INTERMITTENCY, GENERATING FUNCTIONS AND FRACTAL BEHAVIOUR I.

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ABSTRACT

We discuss an interrelation between an underlying probability distribution of particles in a central rapidity region, an intermittency and a fractal behaviour. A new way of looking for a fractal like pattern is proposed.

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1. Introduction

An understanding of large multiplicity events in particle production at high energies needs more detailed description than the studying of average phenomena. Macroscopic properties of a system are derived from a probability distribution which is related to interactions between individual particles. This is an entropy which links the micro- and macro-descriptions of the system. Equilibrium states correspond to the states with the largest entropy and a phase transition is related to the existence of more than one equilibrium state.

From different kinds of experiments we can obtain probability distributions which provide more or less detailed information about the system. Koba-Nielson-Olesen scaling, multiplicity distributions and multiplicity correlations are examples of such general properties. A more and more powerful tool in the last years has become searching for a distribution of particles in small (pseudo)rapidity intervals. Since A. Bialas and R. Peschanski[1] noticed a scaling behaviour of the factorial momenta still new methods of data interpretation were proposed[2, 3, 4]. Moments when measured in small rapidity windows are as a magnifying glass which allows for an observation of sometimes unexpected phenomena.

Let us assume that we know some probability distribution $U(x_1, \dots, x_n)$. The generating function $\mathcal{C}(t_1, \dots, t_n)$ is defined as

$$\mathcal{C}(\mathbf{t}) = \int dx_1 \cdots dx_n U(x_1, \dots, x_n) \exp \sum_{j=1}^n t_j x_j \quad (1.1)$$

From the behaviour on the function $D(\mathbf{t}) = \log \mathcal{C}(\mathbf{t})$ one can obtain information about collective behaviour of the system. Nonuniqueness of the minimum of the function $D(t)$ points out a possible phase transition in the system[5]. If one treats the probability distribution U as a distribution of macrostates of a gas then the function D corresponds to a free energy of the system and its Legendre transformation to its entropy. It is possible that the probability distribution which is obtained from a given experiment does not allow one to observe the phase transition which in fact had taken place. Only in the case of some kind of an ergodic property of the system there is an equivalence of descriptions on different levels.

2. Moments and generating functions

Let us consider the central rapidity region of the size Δ where exactly n charged particles are produced. The last assumption has a technical character and a generalization to any number of produced particles is straightforward. According to the present taking data conditions there is no selection between different kinds of particles. We introduce the probability measure containing the maximal information about the distribution of particles and consistent with experimental conditions. This measure will be used as a basis for a comparison between different moments and related probability distributions.

Our probability space is a n -fold direct product of Δ and the n particle state is represented by the n -dimensional vector $\vec{y} = (y_1, \dots, y_n)$. The probability measure $\mathcal{P}(y_1, \dots, y_n)$ is a probability density to have n produced particles with rapidities $y_1, \dots, y_n$. The function $\mathcal{P}$ is a symmetric in its arguments. This is related to the fact that all charged particles are treated as the same. Let us take a rapidity subinterval $\delta \in \Delta$ and let $\mathcal{X}_\delta$ be a characteristic function of the interval δ i.e.,

$$\mathcal{X}_\delta(x) = \begin{cases} 1 & \text{if } x \in \delta; \\ 0 & \text{if } x \notin \delta; \end{cases}$$

Let us define a function

$$\mathcal{A}_\delta(\vec{y}) = \sum_{i=1}^n \mathcal{X}_\delta(y_i)$$

$\mathcal{A}_\delta(\vec{y})$ is the number of particles having rapidities in the interval δ . This is a random variable with a corresponding probability measure

$$p_\delta(k) = \mathcal{P}(\mathcal{A}_\delta^{-1}(\vec{y}) |_{\mathcal{A}=k})$$

$p_\delta(k)$ is a probability to find k particles in the rapidity interval δ . It can be expressed by the measure $\mathcal{P}$

$$p_\delta(k) = \sum_{i=0}^{n-k} (-1)^i \binom{n}{k} \binom{n-k}{i} \int dy_1 \cdots dy_n \mathcal{P}(y_1, \dots, y_n) \mathcal{X}_\delta(y_1) \cdots \mathcal{X}_\delta(y_{k+i}) \quad (2.1)$$

Let us divide the interval Δ into m subintervals each of the length $\delta = \Delta/m$ and let $\mathcal{X}_j$ be a characteristic function of the corresponding subinterval $j = 1, \dots, m$. The probabilities

to find exactly k particles in the subinterval j will be denoted by p_{jk} and are expressed by the formulae of the type (2.1). The normalized l^{th} factorial moments is defined [1] as

$$\langle F_l \rangle = \frac{1}{m} \left(\frac{m}{n} \right)^l \sum_{j=1}^m \sum_{k=l}^n k(k-1) \cdots (k-l+1) p_{jk} \quad (2.2)$$

A generating function of the factorial moments

$$\mathcal{K}(z) = \sum_{l=0}^{\infty} \langle F_l \rangle z^l / l!$$

can be expressed directly by means of the probability measure $\mathcal{P}(y_1, \dots, y_n)$

$$\mathcal{K}(z) = \frac{1}{m} \sum_{j=1}^m \int dy_1 \cdots dy_n \mathcal{P}(y_1, \dots, y_n) \prod_{i=1}^n \left[1 + \frac{m}{n} \mathcal{X}_j(y_i) z \right] \quad (2.3)$$

We see that only the part of the information about probability distribution which is contained in the function $\mathcal{P}$ is used at the construction of factorial moments. However, the only probabilities which are used to construct factorial moments (2.2) are probabilities to find k particles in a randomly chosen subinterval. These probabilities are

$$P_k = \frac{1}{m} \sum_{j=1}^m p_{jk} \quad (2.4)$$

Moments of the type (2.2) carry rather restricted information about the underlying probability distribution. One can treat (2.2) as a set of linear equations for the unknown P_k . We can write

$$\langle F_l \rangle = \left(\frac{m}{n} \right)^l \sum_{k=l}^n k(k-1) \cdots (k-l+1) P_k \quad (2.5)$$

The solution is of the form

$$P_k = \sum_{l=k}^n (-1)^{k+l} \frac{1}{k!(l-k)!} \left(\frac{n}{m} \right)^l \langle F_l \rangle \quad (2.6)$$

We are not able in practice to recover the probabilities P_k from this formula. The order of the measured moments does not exceed sixth, so there is a substantial ambiguity of the

corresponding probabilities. It was observed [1, 6] that the factorial moments (2.2) rise with decreasing bin size δ and follow a power law

$$\langle F_l \rangle \sim m^{\alpha_l}, \quad m \rightarrow \infty \quad (2.7)$$

This behaviour is called an intermittency [7]. The size of the bin is restricted by the detector's resolution and usually does not decrease below 0.1 in rapidity units. The strength of the intermittency is characterized by the parameter α_l .

The moments (2.5) are related to the one dimensional probability measure

$$U(r) = \sum_{k=0}^n \delta(r - km/n) P_k \quad (2.8)$$

This measure can be expressed in terms of the underlying probability $\mathcal{P}$

$$U(r) = \frac{1}{m} \sum_{j=1}^m \int dy_1 \cdots dy_n \mathcal{P}(y_1, \dots, y_n) \sum_{k=0}^n \delta(r - km/n) \times \\ \times \sum_{l=k}^n (-1)^{l-k} \binom{l}{k} \sum_{i_1 < \dots < i_l} \mathcal{X}_j(y_{i_1}) \cdots \mathcal{X}_j(y_{i_l}) \quad (2.9)$$

As was shown in the previous analysis this measure is rather ambiguous, at least under present experimental conditions. In spite of this ambiguity the problem of finding of a dynamical model which would reproduce an intermittent behaviour waits still for a solution [6, 8, 9].

A generating function is connected with the measure (2.8). We denote it as $\mathcal{G}$ and according to the definition (1.1) it is

$$\mathcal{G}(t) = \sum_{k=0}^n P_k \exp(tkm/n)$$

or expressed by the basic measure $\mathcal{P}$

$$\mathcal{G}(t) = \frac{1}{m} \sum_{j=1}^m \int dy_1 \cdots dy_n \mathcal{P}(y_1, \dots, y_n) \prod_{i=1}^n \left[1 + \mathcal{X}_j(y_i)(e^{tm/n} - 1) \right] \quad (2.10)$$

It is easy to see that this function is simultaneously a generating functional of the usual nonfactorial moments. These moments are normalized in the same manner as factorial ones.

3. Fractals...

The factorial moment or equivalently the set of probabilities $\{P_k\}$ can be considered as an approximation of the underlying probability density. To obtain a better insight in the structure of the underlying dynamics let us look for a possible fractal-like pattern. Fractals are introduced as objects with the non-integer dimension. The fractal dimension d_F is defined as a real number which provides a finite limit

$$0 < \lim_{l \rightarrow 0} N(l)l^{d_F} < \infty \quad (3.1)$$

where $N(l)$ is the minimal number of covering the object hypercubes of a size l .

In the simplest case we can identify l with the size δ of the rapidity window and $N(l)$ with an average number of *nonempty* windows $\langle \hat{N}(\delta) \rangle$. This number is related to the probability P_0 to have no particles in the randomly chosen subinterval

$$\langle \hat{N}(\delta) \rangle = \frac{\Delta}{\delta} (1 - P_0) \quad (3.2)$$

A fractal beviour is obtained if the limit

$$\lim_{\delta \rightarrow 0} \frac{\log(\Delta(1 - P_0)/\delta)}{\log \delta} \quad (3.3)$$

exists. According to the definition (3.1) this limit is equal to the fractal dimension d_F . With decreasing δ the probability P_0 tends to zero. We see than that the fractal related to the proposed measurement would have a dimension always smaller than one. In the experiment one would observe scaling (3.3) only up to some saturation point. The existence of the limit (3.3) is not related to the intermittency behaviour (2.7). However, it can provide a subsidiary information about production processes. A generalization of the situation described by the fractal-like pattern is a picture obtained by a superposition of fractals with different dimensions [10]. Then a more detailed information is needed. This can be realized by the set of probabilities $\{p_{jk}\}$. Let us introduce a corresponding set of new moments. Let $\hat{p}_j$ be a probability to find *at least* one particle in the interval j . This is

$$\hat{p}_j = \frac{1}{n} \sum_{k=1}^n k p_{jk} \quad (3.4)$$

Let us define

$$\langle \mathcal{F}_q \rangle = \sum_{j=1}^m \hat{p}_j^q \quad (3.5)$$

These moments can be related to a fractal-like behaviour in the multiproduction. Scaling

of the moments (3.5) is interpreted in terms of the multifractal patterns[2, 3, 4, 11] . However for more complicated scaling a caution is needed. As a Fourier series does not mean periodical properties of the function, an expansion into fractals does not have to prove fractal-like properties.

According to the definition (3.4) the moment (3.5) can be written as

$$\langle \mathcal{F}_q \rangle = n^{-q} \sum_{j=1}^m \langle n \rangle_j^q \quad (3.6)$$

This gives a direct relation to the directly measured quantities. It proves also that a very rich statistics is needed to obtain data about the moments (3.6). There is no direct relation of the intermittency (2.7) with a scaling of the moments (3.4). It can be shown that the intermittency behaviour does not lead to the scaling in moments (3.4). The more detailed analysis will be a subject of the next publication.

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REFERENCES

1. A. Bialas, R. Peschanski, *Nucl. Phys.* **B273** (1986) 703
2. R.C. Hwa "Fractal Measures in Multiparticle Production" Preprint OITS 407 (1989)
3. P. Lipa, B. Buschbeck "From Strong to Weak Intermittency" Preprint HEPHY-PUB 518/89 (1989)
4. I.M. Dremin "Multifractality of Inelastic Events" Preprint FERMILAB-PUB-89/71-T (1989)
5. O.E. Lanford "Entropy and equilibrium states in classical statistical mechanics" In: *Statistical Mechanics and Mathematical Problems* Ed. by A. Lenard, *Lecture Notes in Physics* **20** Springer, Berlin (1973)
6. A. Bialas, R. Peschanski, *Nucl. Phys.* **B308** (1988) 857
7. Ya.B. Zel'dovich, S.A. Molchanov, A.A. Ruzmaikin, D.D. Sokolov, *Usp. Fiz. Nauk* **152** (1987) 3

8. L. Van Hove "Cold Quark Gluon Plasma and Multiparticle Production" Preprint CERN-TH.5236/88 (1988)
9. H. Satz "Intermittency and Critical Behaviour" Preprint CERN-TH 5312/89 (1989)
10. I.M. Dremin, *Mod.Phys.Lett.* **A3** (1988) 1333
11. P. Carruthers "Multihadron Rapidity Distributions, Fractals and Entropy" Preprint AZPH-TH/89-4 (1989)