/2(\pm 1 \pm e_3)$
$1/2(\pm 1 \pm e_1 \pm e_2 \pm e_3)$	$1/2(\pm e_2 \pm e_3)$	$1/2(\pm e_3 \pm e_1)$	$1/2(\pm e_1 \pm e_2)$

(2)

It is trivial to check that the Hurwitz integers [8] of A_0 form the binary tetrahedral group of order 24. Similarly the 48 roots of F_4 represent the elements of the binary octahedral group provided the short roots underlined by A_1 , A_2 and A_3 are multiplied by $\sqrt{2}$. Since octonions can be regarded as pairs of quaternions it is evident that the following set of octonions describe the root system of E_8 :

$$\begin{aligned} E_8 : \quad [A_0, 0] &= A_0, \quad [0, A_0] = e_7 A_0, \quad [A_1, A_1] = A_1 + e_7 A_1 \\ [A_2, A_3] &= A_2 + e_7 A_3, \quad [A_3, A_2] = A_3 + e_7 A_2 \end{aligned} \quad (3)$$

where e_7 is a new imaginary unit and we use the definitions $e_4 = e_7 e_1$, $e_5 = e_7 e_2$, $e_6 = e_7 e_3$ for the other imaginary units of octonions. A similar construction of the E_8 lattice with quaternions can be made [9]:

$$\begin{aligned} E_8 : \quad (A_0, 0) &= A_0, \quad (0, A_0) = \sigma A_0, \quad (A_1, A_2) = A_1 + \sigma A_2 \\ (A_2, A_3) &= A_2 + \sigma A_3, \quad (A_3, A_1) = A_3 + \sigma A_1 \end{aligned} \quad (4)$$

where $\sigma = 1/2(1 - \sqrt{5})$ with $\sigma^2 = \sigma + 1$. Half of the roots in (4) form the binary icosahedral group of order 120. It is trivial to check that the E_8 lattice represented by (3) and (4) is invariant under the action of $\langle 3, 3, 2 \rangle$ and $\langle 5, 3, 2 \rangle$ respectively. The invariance of (3) under $\langle 3, 3, 2 \rangle$ leads to a natural decomposition of E_8 with respect to $SO(8) \times SO(8)$ where the octonionic roots of this subgroup of E_8 form a closed algebra with chiral-like structures. Therefore the root system of $SO(8)$ describes a 4-dimensional orbifold with a group structure of $\langle 3, 3, 2 \rangle$. This orbifold can be used to obtain the 6-dimensional string [10] from the heterotic string in 10-space-time dimensions. Another 4-dimensional Z_5 - $SU(5)$ orbifold can be obtained in the context of icosianic E_8 lattice where $SU(5) \times SU(5)$ can be embedded in E_8 with Z_5 -symmetry [4]. As we will discuss later there are also 4-dimensional orbifolds with dicyclic group structures representing the root systems of $SU(3)^2$.

In this note we will emphasize more the 6-dimensional orbifolds, Z_3 - E_6 and Z_3 - $SU(3)^3$ which admit further symmetries besides the trivial Z_3 -symmetries. To illustrate this point we give a decomposition, under $SU(3) \times E_6$, of the octonionic roots of E_8 given in (3) :

$$[0, \pm 1, \pm e_1, \pm e_2, \pm e_3] \quad (5)$$

$$\text{SU(3) Roots : } [0, \pm S_0, \pm S_3, \pm S_1, \pm S_2]$$

$$[0, \pm \bar{S}_0, \mp \bar{S}_2, \mp \bar{S}_3, \mp \bar{S}_1]$$

$$[A_1, \pm 1/2(e_2 - e_3), \pm 1/2(e_2 + e_3), \pm 1/2(1 - e_1), \mp 1/2(1 + e_1)]$$

$$\text{E}_6 \text{ Roots : } [A_2, \pm 1/2(e_1 - e_2), \pm 1/2(1 - e_3), \mp 1/2(1 + e_3), \pm 1/2(e_1 + e_2)]$$

$$[A_3, \pm 1/2(e_3 - e_1), \mp 1/2(1 + e_2), \pm 1/2(e_3 + e_1), \pm 1/2(1 - e_2)]$$

$$[A_0, 0]$$

where we denote by $S_0 = 1/2(1 + e_1 + e_2 + e_3)$, $S_1 = 1/2(1 + e_1 - e_2 - e_3)$, $S_2 = 1/2(1 - e_1 + e_2 - e_3)$, $S_3 = 1/2(1 - e_1 - e_2 + e_3)$. (5) should be understood as follows: if the roots of SU(3) are chosen as

$$[0, \pm 1] = \pm e_7, [0, \pm S_0] = \pm 1/2(e_7 + e_4 + e_5 + e_6), [0, \pm \bar{S}_0] = \pm 1/2(e_7 - e_4 - e_5 - e_6) \quad (6)$$

then the corresponding E_6 roots are

$$[A_0, 0], [A_1, \pm 1/2(e_2 - e_3)], [A_2, \pm 1/2(e_1 - e_2)], [A_3, \pm 1/2(e_3 - e_1)] \quad (7)$$

where $A_a (a=0,1,2,3)$ are given by (2). This decomposition is invariant under the transformation $(\pm S_0)R(\pm \bar{S}_0)$ where R represents any root of E_8 . Since $S_0^3 = -1$ the transformation by S_0 leads to a Z_3 -invariance of each column in (5). Another interesting aspect of this decomposition is that we can obtain the E_6 root system with S_0 -invariance in 4 different ways by placing, in turn, the roots of $SO(8)$ $[A_0, 0]$ below the columns 0, 1, 2, 3. Before we show the group structures emerging from the E_6 root system we should briefly explain the method by which one obtains the quaternionic structures in (5). The octonionic roots in (5) can be classified as follows:

- i) Real roots M: ± 1 (roots of SU(2))
- ii) Pure imaginary roots N: (126 roots of E_7)
- iii) Mixed roots P: (112 roots of $E_8/SU(2) \times E_7$)

Each one of the 112 roots satisfies either the relation $P^3 = 1$ or $P^3 = -1$. With the use of any P we can decompose the 240 roots into associative triads satisfying the relations

$$\begin{aligned}
 Q_1 P & \text{ for } \langle Q_1, \bar{P} \rangle = 0 \text{ (42 triads)} \\
 Q_1 Q_2 Q_3 & = \begin{aligned} & 1 \quad \text{for } \langle Q_1, \bar{P} \rangle = 1/2 \text{ (18 triads)} \\ & -1 \quad \text{for } \langle Q_1, \bar{P} \rangle = -1/2 \text{ (18 triads)} \end{aligned}
 \end{aligned} \tag{8}$$

where we defined $Q_2 = PQ_1\bar{P}$, $Q_3 = \bar{P}Q_1P$ and $\langle Q_1, \bar{P} \rangle = 1/2(\bar{Q}_1\bar{P} + PQ_1)$. Since the quaternionic triad (e_1, e_2, e_3) satisfies the relation $e_1 e_2 e_3 = -1$, $e_i^2 = -1$, we can find 18 triads of octonionic roots obeying this condition. Thus we may have 9 sets like $(\pm e_1, \pm e_2, \pm e_3)$. Once we are given a set of quaternionic units we know that they form the binary tetrahedral group which consists of Hurwitz integers A_0 in (2) which can also be written as

$$A_0: \pm 1, \pm e_1, \pm e_2, \pm e_3, \pm S_a, \pm \bar{S}_a \text{ (a=0,1,2,3)} \tag{9}$$

Here S_a describe the vertices of the tetrahedron. Therefore it is likely that the E_8 root system can be decomposed into sets of roots including the 9 sets of Hurwitz integers. Explicit calculations show that this is indeed the case provided the set of $SU(3)$ roots $X = (\pm 1, \pm S_0, \pm \bar{S}_0)$ are combined with each set separately. In other words, 9 sets of Hurwitz integers have an intersection which consists of the roots of X where they play a crucial role in the decomposition. The remaining roots of E_8 consist of the pure imaginary octonions, subsets of which together with X form dicyclic groups of order 12. We have also checked that the octonionic roots of E_7 can be decomposed into 5 sets of Hurwitz integers. In the case of E_6 , as we show below explicitly, this decomposition leads to 3 sets of $\langle 3, 3, 2 \rangle$ and 2 sets of dicyclic groups of order 12. The association of the numbers 3, 5, 9 with E_6 , E_7 , and E_8 has also been noticed by Kac and Peterson in a different context of investigation of simply-laced exceptional groups [11].

The Coxeter-Dynkin diagrams of A_2 and E_6 with octonionic roots are given in Fig.1 and Fig.2. The simple roots are chosen so that they lead to the $SU(3) \times E_6$ decomposition of E_8 , where $SU(3)$ and E_6 roots are displaced under the column denoted by 0 in (5). The threefold symmetry of the diagrams must be understood as follows:

$$\begin{aligned}
 S_0 B_1 \bar{S}_0 & = B_2 \\
 S_0 e_1 \bar{S}_0 & = e_2 \quad \text{(cyclic permutations)} \\
 S_0 (-e_7) \bar{S}_0 & = 1/2(e_7 - e_4 - e_5 - e_6)
 \end{aligned} \tag{10}$$

The inspection of Fig.2 shows that the simple roots e_1, e_2, e_3 and $B_0 = \bar{S}_0$ lead to A_0 , the roots of the $SO(8)$ subalgebra of E_6 . It is clear that the simple roots B_0, B_1, B_2 , and B_3 may lead to the same type of structure. We should notice that B_1, B_2 , and B_3 are not made of pure imaginary roots. However, it is possible to define the following imaginary roots

$$\begin{aligned} E_1 &= -1/2(B_0 - B_1 + B_2 + B_3) = B_1 + \bar{B}_0 \\ E_2 &= -1/2(B_0 + B_1 - B_2 + B_3) = B_2 + \bar{B}_0 \\ E_3 &= -1/2(B_0 + B_1 + B_2 - B_3) = B_3 + \bar{B}_0 \end{aligned} \quad (11)$$

satisfying the relations $E_i^2 = -1$, ($i=1,2,3$) $E_1 E_2 E_3 = -1$. A similar relation is valid between (e_1, e_2, e_3) and S_a ($a=0,1,2,3$) where E_i, B_i , and $-B_0$ are replaced by e_i, S_i , and S_0 in (11). One uses the same formulation in (9) to obtain a different embedding of $SO(8)$ in E_6 :

$$\pm 1, \pm E_1, \pm E_2, \pm E_3, 1/2(\pm 1 \pm E_1 \pm E_2 \pm E_3) \quad (12)$$

This can also be written as

$$\begin{aligned} &\pm 1, \pm B_0, \pm \bar{B}_0 \\ &\pm E_1, \pm B_1, \pm \bar{B}_1 \\ &\pm E_2, \pm B_2, \pm \bar{B}_2 \\ &\pm E_3, \pm B_3, \pm \bar{B}_3 \end{aligned} \quad (13)$$

where $X = (\pm 1, \pm S_0 = \pm \bar{B}_0, \pm \bar{S}_0 = \pm B_0)$ occurs once more as it occurred in A_0 . Now we define the roots $G_i = B_i + e_i$ ($i=1,2,3$), $G_0 = B_0$ satisfying the conditions $G_a^2 = \pm 1$. We define three new quaternionic units F_i replacing B_a in (11) by $\bar{G}_a$. The new set of roots will be in the form of (12) where E_i 's are replaced by F_i 's, or of (13) where B_a 's are also replaced by $\bar{G}_a$'s. In this new embedding of the $SO(8)$ algebra in E_6 the roots of X occur once again. The remaining $72 - (24 + 18 + 18) = 12$ roots of E_6 are made of pure imaginary octonions and can be divided into two sets of orthogonal roots of $Y = (\pm H_i)$ and $Z = (\pm I_i)$ where we define

$$\begin{aligned} H_1 &= E_3 - F_2, \quad H_2 = E_1 - F_3, \quad H_3 = E_2 - F_1 \\ I_1 &= F_3 - E_2, \quad I_2 = F_1 - E_3, \quad I_3 = F_2 - E_1 \end{aligned} \quad (14)$$

They satisfy the relations

$$\begin{aligned}
 S_0 H_1 \bar{S}_0 &= H_2 \\
 S_0 I_1 \bar{S}_0 &= I_2 \\
 H_1 H_2 &= S_0, H_2 H_1 = \bar{S}_0 \quad (\text{cyclic permutations}) \\
 I_1 I_2 &= S_0, I_2 I_1 = \bar{S}_0
 \end{aligned} \tag{15}$$

Three sets of roots X , Y , and Z correspond to the roots of $SU(3)^3 \subset E_6$. Notice that each set is invariant under Z_3 -rotation of S_0 . This embedding of $SU(3)^3$ in E_6 is different from that of $SU(3)^3$ whose roots can be read from Fig.2 :

$$T_i = (\pm B_i, \pm e_i, \pm G_i) \quad (i=1,2,3) \tag{16}$$

Here T_1 , T_2 , and T_3 are cyclically rotated to each other under S_0 . We have already remarked that the set X forms a cyclic group of order 6. One can check that the sets of combined roots $(X+Y)$ and $(X+Z)$ separately form dicyclic groups of order 12, the quaternionic group covering the dihedral group of order 6. What we have shown so far is that the octonionic roots of E_6 can be grouped into 3 sets of 18 roots and 3 sets of 6 roots of X , Y , Z . When X is combined with any set of 18 elements they form the binary tetrahedral group whereas when Y or Z is combined with X the new set is a dyclic group of order 12. Therefore, this octonionic root system of E_6 can be regarded as the Z_3 - E_6 orbifold with additional group structures which cannot be obtained by other means of automorphisms of the E_6 lattice.

One can also use the $SU(3)^3$ orbifold of X , Y , and Z or of T_1 , T_2 , and T_3 for the compactification of the heterotic string. In either case one may exploit the non-Abelian group structures of the orbifolds to reduce the rank of the gauge symmetry left over the compactification. Either of the $(X+Y)$ or $(X+Z)$ root systems of $SU(3)^2$ can also be used as a 4-dimensional orbifold.

A number of additional remarks are in order for the use of the extraordinary power of the octonionic description of E_6 . Let us denote by $W=e_7X$ the roots of $SU(3)$ orthogonal to E_6 . The roots in $(W+X)$ also form a dicyclic group of order 12. Moreover, the set of roots in $(X+Y+Z+W)$ form a closed non-associative algebra of order 24 where the non-associativity occurs in the products like $YZW \subset X$. (Y, Z, W) form a closed algebra of order 18 with a cyclic permutation of the product $YZ \subset W$.

The second remark is about the fact that the octonionic roots of E_8 can be generated, by multiplication, from 3 fundamental roots B_1 , e_1 , and B_0 corresponding to the simple roots of A_3 which is the incidence diagram of the Ising model [12]. This reminds us that there could be a relation between the generation of E_8 roots from the simple roots of A_3 and the extension of the simplest Ising model to a more sophisticated statistical mechanics model, and thereby the related conformal field theory associated with E_8 .

One more important property of the octonionic roots of E_8 concerns the discrete subgroup of the exceptional group G_2 . It is known that the automorphism group of the octonion algebra is G_2 [13]. The discrete subgroup of G_2 of order 6048 preserves the discrete octonion algebra of order 240 [14]. In other words, the discrete subgroup of G_2 of order 6048 is an automorphism group of E_8 lattice. One can construct the 7-dimensional representation of the discrete subgroup of G_2 acting on the 7 imaginary units of octonions with the following simple prescription. We have already stated that the E_8 can be decomposed into 9 sets of binary tetrahedral groups of order 24. Therefore for a given decomposition of the E_8 root system as in (5), where $\pm S_0$ plays the principal role, there exist $9 \times 24 = 216$ 7×7 matrices of elements 0, $\pm 1/2$, and ± 1 . Since we have 28 pairs of roots like $(\pm S_0, \pm \bar{S}_0)$ leaving the octonion algebra invariant, we can decompose the root system of E_8 28 different ways similar to the one in (5). Therefore the order of the discrete subgroup of G_2 , $9 \times 24 \times 28 = 6048$, is in accord with the decomposition of the octonionic root system of E_8 into 9 sets of the binary tetrahedral groups.

When we investigate the similar properties of the icosianic roots of E_8 we observe that the similar group structures of the root systems arise but with restricted possibilities.

Details will be published elsewhere.

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FIGURE CAPTIONS

Figure 1: Extended Coxeter-Dynkin Diagram of $SU(3)$

Figure 2: Extended Coxeter-Dynkin diagram of E_6 . Here $B_1 = 1/2(-1 - e_1 - e_5 + e_6)$, $B_2 = 1/2(-1 - e_2 - e_4 + e_5)$, $B_3 = 1/2(-1 - e_3 + e_4 - e_6)$, $B_0 = \bar{S}_0 = 1/2(1 - e_1 - e_2 - e_3)$