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FINITE – SIZE SCALING, ZAMOLODCHIKOV'S C – THEOREM AND THE S – MATRIX

N.E. Mavromatos ¹⁾ and J.L. Miramontes ²⁾

CERN – Geneva

Abstract

Using finite – size scaling methods, we establish the equivalence of Zamolodchikov's C – function with the generalized central charge and the effective action in coupling constant space of a generic 2D quantum field theory under redefinitions of the couplings.

¹⁾ On leave from Dept. of Theoretical Physics, Univ. of Oxford, and Hertford College, Oxford (U. K.)

²⁾ On leave from Dept. de Particulas, Univ. de Santiago, E-15706 Santiago de Compostela (Spain). Supported in part by CICYT

Zamolodchikov's C-theorem [1] has become an important tool for the study of 2D renormalisable field theories away from their critical points, where they become conformally invariant. This theorem allows for the construction of a scalar function of the couplings (C) which interpolates between different fixed points, where it equals the central charge, in such a way that it never increases under Renormalisation Group (RG) transformations in unitary theories, thus providing an analogue of an entropy function. Consider a renormalisable 2D quantum field theory described by a Lagrangian $\mathcal{L}(g, \alpha, x)$, with $\{g^i\}$ being a set of coupling constants and α an ultraviolet (UV) cut-off. From a geometrical point of view the RG transformation may be interpreted as a coordinate transformation in the (possibly infinite dimensional) coupling constant space, from a system of (bare) coordinates $\{g^i_0\}$ to another (renormalised) $\{g^i(t)\}$. The RG parameter t , given by $-\ln(\mu\alpha)$ with μ the renormalisation scale, plays the rôle of an evolution parameter characterising the RG transformation. The usual way to solve the RG equation $dg/dt = \beta^i(g)$ is to identify the g^i_0 with $g^i(t=0)$, and expand $g(t)$ in power series of g_0 [2]

$$g^i(t) = \exp(\lambda_i t) g^i_0 + [\exp\{(\lambda_j + \lambda_k)t\} - \exp(\lambda_i t)] \frac{a^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} g^j_0 g^k_0 + \dots \quad (1)$$

In this coupling constant expansion energy denominators appear in much the same way as in scattering amplitudes of simple quantum mechanical systems. It is this similarity that motivates the conjecture that the vanishing of the β -functions, when this procedure is applied to first-quantized string theories, may lead to background field equations of motion, obtained from a string effective action [3]. In fact this conjecture has been verified in perturbation theory up to some non-trivial orders in the α' -expansion [4].

Recently a formal proof of the equivalence of the non-linear σ -model (NLSM) conformal invariance conditions and the stationarity of the generalised central charge coefficient, based on perturbative RG techniques, has become available [5]. In particular an off-shell relation between the β -functions and the generalised central charge $\int \tilde{\beta}^\phi$ has been derived. In addition, a detailed proof of a

C-theorem in this context has also been presented [6] by completing previous arguments [7]. Moreover, as shown in [6] the C-function is entirely expressed in terms of the corresponding running generalised central charge, at least in some class of RG schemes. Therefore it is related off-shell to the β -functions of the theory through relations of the form $\partial_i C = D_{ij} \beta^j$, where D_{ij} is a perturbatively invertible tensor in coupling constant space. These relations show that the conformal invariance conditions imply the stationarity of C and vice versa. Perturbative calculations in σ -models [4] and general arguments [8,9] show that the on-shell amplitudes generated by the integrated generalised central charge function $\int \tilde{\beta} \phi$ are the same as those of the corresponding string theory at tree level. However a general proof of this has not become available so far. Such proof requires, first, the establishment of the exact relation of the S-matrix generator (the effective action) to the generalised central charge and Zamolodchikov C-function and, second, the proof of the (off-shell) relation between field variations of such an action and the β -functions.

In this letter we show that such relations can be understood as a consequence of the application of the RG to 2D systems perturbed around the fixed points by a complete set of primary marginal operators, as is the case of all the on-shell string vertices in the Minkowskian target space formulation. Therefore the energy denominator structure of the perturbative expansion in eq.(1) is just a manifestation of the result that the vanishing of the β -functions are the equations of motion of the effective action in coupling constant space, which is nothing else but the free¹energy of the underlying 2D system.

Our basic tool will be finite-size scaling methods. One of the motivations for this is to have an automatic infrared cut-off and, in this way, to avoid infrared divergences which could spoil the existence of the two-point functions entering the definition of the C-function. In addition this method provides a direct connection between the free energy of the system, which generates the S-matrix amplitudes, the generalised central charge and, under coupling constant redefinitions, Zamolodchikov's C-function. The organization of this letter is as follows. We first extend the work of ref.[10] for systems away from criticality. We shall restrict ourselves to the case of a statistical system defined on the plane. Recently this method has been followed in ref.[11] in an attempt to give an alternative proof of

¹ It is the effective action in the sense that its derivatives with respect to the (renormalised) couplings yield the connected correlation functions of the theory, and hence the S-matrix in the string case.

the C-theorem. However the simplified case of a single coupling constant is considered there, which can be always expressed as a gradient flow, instead of the relevant one with many coupling constants where this fact is non-trivial.

Consider a theory defined around the RG Gaussian fixed point (*) on an infinite strip of width L. The free energy is given by

$$F = -\ln \langle \exp(-g^i \int_S O_i) \rangle_* \equiv -\ln \int \mathcal{D}\phi \exp(-I^*[\phi] - g^i \int_S O_i) \quad (2)$$

where the perturbation around * has been expanded in a complete set of local operators $\{O_i\}$, $I(g) = I^* + g^i \int O_i$. For simplicity, and motivated by the string case, we restrict ourselves to theories with real critical exponents and choose the operators in our set to be primary and slightly relevant (with small but positive anomalous dimension $\epsilon_i \rightarrow 0^+$). In this case the theory is renormalisable and Zamolodchikov's metric in coupling constant space is flat at the fixed point [12]. The free energy on the strip may be calculated as a power series in $g^i(t)$ in terms of the correlation functions evaluated at the fixed point. The latter can be obtained from the correlation functions on the plane (bulk) by a conformal transformation $w = (L/2\pi) \ln z$ which maps the plane (z) to the strip (w) of width L [13]. Note that away from the fixed point the free energy is not scale invariant, since the coupling constants (after the subtractions) are scale dependent. It only becomes scale independent at the fixed points, where the β -functions vanish. Since our operators are slightly relevant, we can choose the UV subtractions on the operators (normal ordering prescription) in such a way that the one-point functions $\langle O_i \rangle_g$ on the plane vanish away from * [11,13]. Nevertheless on the strip they will only vanish at the fixed point, as implied by conformal invariance. Following these prescriptions the difference between the free energy on the strip (F) and that on the plane (F_0) can be obtained after straightforward manipulations, and it reads

$$F = F_0 + \frac{V}{L^2} [\Delta(g) + O(\epsilon \ln L)] \quad (3)$$

where V is the strip volume and all the explicit dependence on $\ln L$ is removed in the limit $\epsilon_i \rightarrow 0^+$ which we take at the end. The coefficient of $1/L^2$ in eq.(3) is universal up to $O(\epsilon)$ terms for the per-

turbations considered here. It will be proportional to the generalised central charge (see below). Eq.(3) provides the connection between the effective action on the strip² and Δ , given that $\partial_i F \sim \partial_i \Delta$, since $\partial_i F_0 = 0$ due to our prescriptions. In fact it has the form

$$\Delta(g) \propto \text{const.} - \epsilon^i G_{ij}(s_0) g^i g^j + O(g^3) + O(\epsilon \ln L) \quad (4)$$

showing a decrease along RG trajectories of relevant operators if $G_{ij}(S_0)$, the two-point correlation function at the fixed point, is positive, exactly as predicted by Zamolodchikov, and Ludwig and Cardy [1,14]. Of course in the limit $\epsilon \rightarrow 0^+$ this term vanishes and one has to examine the $O(g^3)$ term which is universal [13,12]

$$C_{ijk} g^i g^j g^k = -g^i G_{ij}(s_0) \alpha_{mn}^j g^m g^n \quad (5)$$

showing that to this order $\partial_i \Delta \sim G_{ij} \alpha_{mn}^j g^m g^n$ and, again, that $d\Delta/dt \leq 0$ if G_{ij} is positive. As we shall discuss below, this positivity is not essential for the equivalence between the β -functions and the equations of motion of the effective action.

After having identified the S-matrix generator with Δ , let us derive a closed expression for the function Δ in terms of the components of the energy-momentum tensor. It is worthwhile to mention that the constant on the RHS of eq.(4) equals the central charge, according to the analysis in ref.[10]. Now consider the following infinitesimal transformation

$$\begin{aligned} u &\longrightarrow u + \lambda_1 u \\ v &\longrightarrow v + \lambda_2 v \end{aligned}, \quad |\lambda_{1,2}| \ll 1 \quad (6)$$

where u and v denote the longitudinal and transversal coordinates on the strip. This is just a coordinate transformation on the strip, and hence the change in the free energy is given in terms of the energy-momentum tensor

² F generates the connected correlation functions on the strip. However at the conformal points it becomes independent of L , which in this case acts just as an infrared cut-off which can be removed ($L \rightarrow \infty$).

$$\delta F = \langle \delta I \rangle_g^s = -\frac{1}{2\pi} \lambda^+ \int_s \langle \Theta \rangle_g^s - \frac{\lambda^-}{\pi} \int_s \langle T + \bar{T} \rangle_g^s \quad (7.a)$$

where $\lambda^\pm = (\lambda_1 \pm \lambda_2)/2$, Θ is the trace of the energy-momentum tensor (which vanishes at the fixed point) and T is the zz -component in complex notation. We assume periodic boundary conditions on the strip, which is equivalent to having the theory on a cylinder, and hence $\langle T \rangle_g = \langle \bar{T} \rangle_g$. This property is certainly true in the conformal limit due to symmetry. In the presence of backgrounds it is maintained for the correlation functions of T and $\bar{T}$, given that these functions are real for the case of Hermitian Lagrangians we consider here. The energy-momentum tensor is not a primary operator, and its v.e.v. on the strip cannot be simply obtained from its value on the plane. There is the conformal anomaly, the central charge, which appears in the transformation. Hence by choosing the UV subtractions (normal ordering prescription) in such a way that $\langle T \rangle_g = 0$ on the plane, we get that $\langle T \rangle_g$ on the strip does not vanish [11,13] and is proportional to L^{-2} up to $O(\epsilon \ln L)$ terms by dimensional analysis.

The change in F given by eq.(7) is due to its scale dependence on the width L and on the (covariant) UV (length) cut-off α . Hence we can write

$$\delta F = -\lambda^+ \left(\frac{\partial F}{\partial \ln L} + \frac{\partial F}{\partial \ln \alpha} \right) + \lambda^- \frac{\partial F}{\partial \ln L} \quad (7.b)$$

By equating the coefficients of $\lambda^\pm$ in eqs.(7) we obtain the relations

$$\frac{\partial F}{\partial \ln L} + \frac{\partial F}{\partial \ln \alpha} = \frac{1}{2\pi} \int_s \langle \Theta \rangle_g^s \quad (8.a)$$

$$\frac{\partial F}{\partial \ln L} = -\frac{1}{\pi} \int_s \langle T \rangle_g^s \quad (8.b)$$

Eq.(8.a) is just the statement that F has zero anomalous dimension, and provides a check of the well-known expansion $(2\pi)^{-1} \int_S \langle \Theta \rangle = \beta^i \int_S \langle O_i \rangle_g = \beta^i \partial F / \partial g^i$. The second equation (8.b) yields the relation between the value of $\langle T \rangle_g$ on the strip and the quantity Δ appearing in eq.(3), which reads

$$\langle T \rangle_g^s = \frac{\pi \Delta(g)}{L^2} + O(\epsilon) \quad (9)$$

which is the extension of the result of ref.[10] in the neighbourhood of a fixed point. To relate Δ with the central charge and, subsequently, with Zamolodchikov's C -function we perform again the coordinate transformation (6) on both sides of eq.(9), taking into account the L -independence of local operators on the strip. The transformation of the LHS reads

$$\begin{aligned} \delta \langle T \rangle_g^s &= -\lambda^+ \left[\frac{\partial \langle T \rangle_g^s}{\partial \ln L} + \frac{\partial \langle T \rangle_g^s}{\partial \ln \alpha} \right] + \lambda^- \frac{\partial \langle T \rangle_g^s}{\partial \ln L} = \\ &= \lambda^+ \left[\frac{2\pi\Delta}{L^2} - \frac{\pi}{L^2} \beta^i \partial_i \Delta \right] + 2\lambda^- \frac{\pi\Delta}{L^2} \end{aligned} \quad (10)$$

where we have used that T and I have zero anomalous dimension, and that I is dimensionless. Equating this result with the transformation of the RHS of eq.(9) one gets

$$\Delta(g) = \frac{L^2}{2\pi^2} \int_S d\sigma^2 \langle T(\sigma) T(0) \rangle_g^s \quad (11)$$

assuming periodic boundary conditions again ($\langle T\bar{T} \rangle_g = 0$ on the strip). This correlator can be computed through a perturbative expansion in g^i

$$\langle T(\sigma) T(0) \rangle_g^s = \langle T(\sigma) T(0) \rangle_*^s + \sum_N \frac{1}{N!} g^{i_1} \dots g^{i_N} \langle T(\sigma) T(0) \int_S O_{i_1} \dots \int_S O_{i_N} \rangle_*^s \quad (12)$$

Using the fact that $\langle T\bar{T} \rangle_g$ does not acquire anomalous dimension (even if its arguments come close to each other [5,6]) and similar arguments as in eq.(3), we finally conclude that, up to $O(\epsilon)$ terms, Δ and, hence, the effective action, is proportional to the generalized central charge in the neighbourhood of the Gaussian fixed point.

Once we have identified the effective action with the generalised central charge, the next question concerns its relation to Zamolodchikov's C -function³

³ There are some technical difficulties concerning extra infinities appearing in the coincidence limit of the arguments of the two-point functions which define C . Because of these infinities the RG invariants that satisfy Zamolodchikov's theorem are the ones in which these infinities are subtracted [6]

$$C = 2z^4 \langle T(\tau) T(0) \rangle_g - 4z^3 \bar{z} \langle T(\tau) \Theta(0) \rangle_g - 6z^2 \bar{z}^2 \langle \Theta(\tau) \Theta(0) \rangle_g \Big|_{\sigma \neq 0} \quad (13)$$

Notice that the first term is proportional to Δ . It has been shown in ref.[15] that, if $C(g)$ is related off-shell to the β -functions, then the coefficients of the last two terms on the RHS of (13) can be arbitrarily varied by redefinitions of the couplings, which amounts to changing the RG subtraction scheme. Such an off-shell relation has been proved in the case of NL ϕ M propagating in massless backgrounds [6] using the results in [5]. The proof can be straightforwardly generalised to our case, where an infinite number of (renormalisable) marginal couplings are considered. The basic procedure is to couple the system to 2D gravity by: (i) extending the flat-space couplings $\{g^i\}$ to 2D-space dependent ones, and removing this dependence only at the very end of the calculations, and (ii) introducing an extra coupling constant ϕ , from thereon being referred to as the dilaton⁴ multiplying the 2D scalar curvature as required by renormalisability. Then the analysis of ref.[5] allows for the construction of a generalised central charge given by $\tilde{\beta}^\phi = \beta^\phi + W_i g^i$, where W_i is a function appearing in the total derivative terms in the expression of the trace of the energy-momentum tensor, which by power counting has the form

$$\Theta = \beta^i [O_i] + \beta^\phi R^{(2)} + \nabla_\mu (W_i \partial^\mu g^i), \quad \mu = 1, 2 \quad (14)$$

W_i can be computed by RG analysis. Then an off-shell relation for $\tilde{\beta}^\phi$ follows: $\partial_i \tilde{\beta}^\phi = K_{ij} \beta^j$, with K_{ij} being a perturbatively invertible (but not positive) tensor in coupling constant space. Extending now the analysis of ref.[6] for the correlators of the components of the energy-momentum tensor to this case, we obtain that all of them are functions of $\tilde{\beta}^\phi$ and its derivatives with respect to $\ln \mu$. Therefore Zamolodchikov's C -function satisfies

$$\partial_i C = D_{ij}(g) \beta^j \quad (15)$$

with $D_{ij}(g)$ being a perturbatively invertible tensor in coupling constant space. Notice that $D_{ij}(g)$ does not need to be positive-definite to satisfy eq.(15). In fact, in the case of Minkowski target space for-

⁴ In general ϕ has to be identified with a certain combination of the couplings $\{g^i\}$ to yield covariant vertex operators with the correct conformal weight on curved space. In the case of NL ϕ M propagating in massless backgrounds, this combination is the trace of the (target-space) graviton, and ϕ is the standard dilaton [16].

mulations of stringy NLM (as is our case) Zamolodchikov's metric is non-positive, due to the existence of the non-unitary X_0 -sector (the time coordinate). Therefore we conclude that Zamolodchikov's C-function does contain the same (on-shell) physical information as the free energy, which is our effective action, for the generic 2D case considered here.

Before closing, let us make some comments on the particular application of our analysis to strings, where the couplings are understood as target space-time dependent fields corresponding to the physical modes of the string. One of the problems is the incorporation of target space-time symmetries, such as diffeomorphism invariance, gauge symmetries, etc. From a RG point of view these symmetries do not affect RG flows and, therefore, must correspond to redundant operators. Their effect is however non-trivial. They result in the modification of eq.(15) by, first, the replacement of β^i by the so-called Weyl anomaly coefficients $\bar{\beta}^i$ [17], and second, by the appearance of extra terms that spoil the off-shell relation. There is one resolution, at least in the case of NLM with massless backgrounds [5]. The generalised central charge is replaced in that case by a (target) space-time integrated version with respect to an appropriately-defined measure which varies under variations of g^i in such a way as to cancel the unwanted terms (such a measure comes from the functional integration over the zero-modes of the 2D Laplacian in I^* , which is responsible for the target space-time dependence of the couplings). By analogy with the massless case, one expects this measure to have the form $\int d^D y \exp(-\phi_\chi) J(g^i)$ in this case, where χ is the Euler characteristic of the 2D-space (the sphere, for the closed string at tree-level). The **Jacobian** $J(g^i)$ will then be determined by the condition that its variation cancels any possible unwanted term which would spoil eq.(15). Whether such a condition can be satisfied is something which is not known exactly, even for the massless case [5], where however there is comforting evidence through perturbative calculations [4]. From our point of view this Jacobian is just a change of the RG scheme for the dilaton field [6], whose exponential plays the rôle of the string coupling constant. We shall come back to these issues in the near future.

However, we note that the off-shell relations are not essential for the equivalence of the conformal invariance conditions with the equations of motion of an action which reproduces scattering amplitudes in coupling constant space. It was only necessary for the equivalence of the central charge ($\sim \Delta$) with Zamolodchikov's C-function, both understood as actions, under coupling redefinitions

which do not change the S -matrix. This result agrees with the one obtained in ref.[9], where the generating functional of 1PI graphs in a generic bosonic string theory was shown to satisfy a relation analogous to Zamolodchikov's in a given scheme. Their approach was based on $SL(2,C)$ invariance and factorization of the string amplitudes, which seems not to be necessary in ours. Nevertheless it has been shown [18] that it is possible to divide by the (infinite) Möbius volume in a RG invariant way in the case of closed strings, by taking the logarithmic derivative of the partition function with respect to the RG scale μ . This results in the representation of the Möbius regulated generating functional of the closed string scattering amplitudes in terms of the generalised central charge, exactly as we got in our approach where the Möbius infinities are implicitly regulated in the definition of the free energy. We consider this as a consistency check of our results, which are, thus, in agreement with the general arguments of Polyakov [8].

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