A remark on the $N \rightarrow \infty$ limit of W_N -algebrasADEL BILAL[★]

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ABSTRACT

The (non-linearly) extended conformal algebras associated with $su(N)$ (W_N -algebras) contain operators $\tilde{W}^k(z)$ of dimensions $k = 2, 3, \dots, N$. We show that in the $N \rightarrow \infty$ limit, the quantum (commutator) algebra of the $\tilde{W}^k(z)$ reduces to their classical (Poisson bracket) algebra. In particular, this proves the closure - in the usual non-linear sense - of the quantum W_∞ algebra.

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Since the discovery of Zamolodchikov and Fateev^[1] of the spin-3 algebra, a lot of work has been devoted to the study of general W_N algebras with generators of conformal spin up to $N^{[2-9]}$. Although the commutator of two W generators does not close in the Lie algebra sense, it is expressible as a (in general non-linear) polynomial of the W generators. Representation theory can be based on generalized Verma modules^[6-7,10] and series of unitary $c < N - 1$ theories have been obtained by coset constructions based on $(A_{N-1})_k \oplus (A_{N-1})_1 / (A_{N-1})_{k+1}$ ^[3]. The coset construction can be extended to any simply-laced Kac-Moody algebra $\hat{g}$. W algebras associated with an arbitrary simple Lie algebra g have been studied by means of the g -Toda field theories^[4-7].

There are several reasons to believe that the $N \rightarrow \infty$ limit of W algebras might be relevant to membrane theories^[11]. To be definite, we consider W algebras associated with $su(N)$, containing operators $W^k(z)$, $k = 2, 3, \dots, N$.

First, membrane theories contain a doubly infinite set of generators $V_{l,n}$ of the area-preserving diffeomorphism algebra^[12,13]. Clearly, in the $N \rightarrow \infty$ limit, the W -algebra also contains a doubly infinite set of generators W_n^k which are the Fourier modes (or Laurent series coefficients) of the $W^k(z)$. In particular, it was shown^[12,13] that a certain linear combination of the $V_{l,n}$ generates the Virasoro algebra, and hence might be identified with W_n^2 .

Second, it was shown^[14-18] that the area-preserving diffeomorphism algebra on the sphere is connected to the $su(\infty)$ Yang-Mills gauge algebra. On the other hand, it is known^[19,20] that self-dual $su(N)$ Yang-Mills theory is closely linked to the $su(N)$ Toda theory. This relation was made even more explicit, when it was shown that the self-dual spherical membrane is an integrable system isomorphic to the $su(\infty)$ Toda theory^[21].

The $su(N)$ Toda field theories give realizations of the W_N algebras^[4-7], and one expects again a correspondence between the $V_{l,n}$ of the area-preserving diffeomorphism algebra and the W_n^k generators.

Third, it was argued in a recent paper^[22] that the W_n^k directly provide a realization of the area-preserving diffeomorphism algebra of the plane, allowing however for modifications of the latter algebra such as extensions by the central term and by the terms which are non-linear in the W generators. In our opinion, it would help if one could clarify further the relation of these modified algebras to the initial area-preserving diffeomorphism algebra. In order to do this one should have some explicit formula, relating a given realisation of the W_n^k (in terms of harmonic oscillators e.g.) in the $N \rightarrow \infty$ limit to the $V_{l,n}$.

The aim of the present note is to make a modest step in this direction. We shall show that in the $N \rightarrow \infty$ limit the structure constants of the W_N algebra are greatly

simplified. In general, the $W^k(z)$ can be expressed as normal-ordered products of free fields $\partial\varphi, \partial^2\varphi, \dots, \partial^{(k)}\varphi$. When evaluating an operator product expansion of $W^k(z)$ and $W^{k'}(z')$ a given structure constant receives contributions from simple and multiple contractions of the $\partial^{(l)}\varphi$ fields. We will show that in the $N \rightarrow \infty$ limit, and for fixed imaginary background charge, i.e. positive, real Planck constant $\hbar$, the contributions from the simple contractions ($O(\hbar)$) are dominating and the multiple contractions ($O(\hbar^p)$, $p \geq 2$) are irrelevant.[†] This amounts to neglecting all loop diagrams and keeping only the tree diagrams. In other words, in the $N \rightarrow \infty$ limit, a Poisson bracket computation of the $\tilde{W}$ -algebra yields the same result as a complete quantum computation. This greatly simplifies the explicit determination of all structure constants. In particular, we know^[5] that the Poisson bracket algebra of all $\tilde{W}_n^k$ closes in their enveloping algebra,^{*} whereas concerning the quantum commutator algebra (for generic finite N), closure is still an unproven conjecture. Hence, our result proves the closure - in the usual non-linear sense - of the quantum W_∞ algebra.

The arguments we will give in the following not only apply to W_N algebras associated with $su(N) = A_{N-1}$, but also to those based on any other simple Lie algebra (see e.g. refs^[5,6]). For simplicity however, we will only consider $su(N)$ explicitly.

Before considering the case of a general W^k , let us look at some simple examples. W^2 is nothing but the energy-momentum tensor, and in the Feigin-Fuchs-type representation in terms of free fields it reads

$$T(z) = -\frac{1}{2} : \partial\varphi(z) \cdot \partial\varphi(z) : + 2i\alpha_0 \partial \cdot \partial^2\varphi(z) \quad (1)$$

where

$$i\partial\varphi_j(z) = \sum_{n=-\infty}^{\infty} a_n^j z^{-n-1}, \quad j = 1, \dots, N-1 \quad (2.a)$$

$$[a_n^i, a_m^j] = \delta^{ij} n \delta_{n,-m} \Rightarrow \langle \partial\varphi_i(z) \partial\varphi_j(w) \rangle = -\delta_{ij} \frac{1}{(z-w)^2} \quad (2.b)$$

are $N-1$ free bosonic fields. It is convenient to introduce N vectors h_a ($a = 1, \dots, N$) obeying $\sum_a h_a = 0$, $h_a \cdot h_b = \delta_{ab} - \frac{1}{N}$, $\sum_a x \cdot h_a \ y \cdot h_a = x \cdot y$. The $N-1$ simple

[†] To be precise, we will show this property for the algebra of the $\tilde{W}^k$. The true primary fields W^k are related to the $\tilde{W}^k$ by adding composite fields: $W^k = \tilde{W}^k + \sum_{r_i < k} \prod \partial^{s_i} \tilde{W}^{r_i}$, see eq. (8) below.

^{*} Closure for the $\tilde{W}_n^k$ immediately implies closure for the W_n^k .

roots of $su(N)$ then are $e_a = h_a - h_{a+1}$, and the fundamental weights are given by $\lambda_i = \sum_{a=1}^i h_a$. The Weyl vector ϱ equals

$$\varrho = \sum_{i=1}^{N-1} \lambda_i = \sum_{a=1}^N (N-a)h_a, \quad \varrho^2 = \frac{N(N-1)(N+1)}{12} \quad (3)$$

The central charge of the Virasoro algebra generated by the Laurent coefficients of the energy-momentum tensor (1) is

$$c = (N-1) - 48\alpha_0^2 \varrho^2 = (N-1) (1 - 4\alpha_0^2 N(N+1)) \quad (4)$$

The second term, proportional to α_0^2 , has its origin in a simple contraction of the $\partial^2 \varphi$ term with itself in the OPE of $T(z)T(z')$. The first term in (4) comes from the double contraction (loop diagram) of $(\partial\varphi)^2$ with itself. Hence $-1/\alpha_0^2$ is a loop counting parameter and plays the role of Planck's constant $\hbar$. We may refer to the second term in the central charge (4) as the classical contribution, already present when computing Poisson brackets, and to the first term as a quantum correction. One sees explicitly that the classical term is of order N^3 while the quantum correction is of order N only: for $N \rightarrow \infty$, only the classical contribution to the central charge, and hence to the structure constants of the Virasoro algebra, survives.*

The same result can explicitly be seen to be true for the commutators of $\tilde{W}_n^3$ with L_m and with $\tilde{W}_m^3$. These commutators have been computed for arbitrary N in ref.^[7].[†] Although the notation used there is slightly different from the present one, it is not difficult to see that the ratio of the contribution to a given structure constant coming from a double (triple) contraction and of the corresponding contribution coming from a simple (double) contraction is of order $\hbar N^{-2} \sim \frac{-1}{\alpha_0^2} N^{-2}$. Again, in the $N \rightarrow \infty$ limit only the simple contractions survive. Finally, looking at the commutator of L_n and of $\tilde{W}_n^3$ with certain basic primary fields ψ_a (the generalizations of the $\phi_{(2,1)}$ fields of BPZ^[23]) one finds again^[6,7] that for $N \rightarrow \infty$ the dominant terms are those of lowest order in $\hbar$ (simple contractions).

* In order that the central charge remains positive when $N \rightarrow \infty$ it is of course necessary that $-1/\alpha_0^2 \sim \hbar > 0$: This is the regime naturally arising in the Toda field theory.^[4-7]

† In ref.^[7] a slight error crept in eq. (3.53): in the contribution of order $\hbar^2$ to the central term one should replace the factor $6n^2 + 1$ by $5Nn^2 + n^2 + 1$.

Having motivated our claim by specific examples, let us now proceed to show it in the general case. The $W^k(z)$ are defined, starting from a differential operator, by

$$\begin{aligned} & : (2\alpha_0\partial - H_N(z))(2\alpha_0\partial - H_{N-1}(z)) \dots (2\alpha_0\partial - H_2(z))(2\alpha_0\partial - H_1(z)) : \\ & = \sum_{k=0}^N (-)^{k+1} \tilde{W}^k(z) (2\alpha_0\partial)^{N-k} \end{aligned} \quad (5)$$

where

$$H_a(z) = i h_a \cdot \partial \varphi(z), \quad \sum_{a=1}^N H_a = 0 \quad (6)$$

One has $\tilde{W}^0 = -1$ and $\tilde{W}^1 = 0$. One readily verifies that $\tilde{W}^2$ coincides with the energy-momentum tensor defined in eq.(1):^{*}

$$-\tilde{W}^2 = \sum_{a>b} H_a H_b - 2\alpha_0 \sum_{a=1}^N (N-a) \partial H_a = \frac{1}{2} (\partial \varphi)^2 - 2i\alpha_0 \varrho \cdot \partial^2 \varphi = -T \quad (7)$$

In general, the $\tilde{W}^k(z)$ so defined are not primary fields of the Virasoro algebra. However, one can define primary fields $W^k(z)$ by adding fields composite in the $\tilde{W}^r(z)$, $r < k$:

$$W^k(z) = \tilde{W}^k(z) + \sum d(s_i, r_i) \times \partial^{s_1} \tilde{W}^{r_1}(z) \dots \partial^{s_l} \tilde{W}^{r_l}(z) \times \quad (8)$$

with well-chosen coefficients $d(s_i, r_i)$. The sum is over $r_i, s_i < k$ such that $\sum s_i + \sum r_i = k$. One has, e.g.,

$$W^3(z) = \tilde{W}^3(z) - (N-2)\alpha_0 \partial \tilde{W}^2(z) \quad (9)$$

The double crosses $\times \dots \times$ in eq.(8) denote a normal ordering with respect to the Fourier modes of the $\tilde{W}^{r_i}$, *not* with respect to the modes of the oscillators a_n^i . In the following we will only consider the $\tilde{W}^k$'s, since once the structure constants for the $\tilde{W}$ algebra are known, it is straightforward to deduce those of the W algebra.[†]

* We do not write $:\dots:$ any longer, normal ordering with respect to the a_n^i oscillator modes is implicitly understood.

† This would not be true if the double cross normal ordering $\times \dots \times$ in eq.(8) was replaced by the normal ordering $:\dots:$ with respect to the oscillators.

From the defining equations (5) we find the explicit form of $\tilde{W}^k$:

$$\begin{aligned}
 -\tilde{W}^k &= \sum_{i_1 > \dots > i_k}^N H_{i_1} \dots H_{i_k} - 2\alpha_0 \sum_{i_1 > \dots > i_{k-1}}^N \sum_{l=1}^{k-1} (N - i_l - l + 1) \partial_{i_l} H_{i_1} \dots H_{i_{k-1}} \\
 &+ (2\alpha_0)^2 \sum_{i_1 > \dots > i_{k-2}}^N \sum_{l,m=1}^{k-2} \frac{1}{2} (N - i_l - l + 1)(N - i_m - m) \partial_{i_l} \partial_{i_m} H_{i_1} \dots H_{i_{k-2}} - \dots + \dots \\
 &= - \sum_{r=0}^{k-1} \tilde{W}^{k,r}
 \end{aligned} \tag{10a}$$

where the indices $i_1, \dots, i_k$ run from 1 to N , and

$$\tilde{W}^{k,r} = (-)^{r+1} (2\alpha_0)^r \sum_{i_1 > \dots > i_{k-r}}^N \sum_{l_1, \dots, l_r=1}^{k-r} \frac{1}{r!} \prod_{s=1}^r (N - i_{l_s} - l_s - s + 2) \partial_{i_{l_1}} \dots \partial_{i_{l_r}} H_{i_1} \dots H_{i_{k-r}} \tag{10b}$$

Here ∂_{i_l} means ∂^z acting on H_{i_l} . $\tilde{W}^{k,r}$ is the sum of all terms containing r derivative operators acting on $k - r$ fields H .

Consider now the OPE of $\tilde{W}^k(z)$ and $\tilde{W}^{k'}(z')$:

$$\tilde{W}^k(z) \tilde{W}^{k'}(z') = \sum_{r=0}^{k-1} \sum_{r'=0}^{k'-1} \tilde{W}^{k,r}(z) \tilde{W}^{k',r'}(z') \tag{11}$$

Using $\langle H_a(z) H_b(z') \rangle = (\delta_{ab} - \frac{1}{N}) / (z - z')^2$, the contributions to the OPE of $\tilde{W}^{k,r}(z)$ and $\tilde{W}^{k',r'}(z')$ from simple contractions are easily seen to be:

$$\begin{aligned}
 &(-)^{r+r'} (2\alpha_0)^{r+r'} \sum_{i_1 > \dots > i_{k-r}}^N \sum_{j_1 > \dots > j_{k'-r'}}^N \sum_{l_1, \dots, l_r=1}^{k-r} \sum_{l'_1, \dots, l'_r=1}^{k'-r'} \frac{1}{r!} \frac{1}{r'!} \\
 &\times \prod_{s=1}^r (N - i_{l_s} - l_s - s + 2) \partial_{i_{l_s}}^z \prod_{s'=1}^{r'} (N - j_{l'_{s'}} - l'_{s'} - s' + 2) \partial_{j_{l'_{s'}}}^{z'} \\
 &\times \sum_{m=1}^{k-r} \sum_{m'=1}^{k'-r'} : \prod_{\substack{p=1 \\ p \neq m}}^{k-r} H_{i_p}(z) \prod_{\substack{q=1 \\ q \neq m'}}^{k'-r'} H_{j_q}(z') : \frac{(\delta_{i_m j_{m'}} - \frac{1}{N})}{(z - z')^2}
 \end{aligned} \tag{12}$$

Again, $\partial_{i_{l_s}}^z$ acts on $H_{i_{l_s}}(z)$ if this field has not been contracted. It acts on $(\delta_{i_m j_{m'}} - \frac{1}{N}) / (z - z')^2$ if the H -field has been contracted ($l_s = m$), and similarly for $\partial_{j_{l'_{s'}}}^{z'}$.

The term (12) contributes to $\tilde{W}^{k+k'-1, r+r'+1}(z')/(z-z')^1$, $\tilde{W}^{k+k'-2, r+r'}(z')/(z-z')^2$, ... $\tilde{W}^{k+k'-l, r+r'-l+2}(z')/(z-z')^l$, ... $\tilde{W}^{k+k'-r-r'-2, 0}(z')/(z-z')^{r+r'+2}$. Other contributions of the same form (same number of H and same number of derivatives) to these operators arise from simple contractions in the OPE of $\tilde{W}^{k, r+q}(z)$ and $\tilde{W}^{k', r'-q}(z')$.

On the other hand, one also obtains the same kind of terms in the OPE of $\tilde{W}^{k, r+q-(n-1)}(z)$ and $\tilde{W}^{k', r'-q-(n-1)}(z')$ by taking n -fold contractions of the H -fields. For $q = 0$ e.g. the result reads ($R \equiv r - n + 1$ and $R' \equiv r' - n + 1$)

$$\begin{aligned}
& (-)^{r+r'} (2\alpha_0)^{R+R'} \sum_{i_1 > \dots > i_{k-R}}^N \sum_{j_1 > \dots > j_{k'-R'}}^N \sum_{l_1, \dots, l_R=1}^{k-R} \sum_{l'_1, \dots, l_{R'}=1}^{k'-R'} \frac{1}{R!} \frac{1}{R'!} \\
& \times \prod_{s=1}^R (N - i_{l_s} - l_s - s + 2) \partial_{i_{l_s}}^z \prod_{s'=1}^{R'} (N - j_{l'_{s'}} - l'_{s'} - s' + 2) \partial_{j_{l'_{s'}}}^{z'} \\
& \times \sum_{m_1, \dots, m_n=1}^{k-R} \sum_{m'_1, \dots, m'_n=1}^{k'-R'} : \prod_{\substack{p=1 \\ p \neq m_1, \dots, m_n}}^{k-R} H_{i_p}(z) \prod_{\substack{q=1 \\ q \neq m'_1, \dots, m'_n}}^{k'-R'} H_{j_q}(z') : \prod_{t=1}^n \frac{\left(\delta_{i_{m_t} j_{m'_t}} - \frac{1}{N} \right)}{(z-z')^2} \quad (13)
\end{aligned}$$

Only the first two sums range from 1 to N . There are $k - R + k' - R' = k - r + n - 1 + k' - r' + n' - 1$ indices summed from 1 to N . However, they are constrained by n Kronecker deltas $\delta_{i_{m_t} j_{m'_t}}$ leaving $k + k' - r - r' + n - 2$ summations. Since there are $k + k' - r - r' - 2$ fields H , there are n sums over "free" indices, resulting in a factor N^n . There are also $R = r - n + 1$ factors $(N - i_{l_s} - l_s - s + 2)$ and $R' = r' - n + 1$ factors $(N - j_{l'_{s'}} - l'_{s'} - s' + 2)$. Altogether this is of order $N^{n+r-n+1+r'-n+1} = N^{r+r'+2-n}$. The same result is obtained for $q \neq 0$.

We conclude that the contributions to a given $\tilde{W}^{k+k'-l, r+r'-l+2}(z')/(z-z')^l$ arising from the n -fold contractions in the OPE of $\tilde{W}^{k, r+q-(n-1)}(z)$ and $\tilde{W}^{k', r'-q-(n-1)}(z')$ are proportional to $(2\alpha_0)^{r+r'-2(n-1)}$ and are of order $N^{r+r'+2-n}$. Hence the ratio of the contributions of n -fold contractions to those of single contractions is

$$\frac{n\text{-fold contractions}}{\text{single contractions}} \sim (2\alpha_0)^{-2(n-1)} N^{1-n} \quad (14)$$

In fact, all explicit computations indicate, that due to the identity $\sum_{a=1}^N H_a = 0$, there are indeed only $\star \left[\frac{k-r}{2} \right]$ independent summations over indices in $\tilde{W}^{k, r}$, and that

$\star [x]$ denotes the integer part of x .

(13) contains only $\left[\frac{n}{2}\right]$ summations over "free" indices. This tends to show, that one could even establish a more severe result, replacing N^{1-n} in eq. (14) by $N^{1-n-[n/2]}$. In any case, equation (14) is enough to establish the property claimed above:

$(2\alpha_0)^{-2}$ is a loop-counting constant ($\sim \hbar$), and in the $N \rightarrow \infty$ limit only the contributions of the simple contractions (no loop) to a given structure constant of the $\tilde{W}$ -algebra survive. This allows to obtain all $\tilde{W}$ -algebra structure constants by Poisson bracket computations. In particular, the classical result of ref.^[5] shows that in the $N \rightarrow \infty$ limit the quantum $\tilde{W}$ -algebra, and hence also the quantum W -algebra, closes - in the usual non-linear sense.

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