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REGULARIZING THE FUNCTIONAL INTEGRAL IN 2D - QUANTUM GRAVITY

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Abstract

We show that the usual conjectures made to understand Polyakov's quantization of 2D - gravity in the functional integral approach arise from an explicit calculation using heat kernel regularization. The connection of these results with string theory is briefly commented upon.

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Polyakov [1] has proposed an exact solution for the quantization of 2D-gravity, given by the old Liouville Lagrangian, completing earlier results by other groups [2,3]. His original method uses the light-cone gauge to fix the invariance under 2D-diffeomorphisms. Later on, however, it has been extended to the covariant gauge as well [4,5], where the results are derived in an easier way. 2D-quantum gravity coupled to matter appears naturally when string theories are formulated in the Polyakov functional integral approach [6]. Therefore a full understanding of Polyakov's results on the quantization of 2D-gravity in the functional integral is particularly desirable. In this direction, it has been shown that under some **plausible conjectures** the results of ref.[1] can be obtained using functional integrals [5,7,8]. In this letter we show that these conjectures can in fact be demonstrated using the standard heat kernel regularization.

Consider, for definiteness, the case of the bosonic closed string.¹ The contribution to the partition function of closed world sheets with h handles is proportional to [6]

$$Z_h = \int \frac{[Dg][DX]}{V} e^{-S_m(g,X) - S_G} \quad (1)$$

where $S_m(X,g)$ is the matter action coupled to 2D-gravity because of covariance, S_G is a counterterm whose form will be fixed below, and V is the volume of the symmetry group which includes the world sheet diffeomorphisms. The functional measures $[Dg]$ and $[DX]$ can be defined once an inner product for arbitrary variations of the fields is given. In our case these inner products are dictated by the requirements of world sheet reparametrization invariance and locality

$$\begin{aligned} (\delta_1 g, \delta_2 g) = & A \int d^2\xi \sqrt{g} g^{ab} g^{rs} \delta g_{ab} \delta g_{rs} + \\ & + B \int d^2\xi \sqrt{g} (g^{ar} g^{bs} - \frac{1}{2} g^{ab} g^{rs}) \delta g_{ab} \delta g_{rs} \end{aligned} \quad (2)$$

with A and B arbitrary positive numbers, and a similar expression for $(\delta_1 X, \delta_2 X)$. As usual, we can parametrize a general metric g by a diffeomorphism and a Weyl scaling acting on a family of fiducial metrics $\hat{g}(\tau)$, depending on the moduli τ . Therefore, if we use a reparametrization invariant regulariza-

¹ In general it can be any conformal matter field theory defined on the world sheet.

tion, the functional integral over g is reduced to an integration over the moduli and over the Weyl scaling parameter ϕ

$$\frac{[Dg]}{\mathcal{V}} = \frac{[d\tau]}{\mathcal{V}'} [D\phi]_{\hat{g}e\phi} [Dx]_{\hat{g}e\phi} \Delta_{FP}(\hat{g}e\phi) \quad (3)$$

where $\Delta_{FP}(g)$ is the Fadeev–Popov Jacobian coming from integrating out the 2D diffeomorphisms.

In (3) we have explicitly indicated that the functional measures of X and g depend on g in a non-trivial way. On the contrary, $[d\tau]$ can be chosen to be ϕ -independent [9].² It is well known that Δ_{FP} and the matter part have well-defined transformations under Weyl rescalings of $\hat{g}$ [6]

$$\Delta_{FP}(\hat{g}e\phi) = e^{-26S_L(\hat{g},\phi)} \Delta_{FP}(\hat{g})$$

$$\int [Dx]_{\hat{g}e\phi} e^{-S_m(\hat{g}e\phi, x)} = e^{C_m S_L(\hat{g},\phi)} \int [Dx]_{\hat{g}} e^{-S_m(\hat{g}, x)} \quad (4)$$

where S_L is the Liouville action

$$S_L(\hat{g}, \phi) = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \phi \partial_b \phi + \hat{R} \phi + \mu e^\phi \right) \quad (5)$$

C_m is the central charge of the matter action, assumed to be in a conformally invariant configuration.

Notice that C_m has not to coincide in general with the dimension of the string target space because of the value of the dilaton β -function, which is forced to be constant [10], but not necessarily vanishing.

Therefore eq.(1) reads

$$Z_h = \int \frac{[d\tau]}{\mathcal{V}'} \Delta_{FP}(\hat{g}) \int [Dx]_{\hat{g}} e^{-S_m(\hat{g}, x)} \times$$

$$\times \int [D\phi]_{\hat{g}e\phi} e^{x\rho - \left\{ (26 - C_m) S_L(\hat{g}, \phi) + S_G \right\}} \quad (6)$$

and, when the theory is not Weyl invariant on the world-sheet ($C_m \neq 26$), we have two decoupled field theories depending on the moduli, one for the matter X , and one for the Liouville mode ϕ .

² A related problem which has not been discussed in the literature so far in this context is the regularization of the integration over the moduli, which could result in extra ϕ -dependences. This, we hope, will be not a problem for a conformal theory, but, in any case, the results are safe for $h=0$, the spherical topology.

The main difficulty with the functional integral over ϕ is that the inner product which defines its functional measure, induced by (2), depends on ϕ itself

$$(\delta_1 \phi, \delta_2 \phi) = \int d^2 \xi \sqrt{\hat{g}} e^{\phi} \delta_1 \phi \delta_2 \phi \quad (7)$$

It would be convenient to write (6) in terms of a different measure defined by

$$(\delta_1 \phi, \delta_2 \phi)^{\dagger} = \int d^2 \xi \sqrt{\hat{g}} \delta_1 \phi \delta_2 \phi \quad (8)$$

The relationship between the measures corresponding to (7) and (8) is the main point in refs.[5,7,8] and, implicitly, in [4], for which they assume the expression

$$[D\phi]_{\hat{g}} e^{\phi} = [D\phi]_{\hat{g}} e^{\tilde{S}(\hat{g}, \phi)} \quad (9)$$

with $\tilde{S}(\hat{g}, \phi)$ being a renormalizable local action, without any proof.

We shall demonstrate that the conjecture (9) can be explicitly deduced by computing the Jacobian which relates the measures defined by (7) and (8) with some regularization procedure. The regularization method has to preserve 2D-reparametrization invariance, and we will use the heat kernel. From (7) and (8) one can write formally [11,12]

$$[D\phi]_{\hat{g}} e^{\phi} = [D\phi]_{\hat{g}} \sqrt{\text{Det}(\tilde{L})} \quad (10)$$

where $\tilde{L}$ is an operator that is diagonal in world-sheet coordinates

$$\tilde{L} = e^{\phi(\xi_1)} \delta^2(\xi_1 - \xi_2) \quad (11)$$

This determinant needs to be regularized and we will do so via the standard expression for its variation

$$\delta \ln \text{Det}(\tilde{L}) = \delta \text{Tr} \ln(\tilde{L}) = \int d^2 \xi \delta^2(0) \delta \phi(\xi) \quad (12)$$

$\delta^2(0)$ is meaningless, and we replace it by $G_{\epsilon}(\xi, \xi)$, where $G_{\epsilon}(\xi_1, \xi_2)$ is the heat kernel for the covariant

Laplacian Δ (depending on g) acting on world-sheet scalars like $\delta\phi$, whose boundary condition is $G_0(\xi_1, \xi_2) = \delta^2(\xi_1 - \xi_2)$. The small-time expansion of G is

$$G_{\epsilon \rightarrow 0}(\xi, \xi) = \frac{\sqrt{g}}{4\pi\epsilon} + \frac{\sqrt{g} R}{24\pi} + O(\epsilon) \quad (13)$$

from which the regularized expression for the determinant follows. Putting the result back in (10), we obtain the Jacobian which relates the two functional measures, and demonstrate the conjecture in (9). In this way $\tilde{S}$ turns up to be ($g = \hat{g}e^\phi$, $R = e^{-\phi}(\hat{R} + \hat{\Delta}\phi)$)

$$\begin{aligned} \tilde{S}(\hat{g}, \phi) = & \frac{1}{48\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \phi \partial_b \phi + \hat{R} \phi \right) \\ & + \frac{1}{8\pi\epsilon} \int d^2\xi \sqrt{\hat{g}} e^\phi \end{aligned} \quad (14)$$

which is nothing other than the Liouville action (5). Notice that the cosmological constant piece is divergent (in fact μ in eqs.(4-6) is proportional to ϵ^{-1} as well, if one uses the same renormalization method [11]). Therefore the final expression for the Liouville functional integral reads

$$\int [D\phi]_{\hat{g}} \exp - \left\{ (25 - C_m) S_L(\hat{g}, \phi) + S_G \right\} \quad (15)$$

and hence we obtain explicitly the expected **renormalization** of the critical value of the matter central charge ($C_m = 25$) [1,2,4,5,7,8].

The action in (15) is in fact infinite because of the divergent cosmological constant pieces in S_L . These non-logarithmic divergences, which do not contribute to the renormalization group, have to be removed explicitly by the counterterm S_G , which we take as

$$S_G = \frac{\mu_0}{\epsilon} \int d^2\xi \sqrt{\hat{g}} e^\phi + S_S(\hat{g}, \phi) \quad (16)$$

where μ_0 is fixed by the requirement of cancelling all the $1/\epsilon$ terms. The finite (arbitrary) counterterm has to be chosen in such a way so as not to violate the invariance of Z_h under conformal changes of $\hat{g}$ at the quantum level, whose origin is the following. Z_h was defined as an integral over the metric on the world sheet $g = \hat{g}e^\phi$, so it is invariant under the simultaneous transformation $\hat{g} \rightarrow \hat{g}e^\sigma$ and $\phi \rightarrow \phi - \sigma$.

Since the measure over ϕ defined by (8) is invariant under local shifts of ϕ , this implies the invariance of Z_h under conformal changes of $\hat{g}$. This property³ is implicit in our method, and it is equivalent to the cancellation of the total central charge which was imposed ad hoc in refs.[4,5,7,8] to obtain (15). In order to preserve this invariance, δS has to verify

$$\delta S(\hat{g} e^\sigma, \phi - \sigma) = \delta S(\hat{g}, \phi) \quad (17)$$

Therefore it has to be constructed from (1,1) conformal fields in the total matter–Liouville theory [4,5]

$$\delta S(\hat{g}, \phi) = \sum_i u_i \int d^2 \xi \sqrt{\hat{g}} \varphi_i(\xi) e^{A_i \phi}$$

$$A_i = \frac{25 - C_m}{12} \left[1 - \sqrt{1 + \frac{24(\Delta_i - 1)}{25 - C_m}} \right] \quad (18)$$

where φ_i is a spinless primary field with conformal dimension Δ_i . In (18) we exhibit only the solution compatible with the semiclassical (formal) limit $C_m \rightarrow -\infty$ [13]. The simplest choice is to include just the term corresponding to the identity operator, with $\Delta_i = 0$. This contribution plays the rôle of a re-normalized 2D–cosmological constant term. Notice that, because it is generated as a finite counter-term, its coefficient is arbitrary and not necessarily vanishing, which is useful in defining properly the behaviour of the system as a function of the area [5]. Including all the tower of contributions in (18), we have a theory defined outside the original critical point as a function of the couplings u_i , where the matter is coupled to the Liouville mode. This is similar to the standard method of perturbing a Renormalization Group fixed point with marginal operators [14], which defines an off–shell theory.

The above construction is known to work when $C_m < 1$ or $C_m > 25$ if one requires just the re-normalized cosmological constant term to be real (A_0 real), but in the case $C_m > 25$ not all the critical exponents of the theory are real [1,4,5]. The case $C_m > 25$ has another peculiarity; the kinetic term of ϕ is negative. In string theories it has been suggested that ϕ could be identified with the (Minkowskian)

³ Because it can be confusing, it is worthwhile to stress that this built–in symmetry is different from the requirement of invariance under conformal transformations of the world sheet metric $g \rightarrow g e^\sigma$ ($g = \hat{g} e^\phi$) which one imposes in string theory in order to decouple the negative norm states, and which selects the value $C_m = 26$.

time if the d -dimensional target space corresponding to the matter theory is Euclidean. Then the coupling $\hat{R}\phi$ would be interpreted as a (non-covariant in target space-time) contribution to the dilaton background linear in the time [7,8], in agreement with certain cosmological models [15]. A necessary requirement for this identification is that the $(1,1)$ conformal operators of the theory (used to construct δS in eq.(18)) contain the (on-shell) vertex operators of the string in a $(d+1)$ -dimensional target space-time. This is known to happen in the limit $d \rightarrow 25^+$, with $C_m = d$ and keeping $\sqrt{|d-25|}\phi \sim X^0$ finite [2,4,16] (if $d \rightarrow 25^-$, ϕ corresponds to the $d+1$ coordinate of a $(d+1)$ -Euclidean target space). In this limit the coupling between $\hat{R}$ and X^0 vanishes and the symmetry property in (17) becomes just the conformal invariance under $\hat{g} \rightarrow \hat{g}e^\sigma$, which makes the identification of ϕ with the time much more natural. Therefore, in this limit one gets the standard bosonic string theory in $D = 26$ (with either Minkowskian or Euclidean target space-time). Of course a dilaton background linear in X^0 is always a possible (non-covariant) solution of the standard conformal invariance conditions [15], but we think that it is not a necessary consequence of the quantization of 2D-gravity.

A last remark concerns the dependence of the result on the inner product (2). It has been shown [12] that, in general, one can allow A and B to be scalar dimensionless functions of X and g in principle (they have to depend on $g = \hat{g}e^\phi$, and not on $\hat{g}$ and ϕ , in order to define a functional measure over g). Nevertheless the requirement of locality restricts them to be functions of X only. In this case A and B can be interpreted as (renormalizable) backgrounds [12] which have been already taken into account in the value of C_m (see below eq.(5)). Therefore the functional integral result does not suffer from ambiguities of this kind.

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