CERN-TH.5407/89

Charged Boson Stars

Ph. Jetzer

J. J. van der Bij

CERN, CH-1211 Geneva, Switzerland

ABSTRACT

We study the static spherically symmetric gravitational equilibria of scalars coupled to a $U(1)$ gauge field and with a possible $\frac{\lambda}{2}(\phi^*\phi)^2$ self interaction. We solve numerically the coupled Einstein-Maxwell-Klein-Gordon equations for non-singular and asymptotically flat solutions. Static solutions only exist for $e^2/4\pi < G_N m^2 = e_{crit}^2$. The maximum mass of the boson star increases with the gauge coupling constant as $(e_{crit} - e)^{-1/2}$ for a charge close to the critical charge. We study the behaviour of the solutions as a function of the node number of the scalar field and as a function of the self coupling λ .

CERN-TH.5407/89

June 1989

Recent developments in particle physics and cosmology suggest that evolving scalar fields may have played an important role in the evolution of the early universe, for instance in primordial phase transitions, and that they may make up the missing dark matter. Models for galaxy formation using cold dark matter and the inflationary scenario suggest that the ratio of baryonic (luminous) matter to dark matter can be of the order of 10 %. These facts naturally raise the question whether cold gravitational equilibrium configurations of massive scalar fields - Bose stars - may exist and whether such configurations are dynamically stable. For bosonic fields interacting only via gravity spherically symmetric equilibrium solutions were found by Kaup [1] as well as Ruffini and Bonazzola [2] by solving the coupled Einstein-Klein-Gordon equations. They analyzed only the zero-node solutions, corresponding to the lowest energy state. The results of ref.[1,2] have been confirmed and extended later on [3,4]. In particular Friedberg, Lee and Pang have computed radial excitations with arbitrary node number. It was found that for a fixed node-number there is always a critical mass and particle number above which gravitational collapse occurs. For the zero-node solution these are $M_{crit} = 0.633M_{Pl}^2/m$ and $N_{crit} = 0.653M_{Pl}^2/m^2$ where $M_{Pl} = G_N^{-1/2}$ is the Planck mass and m is the mass of the scalar particle. For the excited states the critical mass as well as the critical particle number grow approximately linear with the node number [4].

The extension to the self interacting case with a potential $V(\phi) = \frac{\lambda}{2}(\phi^*\phi)^2$ ($\lambda > 0$) has been considered in the work of Colpi, Shapiro and Wasserman [5]. The consequence of switching on a quartic self interaction between the scalar particles is to increase the above limits on M_{crit} and N_{crit} . Other models, including soliton stars[6] and boson stars with non-minimal energy momentum tensor [7] show the same tendency to increase the critical mass and particle number. If one plots the mass for equilibrium configurations of Bose stars against their central density one finds a behaviour very similar to neutron stars. The mass quickly rises to a maximum for $\rho = \rho_{crit}$, drops a little, oscillates and approaches an asymptotic value at large central densities. A similar behaviour is found for the particle number. The stability behaviour for these equilibrium configurations has been studied by Jetzer [8] and Lee and Pang [9]. It turns out that the stability behaviour is the same as in the fermionic case. The boundary between stable and unstable configurations is given by $\rho = \rho_{crit}$.

In this paper we extend the analysis of the static spherically symmetric solutions to a system of complex scalar fields coupled to a U(1) gauge field. We also allow for the possibility of a quartic self interaction. Previously solutions for charged multi boson configurations in flat space-time have been considered [10]. Also solutions to the Einstein-Yang-Mills equations with gauge group SU(2) have been studied [11]. As expected the maximum mass increases with increasing gauge coupling constant e . At the same time the binding energy per particle decreases. If $e^2/4\pi \geq G_N m^2$ it is no more possible to have bounded equilibrium solutions, because now the Coulomb repulsion between the particles is bigger than their gravitational attraction. Besides the simple zero-node case we study solutions with nodes, corresponding to excited states. We also consider the limit where the quartic self coupling becomes large.

As in the case without charge [4] we can easily see that the particle number and the mass have their extrema, in particular their maximum, at the same value of the central density. From this fact it follows, as pointed out in ref.[9], that the pulsation equation which governs the time evolution of the oscillations around the equilibrium configurations has a zero eigenvalue where M and N have their extrema. This leads to the conclusion that the critical density ρ_{crit} corresponding to the maximum mass is the border between stable and unstable configurations.

The system we consider is a complex scalar field with a U(1) charge and coupled to gravity, whose action is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} + g^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) - m^2 |\phi|^2 - \frac{\lambda}{2} |\phi|^4 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right), \quad (1)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and

$$D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$$

The action is invariant under a local U(1) gauge transformation, thus the total particle number is conserved. The corresponding conserved current is given by

$$J^\mu = \sqrt{-g} g^{\mu\nu} \{ ie [\phi^* \partial_\nu \phi - \phi \partial_\nu \phi^*] - 2e^2 A_\nu |\phi|^2 \} \quad (2)$$

We take all the fields as functions of r and t alone since we consider only spherically symmetric equilibrium configurations. We express the metric in Schwarzschild coordinates

$$ds^2 = Bdt^2 - A dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (3)$$

where the metric functions A and B depend only on r and t . We choose the gauge field A_μ such that we have only electric charges and no magnetic ones, therefore we set $A_\mu = (C_0(r, t), \vec{0})$. At the equilibrium configuration we set $\phi(r, t) = \phi_0(r)e^{i\omega t}$ and all the other fields and metric functions are time independent. The coupled system of equations, which can be derived from the above action eq.(1) by varying with respect to the various fields, is given by the two Einstein equations :

$$\frac{A'}{A^2 r} + \frac{1}{r^2}(1 - \frac{1}{A}) = 8\pi G_N [(\frac{(\omega + eC_0)^2}{B} + m^2)\phi_0^2 + \frac{\lambda}{2}\phi_0^4 + \frac{\phi_0'^2}{A} + \frac{C_0'^2}{2AB}], \quad (4)$$

$$\frac{B'}{ABr} - \frac{1}{r^2}(1 - \frac{1}{A}) = 8\pi G_N [(\frac{(\omega + eC_0)^2}{B} - m^2)\phi_0^2 - \frac{\lambda}{2}\phi_0^4 + \frac{\phi_0'^2}{A} - \frac{C_0'^2}{2AB}], \quad (5)$$

the Maxwell equation :

$$C_0'' + (\frac{2}{r} - \frac{A'}{2A} - \frac{B'}{2B})C_0' - 2e\phi_0^2 A(\omega + eC_0) = 0, \quad (6)$$

and the scalar wave equation :

$$\phi_0'' + (\frac{2}{r} + \frac{B'}{2B} - \frac{A'}{2A})\phi_0' + A\phi_0(\frac{(\omega + eC_0)^2}{B} - m^2 - \lambda\phi_0^2) = 0. \quad (7)$$

As boundary conditions we impose $\phi_0(0) = \text{const}$, $\phi_0'(0) = 0$ and $\phi(\infty) = \phi_0'(\infty) = 0$ in order to have a localized particle distribution. We allow the possibility of ϕ_0 to have nodes, giving rise to excited states. Furthermore we demand the electric field to be absent at the origin giving $C_0'(0) = 0$ and we normalize the potential by $C_0(\infty) = 0$. The condition $B(\infty) = 1$ is imposed to get the ordinary Minkowski metric at asymptotia and the condition $A(0) = 1$ is a regularity condition. With these boundary conditions the

equations (4-7) form an eigenvalue equation for ω . In order to get the total mass of the Bose star we can use the asymptotic relation:

$$A(r) = (1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{4\pi r^2})^{-1} \quad (r \rightarrow \infty). \quad (8)$$

where M and Q are the total mass and electric charge of the configuration. In order to use eq.(8) one needs to determine the charge of the system independently. This can be done by using the asymptotic formula:

$$E(r) = C'_0 = \frac{Q}{4\pi r^2} \quad (r \rightarrow \infty). \quad (9)$$

Alternatively one can integrate the time component of the conserved current (2) over all space :

$$Q = \int d^3x J^0 = 8\pi e \int_0^\infty dr r^2 (\omega + eC_0) \phi_0^2 \sqrt{\frac{A}{B}}, \quad (10)$$

The total particle number N is related to the total charge by $Q = eN$. The mass of the system can also be determined by the integral:

$$M = \int d^3x T^0_0 = 4\pi \int_0^\infty dr r^2 \left(\left[\frac{(\omega + eC_0)^2}{B} + m^2 \right] \phi_0^2 + \frac{\lambda}{2} \phi_0^4 + \frac{\phi_0'^2}{A} + \frac{C_0'^2}{2AB} \right). \quad (11)$$

This equation can be derived by taking eq.(8) to be valid for all r but with M and Q r -dependent. Differentiation to r and use of eq.(4) results in eq.(11).

We define the radius of the Bose star as:

$$R = \frac{1}{eN} \int d^3x r J^0 = \frac{8\pi}{N} \int_0^\infty dr r^3 (\omega + eC_0) \phi_0^2 \sqrt{\frac{A}{B}}. \quad (12)$$

We choose to define the effective radius via the particle number density rather than via the mass density T^0_0 , as done in Ref.[4]. This is because the long range nature of the electromagnetic force would lead to an ∞ radius if defined via the mass density. The electric field is given by $E(r) = C'_0$, and far outside the Bose star we expect C_0 to behave like $1/r$, therefore the last term in the mass integral eq.(11) $C_0'^2/2AB \sim 1/r^4$ for $r \rightarrow \infty$. Notice that inside the Bose star the C_0 field acquires an effective mass due to the $\phi_0^2 C_0$

term in eq.(6), and thus due to this Higgs mechanism there is a screening of the electromagnetic force. From the behaviour at $r \rightarrow \infty$ we see that a radius definition via the mass density integral would lead to a contribution $\int^\infty dr r^3 \frac{1}{r^4} \rightarrow \infty$. Whereas all the terms which contain ϕ_0 drop exponentially for $r \rightarrow \infty$, since ϕ_0 behaves asymptotically as (for $C_0 \sim \frac{1}{r}, r \rightarrow \infty$)

$$\phi_0(r) \sim e^{-kr} r^\alpha \quad (13)$$

with $k = \sqrt{m^2 - \omega^2}$ and $\alpha = -1 - \frac{M}{k}(1 - 2\omega^2 + \frac{eQ\omega}{4\pi M})$. Therefore the radius definition via the particle number density gives a finite result. It turns out that for $e = 0$ the two definitions give practically the same numerical result.

In order to simplify the integrations of eq.(4-7) we made a rescaling of variables :

$$\begin{aligned} r &\rightarrow \tilde{r} = mr \\ \phi_0 &\rightarrow \tilde{\phi}_0 = (8\pi G_N)^{\frac{1}{2}} \phi_0 \\ C_0 &\rightarrow \tilde{C}_0 = (\omega + eC_0) \\ B &\rightarrow \tilde{B} = m^2 B \\ e^2 &\rightarrow \tilde{e}^2 = \frac{e^2 M_{Pl}^2}{8\pi m^2} \\ \lambda &\rightarrow \Lambda = \frac{\lambda M_{Pl}^2}{8\pi m^2} \end{aligned} \quad (14)$$

with $M_{Pl}^2 = G_N^{-1}$. In terms of these new variables we require $\tilde{e}^2 < \frac{1}{2}$ in order to have bounded equilibrium solutions ; in fact for $\tilde{e}^2 > \frac{1}{2}$ the repulsive Coulomb force is bigger than the gravitational attraction. Finally we used the remaining invariance of the equations under $\tilde{C}_0 \rightarrow f\tilde{C}_0$ and $\tilde{B} \rightarrow f^2\tilde{B}$ to make $\tilde{C}_0(0) = 1$. In this way the equations become an eigenvalue equation for $\tilde{B}(0)$ instead of ω .

It is now straightforward to solve numerically the equations (4-7). We first consider the case where the bosons have no $\lambda|\phi|^4$ interaction. For all values of $\phi(0)$ one can solve the equations and find a value for M and N. The results for different values of the charge are given in fig. 1. This figure contains only zero node solutions. The behaviour of M and N as a function of $\phi(0)$ is well known for the $e=0$ case ref.[1-4]; the mass and particle number rise with increasing $\phi(0)$ to a maximum of $M_{max} = 0.633M_{Pl}^2/m$ and

$N_{max} = 0.653M_{Pl}^2/m^2$. After this M and N drop, oscillate a bit and finally approach a constant independent of $\phi(0)$. We are here mostly interested in the approach to the maximum and not so much with the behaviour at very large $\phi(0)$ and we do not draw this region. From the figure it is clear that the introduction of a charge does not change the behaviour qualitatively. The location of the maximum shifts to lower values of $\phi(0)$ while the maximum mass and particle number increase with the charge. The location of the maxima of M and N are drawn as a separate curve in the figure. For a value of $e \rightarrow e_{crit} = \sqrt{1/2}$ (the charge is now measured in units of $M_{Pl}/m\sqrt{8\pi}$) the value of this central density goes to zero while the mass becomes infinite. For values of $e > e_{crit}$ the behaviour of the solutions changes. It is no more possible to satisfy the boundary conditions at the origin and require that $\phi(r)$ is a monotonically decreasing function of r . In fact, while the first derivative of ϕ can still be zero the second derivative will be positive and ϕ will start to increase going to larger r . If one tries to solve the equations anyway one finds in the end a singularity in the metric at a finite value of r , where both A and B blow up. Therefore such solutions clearly do not correspond to a physical situation. The reason for this change of behaviour is intuitively very simple. For $e > e_{crit}$ the Coulomb repulsion between two particles becomes bigger than the gravitational attraction. This is also clear from the fact that the binding energy per particle decreases if one increases the charge of the field.

The behaviour of the mass and particle number as a function of $\phi(0)$ suggest that the solutions with $\phi(0)$ smaller than $\phi(0)$ at the maximum are stable and the others unstable. In particular this would imply that, even if there are solutions for $e > e_{crit}$, these would be unstable. This statement can be shown to be true by the following argument. As in the case without electrical charge (see Ref.[4]) we can write the total energy of the system $E(\equiv M)$ as

$$M \equiv E = N\omega - L, \quad (15)$$

where L is the total lagrangian, which includes a matter part $L(m)$ and a gravity part $L(g)$ and N is the total particle number. Therefore as a consequence of eq.(15) by regarding the mass M as a function of the particle number N we have

$$\frac{dM}{dN} = \left(\frac{\partial E}{\partial N}\right)_{A,B,\phi_0,C_0} = \omega. \quad (16)$$

For the bounded solutions we have moreover $\omega^2 \leq m^2$. From the relation eq.(16) it follows that the mass M and the particle number N as a function of the central density ρ are stationary at the same points, in particular at $\rho = \rho_{crit}$ where the mass is maximal. From this fact, as discussed in Ref.[9], it follows that at the stationary points the pulsation equation, which governs the small radial perturbations, has a zero mode. Also with a charge the maximum mass corresponds to the first stationary point. We expect therefore the central density ρ_{crit} , corresponding to the maximum mass, to be the boundary between stable, against small radial perturbations, and unstable equilibrium configurations. We notice that in order to have a complete proof one has still to analyse the eigenvalues of the pulsation equation in order to see if they are real and positive for central densities smaller than ρ_{crit} . As in the case with no charge (see Ref.[8]) we expect that the stability behaviour does not change at stationary points of the functions M and N corresponding to central densities bigger than ρ_{crit} . This fact is also suggested from the behaviour of the particle number (or mass) versus radius diagrams, which in both charged and chargeless cases are similar (see figure 2). In fact in both cases the shape of the particle number versus radius diagram is bent counterclockwise at the critical points. We notice that a rigorous proof, that from this behaviour it follows that the system stays unstable, exists only if the pulsation equation is a 1-component equation. In the cases where the pulsation equation consist of two or more coupled differential equations, like in the chargeless case [8], this is still an open point.

The behaviour of M and N as a function of $\phi(0)$ for solutions with nodes is qualitatively not different from the zero node solution. The maximum particle number increases roughly linear with the node number as shown in fig.(3). For values of the charge close to the critical charge e_{crit} the mass, particle, central density and radius of the boson star at ρ_{crit} have the following asymptotic behaviour:

$$\begin{aligned}\phi(0) &\approx \alpha_\phi (e_{crit} - e)^{1/2} \\ N_{max} &\approx M_{max} \approx \alpha_N (e_{crit} - e)^{-1/2} \\ R &\approx \alpha_R (e_{crit} - e)^{-1/2}\end{aligned}\tag{17}$$

We list the coefficients α_ϕ , α_N and α_R in table 1 for different node num-

bers. Note that M is given in units of M_{Pl}^2/m and N in units of M_{Pl}^2/m^2 .

Table 1 : coefficients of behaviour near e_{crit}

	α_ϕ	α_N	α_R
n=0	$6.7 \cdot 10^{-2}$	$4.4 \cdot 10^{-1}$	1.5
n=1	$7.2 \cdot 10^{-2}$	$9.8 \cdot 10^{-1}$	3.6
n=2	$7.6 \cdot 10^{-2}$	1.5	5.6
n=3	$8.1 \cdot 10^{-2}$	2.1	7.7
n=4	$8.1 \cdot 10^{-2}$	2.6	9.8
n=5	$8.2 \cdot 10^{-2}$	3.1	11.9

The effect of introducing a self interaction $\frac{\lambda}{2}|\phi|^4$ has been studied in ref.[5] for the $e=0$ case. The presence of a charge does not qualitatively affect the dependence on λ as found in ref.[5]. Increasing λ increases the value of the maximum mass and particle number. For very large $\Lambda = \frac{\lambda M_{Pl}^2}{8\pi m}$ one finds $M, N \sim \Lambda^{1/2}$. In fact one has $\Lambda \gg 1$ even for small λ because the typical mass of the bosons is much smaller than the Planck mass. As in ref.[5] one can analyze the limit $\Lambda \gg 1$ by scaling the variables and fields in the following way :

$$\begin{aligned}
\tilde{\phi}_0 &\rightarrow \phi_{0*} = \tilde{\phi}_0 \Lambda^{1/2} \\
\tilde{r} &\rightarrow r_* = \tilde{r} \Lambda^{-1/2} \\
M &\rightarrow M_* = M \Lambda^{-1/2} \\
N &\rightarrow N_* = N \Lambda^{-1/2}
\end{aligned} \tag{18}$$

The other fields remain unchanged. Ignoring terms $O(\Lambda^{-1})$ one has the following simplified set of equations, where the scalar wave equation is reduced to an algebraic equation:

$$\phi_{0*} = \left(\frac{\tilde{C}_0^2}{\tilde{B}} - 1 \right)^{\frac{1}{2}} \tag{19}$$

$$\frac{A'}{A^2 r_*} + \frac{1}{r_*^2} \left(1 - \frac{1}{A} \right) = 2\phi_{0*}^2 + \frac{3}{2}\phi_{0*}^4 + \frac{\tilde{C}_0'^2}{2\tilde{e}^2 A \tilde{B}} \tag{20}$$

$$\frac{\tilde{B}'}{A \tilde{B} r_*} - \frac{1}{r_*^2} \left(1 - \frac{1}{A} \right) = \frac{1}{2}\phi_{0*}^4 - \frac{\tilde{C}_0'^2}{2\tilde{e}^2 A \tilde{B}} \tag{21}$$

$$\tilde{C}_0'' + \left(\frac{2}{r_*} - \frac{A'}{2A} - \frac{\tilde{B}'}{2\tilde{B}} \right) \tilde{C}_0' - 2\tilde{e}^2 \phi_{0*}^2 A \tilde{C}_0 = 0 \quad (22)$$

The approach to the asymptotic behaviour for large Λ is shown in fig.(4), where we plot configurations with the same value of ϕ_{0*} but different values of Λ . The solution for $\Lambda \rightarrow \infty$ agrees well with the exact solution except for large values of r . For large values of Λ this region becomes less important and masses and particle numbers are well described by the asymptotic formulae. The behaviour of mass and particle number as a function of ϕ_{0*} and e are given in fig.(5). The effect of introducing a charge is similar to the $\lambda = 0$ case. The maximum values of mass and particle number increase with e and the location of the maximum shifts to smaller values of ϕ_{0*} . For values of the charge close to $e_{crit} = \sqrt{1/2}$ the asymptotic behaviour at the maximum is given by $N_* = M_* = 0.226(e_{crit} - e)^{-1/2}$, $R_* = 0.415(e_{crit} - e)^{-1/2}$ and $\phi_{0*}(0) = 2.43(e_{crit} - e)^{1/2}$. As far as the stability is concerned the same discussion and conclusions hold as for the $\lambda = 0$ case.

Finally we note that equations (19-22) do not have higher node solutions. This is paradoxical because the rescalings (18) do not refer to the node number and one would expect different solutions to exist in the $\Lambda \rightarrow \infty$ limit for different node number. The resolution to this point is that the particle number grows roughly linear with the node number with a proportionality constant independent of Λ . As a consequence the asymptotic behaviour $N \sim \Lambda^{1/2}$ is correct for all finite fixed n with a relative error of order $n/\Lambda^{1/2}$. For $n > \Lambda^{1/2}$ the asymptotic formulas break down due to the fact that for a large range of r the solution is dominated by low values of $\phi_0(r)$ where ϕ_0^2 cannot be ignored compared to $\Lambda \phi_0^4$. For $n \gg \Lambda^{1/2}$ we find that the location of the maximum particle number N behaves asymptotically as $N(\Lambda, n) = N(0, n) + f(\Lambda)$, where $f(\Lambda) \sim \Lambda^{1/2}$ for large Λ .

Acknowledgement

We thank A.Martin for a useful discussion.

References

- [1] D. J. Kaup, Phys. Rev. 172 (1968), 1331.
- [2] R. Ruffini and S. Bonazzola, Phys. Rev. 187 (1969), 1767.
- [3] W. Thirring Phys. Lett. 127B (1983), 27;
E. Takasugi and M. Yoshimura, Z. Phys. C26 (1984), 241;
J. D. Breit, S. Gupta and A. Zaks, Phys. Lett. 140B (1984), 329.
- [4] T. D. Lee, Phys. Rev. D35 (1987), 3637;
R. Friedberg, T. D. Lee and Y. Pang, Phys. Rev. D35 (1987), 3640.
- [5] M. Colpi, S. L. Shapiro and I. Wasserman, Phys. Rev. Lett. 57 (1986), 2485.
- [6] R. Friedberg, T. D. Lee and Y. Pang, Phys. Rev. D35 (1987), 3658.
- [7] J. J. van der Bij and M. Gleiser, Phys. Lett. 194B (1987), 482.
- [8] Ph. Jetzer, Nucl. Phys. B316 (1989), 411;
Fermilab preprint 88A (1988), to appear in the proceedings of the
Fifth Marcel Grossmann meeting, Perth (August 1988);
Phys. Lett. 222B (1989), 447.
- [9] T. D. Lee and Y. Pang, Nucl. Phys. B315 (1989), 477.
- [10] K. Lee, J. Stein-Schabes, R. Watkins and L. M. Widrow, Fermilab
Pub. 139A (1988).
- [11] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61 (1988), 141.

Figure Captions

Fig. 1: Boson star mass in units of M_{Pl}^2/m (solid line) and particle number in units of M_{Pl}^2/m^2 (broken line) as a function of $\phi_0(0)$. The charge e is given in units of $M_{Pl}/(\sqrt{8\pi}m)$. No quartic self-coupling is present. Also the lines going through the maxima of mass and particle number are drawn.

- Fig. 2: The particle number in units of M_{Pl}^2/m^2 as a function of the effective radius R (in units of $1/m$) for the $\lambda = 0$ case for two different values of the charge e (in units of $M_{Pl}/(\sqrt{8\pi}m)$).
- Fig. 3: The value of the maximum particle number in units of M_{Pl}^2/m^2 as a function of the charge e in units of $M_{Pl}/(\sqrt{8\pi}m)$ for different node numbers $n=0,\dots,5$ in the scalar field.
- Fig. 4: The behaviour of the scalar field as a function of the distance for different values of Λ and the $\Lambda \rightarrow \infty$ limit. The radius is scaled as $r/\sqrt{\Lambda}$ and the value of $\phi_0(0)$ is taken to be $1/(2\sqrt{\Lambda})$.
- Fig. 5: Boson star mass in units of $\Lambda^{\frac{1}{2}}M_{Pl}^2/m$ (solid line) and particle number in units of $\Lambda^{\frac{1}{2}}M_{Pl}^2/m^2$ (broken line) as a function of $\phi_0(0)\Lambda^{\frac{1}{2}}$ for different values of e for the case $\Lambda \rightarrow \infty$. Also the lines going through the maxima of mass and particle number are drawn.

FIGURE 1

FIGURE 2

FIGURE 3

$e = 0.5$

FIGURE 4

FIGURE 5