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MODULAR INVARIANTS FROM SIMPLE CURRENTS: AN EXPLICIT PROOF

A. N. Schellekens

and

S. Yankielowicz^{* †}

CERN, 1211 Geneva 23, Switzerland

ABSTRACT

In a previous paper an orbifold construction was used to demonstrate that the existence of primary fields with simple fusion rules in a conformal field theory implies the existence of non-diagonal modular invariant partition functions. Here we present a direct and explicit proof of modular invariance, which also covers a few cases that could not be obtained with the orbifold method. We also give a very simple general formula for the modular matrix M .

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† Permanent address: School of Physics and Astronomy, Raymond and Beverley Sacler Faculty of Exact Sciences, Tel-Aviv University, Israel.

In [1] a very general method for constructing non-diagonal modular invariant partition functions for many conformal field theories was presented. The method makes use of the presence of certain primary fields with the special property that their fusion rule with any other primary fields contains just one term. Such “simple currents” are present in most conformal field theories we know. Indeed, apart from a handful of exceptions, all the constructions of modular invariant partition functions that one can find in the present literature can be understood in terms of this method. Furthermore a more systematic exploration of a large class of modular invariants of a given conformal field theory is now possible, and in many cases previously unknown invariants can easily be found.

The main advantage of our procedure is that one only needs to know the fusion rules of a few of the primary fields, and that to prove modular invariance it is no longer necessary to write down explicit expressions for the modular transformation S . In [1] we derived the partition functions using an orbifold-inspired procedure. Although widely accepted, this does have several drawbacks. If this method is to be used as the foundation for constructing many new modular invariant theories without ever checking modular invariance explicitly, it would be more comforting to have an explicit proof of modular invariance instead of the intuitive, but not mathematically rigorous orbifold procedure. More importantly, there is a small set of partition functions which we know to be modular invariant, but for which we do not know an orbifold procedure (namely for N even and maximal and fractional conformal spin, as discussed in [1]). The purpose of this paper is to solve these problems.

In addition we will give an even simpler description of the modular matrix M than the one of [1], which we know to be very suitable for example for explicit calculations of spectra of four-dimensional string models.

Our goal is to prove the following

Theorem. Consider a rational conformal field theory, and let J be a simple current with order N and monodromy parameter r . Then J can be used to build a non-diagonal modular invariant partition function unless N is even and r is odd.

We remind the reader that N is the smallest positive integer so that $J^N = 1$ (in the sense of fusion rules), and r is defined by the formula

$$h(J) = \frac{r(N-1)}{2N} \bmod 1 .$$

Note that this defines r modulo N if N is odd, and modulo $2N$ if N is even. The exact

expression of the modular invariant partition function in terms of J will be given below.

Proof.

Consider the following formula, due to Verlinde [2]:

$$N_{jlk} = \sum_n \frac{S_{jn} S_{ln} S_{nk}^*}{S_{0n}}. \quad (1)$$

Let the index j represent the simple current J , and l some other primary field. Simple fusion implies that there is just one value of k for which N_{jkl} is non-vanishing. Consider that value of k , and regard the combination $V_n = S_{jn} S_{ln} / S_{0n}$ as a vector with vector index n . Then V_n must be orthogonal to all $S_{nk'}^*$, for all $k' \neq k$, and it must have inner product one with S_{nk} . Since the vectors $S_{nk'}$ form a unitary matrix, this implies that $V_n = S_{nk}$. Thus we find

$$S_{nk} = \left(\frac{S_{jn}}{S_{0n}} \right) S_{ln}. \quad (2)$$

This relation may be used recursively to relate the matrix elements of S within each orbit to each other. Let us first introduce some notation. For each orbit we choose a representative, which may be any field on the orbit. The representatives will be labelled by roman letters $a, b, c, \dots$. In the following we will often denote the primary fields Φ_a simply by their labels. The other fields on an orbit can be obtained from the representative by acting with the current. We will denote these fields as $j^\alpha a$, where α is an integer defined modulo N . The label $j^\alpha a$ belongs to the field obtained from Φ_a by acting α times with the current. Note that it may happen that orbits are shorter than N ; in general their length may be a divisor d of N . In that case one should allow only d values of α so as not to overcount the primary fields.

Using (2) recursively we get

$$S_{j^\alpha a, n} = \left(\frac{S_{jn}}{S_{0n}} \right)^\alpha S_{an} \quad (3)$$

(we separate the subscripts of S by a comma whenever necessary to clarify the notation). Applying a similar procedure to the subscript n and using the fact that S is symmetric we get

$$S_{j^\alpha a, j^\beta b} = \left(\frac{S_{jj}}{S_{0j}} \right)^{pq} \left(\frac{S_{00}}{S_{0j}} \right)^{pq} \left(\frac{S_{bj}}{S_{b0}} \right)^p \left(\frac{S_{aj}}{S_{a0}} \right)^q S_{ab}. \quad (4)$$

To compute the four ratios of elements of S we make use of a formula derived in [3]. In this

paper a general formula is given for the ratio of S -elements S_{ij}/S_{0j} , which depends on the fusion rule $i \times j$. If i and j have simple fusion, $i \times j = k$, this formula reduces to

$$\lambda_i^{(j)} = \frac{S_{ij}}{S_{0j}} = e^{2\pi i(h_i+h_j-h_k)} , \quad (5)$$

where h_i is the conformal weight of the field Φ_i . Substituting this expression into (4) we get ^{*}

$$S_{j^{\alpha_a}, j^{\beta_b}} = e^{2\pi i \beta Q(a)} e^{2\pi i \alpha Q(b)} e^{2\pi i \frac{\alpha \beta r}{N}} S_{ab} , \quad (6)$$

where we have used the definition of the charge Q of the field Φ_a with respect to the current j ($Q(a) = h_a + h_j - h_{j \times a}$) and the expression derived in [1] for the conformal weights of the currents,

$$h_{j^n} = \frac{rn(N-n)}{2N} (\text{mod } 1),$$

where r is the monodromy parameter of the algebra.

Note that for $r = 0$ (*i.e.* for currents of integer spin) (6) has a very simple form which makes the existence of a new modular partition function almost obvious. Indeed, it is clear that if one takes all orbits with $Q = 0$ and defines new characters by simply summing the characters on each orbit with a proper normalization

$$\mathcal{X}_{j\text{-orbit of } a} = \sqrt{\frac{N}{d}} \sum_{\alpha=1}^d \mathcal{X}_{j^{\alpha_a}} , \quad (7)$$

then the new characters form an invariant subspace of S , on which a new diagonal invariant can be defined. This argument appears also in [4], where only the integer spin modular invariants are discussed. In the following we will present a proof that deals with all possible values of r for which modular invariants were constructed in [1].

In (6) S_{ab} represents the matrix elements between different orbits, whereas the remaining factors relate matrix elements within the orbits to each other. The modular invariants discussed in [1] are off-diagonal, but they never have non-vanishing matrix elements between different orbits. Thus they are diagonal in S_{ab} . Hence one may expect to be able to check modular invariance without explicitly knowing S_{ab} , just as one does not have to know S

^{*} While completing this work we received [4], in which a formula for S is obtained which agrees with ours.

explicitly to prove that the diagonal invariant is indeed modular invariant. All we expect to need is a relation among the matrix elements S_{ab} which follows from orthogonality of the original matrix S_{AB} (here we use A and B to denote the original fields, not yet organized into orbits). The orthogonality relation, $S_{AB}S_{BC}^* = \delta_{AB}$ reads in terms of the orbit basis introduced above

$$\sum_{(\beta, b)} e^{2\pi i \beta Q(a)} e^{2\pi i \alpha Q(b)} e^{2\pi i \frac{\alpha \beta r}{N}} e^{-2\pi i \gamma Q(b)} e^{-2\pi i \beta Q(c)} e^{-2\pi i \frac{\beta \gamma r}{N}} S_{ab} S_{bc}^* = \delta_{ac} \delta_{\alpha \gamma} . \quad (8)$$

Here the sum on b is over all orbits, and the sum on β is over all values modulo the length of each orbit, *i.e.* N or a divisor of N .

There exists a very simple expression for the matrix M that one obtains for a certain simple current. We will first consider odd values of N . If N is odd, the following expression for the modular matrix holds for integral as well as fractional spin currents

$$M_{AB} = \sum_{l=1}^N \delta(B, j^l A) \delta^1(Q(A) + Q(B)) \quad (9)$$

Here $\delta^M(x)$ is defined to be equal to 1 if $x = 0 \bmod M$ and equal to 0 otherwise. This formula has a very simple interpretation. To find all fields with which a given field A has non-vanishing matrix elements, one simply looks for all fields on the j -orbit of A whose charge completes $Q(A)$ to an integer. A formula for the charge on an orbit was given in [1]

$$Q(j^\alpha a) = Q(a) + \frac{r\alpha}{N} , \quad (10)$$

where $Q(a) = \frac{t}{N}$ for some integer t . For $r = 0 \bmod N$ (9) tell us to keep only orbits with $Q(a)$ integer, and to pair all fields in such an orbit with every other one in the same orbit, including itself. This is an integer spin modular invariant. Note that the label l in (9) must be summed over all N , even on shorter orbits. This implies that shorter orbits in integer spin invariants appear automatically with a multiplicity N/d , where d is the length of the orbit, a phenomenon that is familiar from the D_{2n} modular invariants of $SU(2)$. As we will see in a moment, these enhanced multiplicities are essential for modular invariance. For $r \neq 0 \bmod N$ and N prime (9) pairs each field in each orbit with precisely one field in the same orbit. This is a fractional spin modular invariant as described in [1]. One may

check that the more complicated cases where N has several prime factors are also correctly reproduced by (9), but of course it is far more important to check that (9) is indeed modular invariant.

Consider first the argument of the second δ -function in (9). Substituting $A \rightarrow J^\alpha a$ and $B = j^l A$ we get

$$\begin{aligned} Q(A) + Q(B) &= Q(j^\alpha a) + Q(j^{\alpha+l} a) \\ &= 2Q(a) + \frac{2\alpha + l}{N} r . \end{aligned}$$

Observe that if r is even, this expression is equal to $\frac{2t}{N}$ for some integer t . If N is odd, requiring that $\frac{2t}{N} \in \mathbf{Z}$ is equivalent to requiring that $\frac{t}{N} \in \mathbf{Z}$, so that we may replace in (9) $\delta^1(Q(A) + Q(B))$ by $\delta^1(Q(a) + \frac{2\alpha+l}{2N} r)$. Now if N is odd, r is defined modulo N , and we can always require r to be even without loss of generality. Hence we see that (9) is equivalent to

$$M_{j^\alpha a, j^\beta b} = \sum_{l=1}^N \delta_{ab} \delta_{\beta, \alpha+l} \delta^1\left[Q(a) + \frac{2\alpha + l}{2N} r\right] , \quad (11)$$

for N odd and r even. The purpose of this manipulation will become clear in a moment.

To prove modular invariance, we write the δ function as follows

$$\delta^1\left(Q(a) + \frac{2\alpha + l}{2N} r\right) = \frac{1}{N} \sum_{p=1}^N \delta_{ab} e^{2\pi i p \left[Q(a) + \frac{2\alpha+l}{2N} r\right]} . \quad (12)$$

Note that this representation of the δ -function is valid only if r is even; a similar expression for odd r would require us to double the range of p . Substituting (11) into $S_{AB} M_{BC} S_{CD}^*$ with $A \rightarrow j^\alpha a, \dots, D \rightarrow j^\delta d$, and using (12) we get

$$\begin{aligned} S_{AB} M_{BC} S_{CD}^* &= \sum_{(\beta, b)} e^{2\pi i \beta Q(a)} e^{2\pi i \alpha Q(b)} e^{2\pi i \frac{\alpha \beta r}{N}} S_{ab} S_{bd}^* \\ &\times \sum_{l=1}^N \sum_{p=1}^N e^{-2\pi i (\beta+l) Q(d)} e^{-2\pi i (\delta-p) Q(b)} e^{-2\pi i \frac{(\delta-p)(\beta+l)r}{N}} e^{-2\pi i \frac{p r l}{2N}} . \end{aligned}$$

We can now use (8) to write this as follows

$$S_{AB} M_{BC} S_{CD}^* = \sum_{l=1}^N \sum_{p=1}^N \delta_{ad} \delta_{\alpha, \delta-p}^N e^{-2\pi i l \left[Q(d) + \frac{(2\delta-p)r}{2N}\right]} = M_{AD} ,$$

which proves modular invariance under S .

Note that the rôle of p and l is interchanged in the calculation. This has two important implications. First of all it tells us that even on orbits of length $d < N$ we should sum over all $l, 1 \leq l \leq N$, in order for the ranges of l and p to be the same. Secondly it shows that the proof does not go through for odd values of r , since in the latter case one has to double the range of p . This remark is irrelevant for odd N , but will turn out to be very important for even N .

The discussion of invariance under S for even N is now in fact already completed, since nothing in the derivation starting from (11) depends on whether N is even or odd. The only part in the foregoing discussion that is dependent on that is the derivation of (11) from (9), which is certainly not valid for even N . However, for even N we can simply take (11) as the definition of the matrix M_{AB} , and forget about (9).

An important difference is that whereas odd and even values of r are equivalent for odd N , they are not equivalent for even N . Hence we do not find a modular invariant for N even and r even. This agrees with [1], where the same conclusion was reached on the basis of a level matching condition for the orbifold procedure used in that paper.

A remaining question is whether (11) does indeed reproduce the results of [1] for even N . Since the most important issue, that of modular invariance, has already been settled, we will not present a detailed analysis here, but we will illustrate the relation by considering $N = 2$. The description of the modular invariant partition function obtained from (11) can be done in the same way as for (9), except that instead of Q one should use the charge

$$\hat{Q}(j^\alpha a) = \frac{1}{2}Q(a) + \frac{\alpha r}{2N} .$$

This expression is to be evaluated modulo integers. Note that since $Q(a)$ itself is defined modulo integers, the definition of $\hat{Q}$ is ambiguous by half-integers. However, this does not affect the result as long as one shifts $\hat{Q}$ on the *entire* orbit by a half-integer. Thus the charge $\hat{Q}$ can be viewed as a kind of double covering of the charge Q .

Consider now first $r = 4$. Then the current has integer spin, and there are two types of orbits, with charges Q equal to $(0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$ (always modulo 1) respectively. The charges $\hat{Q}$ on these orbits are $(0, 0)$ and $(\frac{1}{4}, \frac{1}{4})$, so that the latter kind of orbit is projected out, while the first appears as a square in the partition function. If $r = 2$, the current has half-integer spin, and Q takes the values $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ in the Neveu-Schwarz and Ramond sector

respectively. For $\hat{Q}$ we find now $(0, \frac{1}{2})$ and $(\frac{1}{4}, \frac{3}{4})$, so that in M_{ab} the NS-fields are paired with themselves, and the R-fields with each other. Finally if $r = 1$ the Q-charges are $(\frac{1}{4}, \frac{3}{4})$ on any orbit, and $\hat{Q}$ is $(\frac{1}{8}, \frac{3}{8})$. In this case there is no pairing that satisfies $\hat{Q}(a) + \hat{Q}(b) \in \mathbf{Z}$, and indeed in [1] no modular invariant was expected if r is odd and N even.

To complete the discussion we should discuss invariance under the modular transformation T . In [1] a formula is given for the conformal weights on an orbit. It is straightforward to show that if the charges Q or $\hat{Q}$ of two fields on the same orbit add up to an integer, then the conformal weights of those fields differ by an integer. This completes the proof.

Thus we find that all modular invariant partition functions discussed in [1] can be summarized by one simple formula, (11), which is valid for integral as well as fractional spin currents and for even as well as odd N , provided that the monodromy parameter r is even (which can always be achieved if N is odd). For odd N a slightly more convenient formula, (9), may also be used.

An interesting remaining problem is the construction of the new S -matrix for the integral spin invariants. This matrix is not simply the restriction of the original S -matrix to the new characters (7), since in general some characters appear in the partition function with multiplicities larger than 1. This always implies the existence of several distinct fields, distinguished by the extended algebra [5]. The matrix elements of the new S -matrix between such distinct fields do not directly follow from the original S matrix, but can be determined by requiring that the relations $S^2 = C$, $(ST)^3 = C$ are satisfied and the new fusion rules have integer coefficients. It would be interesting to derive an explicit formula for this matrix.

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