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Learning the abc of light-cone harmonic space

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Abstract

Light-cone harmonic space is the key to the recently proposed off-shell formulation of $N = 4$ super-Yang-Mills theory. Some of the most unusual features of that space are examined in great detail on the simplified example of a 3-dimensional gauge theory. It is shown that the description of that theory in terms of an analytic harmonic field is equivalent to the usual one. A gauge fixing for the analytic gauge field is found and propagators are constructed.

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0. Introduction

Recently a solution to the long-standing problem of finding an off-shell formulation of the $N = 4$ supersymmetric Yang-Mills theory was proposed [1]. The main idea was to use light-cone (or twistor-like) harmonic variables [2]. With their help one could interpret the on-shell constraints of the $N = 4$ theory as integrability conditions for the existence of the so-called harmonic analytic superfields. Further, the constraints were rewritten as an equivalent set of conditions on the newly introduced analytic gauge connections. Finally, it turned out that those conditions could be derived as **equations of motion from an action**. The action was given by an integral over the analytic harmonic superspace and had the form of a Chern-Simons term. Thus an off-shell formulation of the theory was found. It turned out that the construction was possible only in 5-dimensional space-time.

It should be stressed that the off-shell formulation of Ref. [1] has a number of very unusual, even puzzling features. First of all, the off-shell superfields are analytic, which means in particular that they depend on 4 of the 5 space-time variables (the light-cone projections x^{++} and x^i , but not x^{--}), and yet it is claimed that their on-shell gauge-independent content is the same as that of the ordinary 5-dimensional superfields. The explanation is that the analytic superfields depend on 3 complex (or 6 real) additional harmonic variables, which substitute for the missing space-time one. Second, the Chern-Simons off-shell action is radically different from the familiar Yang-Mills one, up to the point that the gauge coupling constant is dimensionless, contrary to the expectations in a 5-dimensional Yang-Mills theory. Actually, this point raises hopes that the off-shell $N = 4$ theory in 5 dimensions may turn out renormalizable, even finite. Such a major deviation from the habitual description of Yang-Mills theory has to do with the circumstance that the theory defined by the on-shell constraints and the theory given by the off-shell action are equivalent only on-shell, i.e. they have the same equations of motion.

In view of all that we find it worthwhile to clarify some of those new unusual features on a much simpler example. In the present paper we consider a 3-dimensional Yang-Mills theory (non-supersymmetric). Following the main lines of Ref. [1], we introduce light-cone harmonic variables which reduce the Lorentz group $SO(3)$ (Euclidean case) or $SO(1,2)$ (Minkowski case) to its light-cone subgroup $SO(2)$ or $SO(1,1)$. This allows us to define an analytic gauge field V^{++} for the covariant harmonic derivative D^{++} . A detailed investigation

shows that the gauge-invariant content of V^{++} coincides with that of the original gauge field $A_\mu(x)$. In this simplified example the equivalence between V^{++} and $A_\mu(x)$ takes place even off-shell, not only on-shell, as is the case in the 5-dimensional theory of Ref. [1]. The reason is that in the 3-dimensional case the harmonic variables parametrizing the coset, e.g., $SO(3)/SO(2)$ supply just one (complex) new degree of freedom. The latter exactly compensates for the missing space-time variable x^{--} . In the 5-dimensional case there are 3 harmonic (complex) degrees of freedom (parametrizing the coset $SO(5)/SO(2) \times SO(3)$) instead of just one x^{--} . Therefore in that case one needs further constraints on V^{++} (the on-shell equations), which restrict its harmonic dependence and reduce its content to that of the ordinary 5-dimensional on-shell theory.

After establishing the equivalence between the $A_\mu(x)$ and V^{++} descriptions of the 3-dimensional gauge theory, we proceed by finding a suitable gauge-fixing condition for the specific new gauge invariance which comes with V^{++} . This allows us to construct propagators for V^{++} .

Throughout most of the paper we study the Euclidean (EC) and Minkowski (MC) cases in parallel. The main advantage of EC is the possibility to use harmonic expansions, and that of MC is the fact that the light-cone projections x^{++} and x^{--} are real. Only in Section 4, when varying the analytic field V^{++} in the action, we will have to make a decision in favor of MC. Note, however, that the action in the 5-dimensional case [1] is manifestly analytic, so no such discrimination of EC seems necessary there.

1. Light-cone harmonic space-time

In the present paper we shall study a 3-dimensional Abelian gauge theory with a Chern-Simons action:

$$S = \frac{1}{2} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad (1.1)$$

$$\delta A_\mu(x) = \partial_\mu \tau(x). \quad (1.2)$$

The above assumptions (Abelian theory and Chern-Simons action) are not essential, we just want to simplify the situation as much as possible. The 3-dimensional space where $A_\mu(x)$ is defined can be of Euclidean or Minkowski type, with Lorentz groups $SO(3)$ and $SO(1,2)$, correspondingly. The main idea is to introduce harmonic variables u [3,2], which will help us to covariantly reduce the full Lorentz symmetry to a subgroup ($SO(2) \subset SO(3)$ or $SO(1,1) \subset SO(1,2)$). Then it will become possible to replace the gauge vector field $A_\mu(x)$ by an analytic scalar gauge field $V^{++}(x, u)$. Our main goal will be to study the relationship between the familiar object $A_\mu(x)$ and the unusual one $V^{++}(x, u)$, and to show that the theory admits an equivalent formulation in terms of the latter.

We begin by defining the harmonic variables u . They form a matrix:

$$(u^{\pm\alpha}) = \begin{pmatrix} u^{+1} & u^{-1} \\ u^{+2} & u^{-2} \end{pmatrix}, \quad (1.3)$$

which belongs to $SU(2) \sim SO(3)$ (EC) or to $SL(2, R) \sim SO(1,2)$ (MC). The index α is an $SU(2)$ or $SL(2, R)$ spinor index, and $\pm$ refer to the subgroup $U(1) \sim SO(2)$ or $R \sim SO(1,1)$, correspondingly. In the Euclidean case (EC) $u^{\pm\alpha}$ satisfy the unitarity condition:

$$EC : \quad (u^{\pm\alpha})^* = \pm u_\alpha^\mp, \quad u_\alpha^\pm = \epsilon_{\alpha\beta} u^{\pm\beta}, \quad \epsilon_{12} = \epsilon^{21} = 1; \quad (1.4)$$

in the Minkowski case (MC) they are real:

$$MC : \quad (u^{\pm\alpha})^* = u^{\pm\alpha}. \quad (1.5)$$

In both cases they satisfy the unimodularity condition:

$$u^{+\alpha} u_\alpha^- = u^{+\alpha} \epsilon_{\alpha\beta} u^{-\beta} = 1. \quad (1.6)$$

Note that in EC one can define a second conjugation (the antipodal map on S^2), which allows one to define real functions of the complex variables u [3].

Clearly, $u^{\pm\alpha}$ depend on 3 real parameters, so there exist 3 independent derivatives preserving condition (1.6):

$$D^{++} = u^{+\alpha} \frac{\partial}{\partial u^{-\alpha}} \equiv \partial_u^{++}, \quad D^{--} = u^{-\alpha} \frac{\partial}{\partial u^{+\alpha}} \equiv \partial_u^{--}, \quad D^0 = u^{+\alpha} \frac{\partial}{\partial u^{+\alpha}} - u^{-\alpha} \frac{\partial}{\partial u^{-\alpha}}. \quad (1.7)$$

Their algebra is that of the groups $SU(2)$ or $SL(2, R)$ acting on the indices $\pm$.

We are going to use the variables $u^{\pm\alpha}$ to describe the coset spaces $SU(2)/U(1)$ or $SL(2, R)/R$. To this end we shall assume [3] that only functions $f(u)$ with definite $U(1)$ or R charge are considered:

$$D^0 f^{(q)}(u) = q f^{(q)}(u). \quad (1.8)$$

This means that those functions are homogeneous with respect to $U(1)$ or R transformations, so they effectively depend on 2 rather than 3 real variables. In other words, one can use the $U(1)$ or R freedom in $u^{\pm\alpha}$ and fix a gauge in which only the 2 parameters of the coset spaces $SU(2)/U(1)$ or $SL(2, R)/R$ remain. For example, such explicit parametrizations are [4]:

$$EC : \quad \begin{pmatrix} \cos \theta e^{i\phi} & -\sin \theta e^{i\phi} \\ \sin \theta e^{-i\phi} & \cos \theta e^{-i\phi} \end{pmatrix} \in \frac{SU(2)}{U(1)} \\ 0 < \theta < \frac{\pi}{2}, \quad 0 \leq \phi < \pi; \quad (1.9)$$

$$MC : \quad \begin{pmatrix} \cos \theta e^\rho & -\sin \theta e^\rho \\ \sin \theta e^{-\rho} & \cos \theta e^{-\rho} \end{pmatrix} \in \frac{SL(2, R)}{R} \\ -\frac{\pi}{4} < \theta < \frac{\pi}{4}, \quad 0 \leq \rho < \infty. \quad (1.10)$$

(in MC the space $SL(2, R)/R$ consists of several sectors, one of which is shown in eq.(1.10) [4]). In the harmonic approach [3] one usually avoids referring to parametrizations like (1.9) and (1.10), instead one deals with the covariant variables $u^{\pm\alpha}$, assuming that the charge defined in (1.8) is always preserved. The advantage is that the symmetry (Lorentz in our case) is kept manifest.

In EC the space (1.9) is compact and coincides with S^2 . The functions (1.8) can be given by their harmonic expansions, which are just expansions on S^2 (take, e.g., $q \geq 0$):

$$EC : \quad f^{(q)}(u) = \sum_{n=0}^{\infty} f^{(\alpha_1 \dots \alpha_{2n+q})} u_{(\alpha_1}^+ \dots u_{\alpha_{n+q}}^+ u_{\alpha_{n+q+1}}^- \dots u_{\alpha_{2n+q}}^-). \quad (1.11)$$

The coefficients $f^{(\alpha_1 \dots \alpha_{2n+q})}$ are irreducible representations of $SU(2)$.

In MC we are not able to define harmonic expansions like (1.11). The reason is not just the non-compactness of the space, but the fact that we have divided $SL(2, R)$ by the non-compact subgroup R rather than the compact subgroup $U(1)$. Our motivation in choosing this particular space is that the variables $u^{\pm\alpha}$ are real ($u^{\pm\alpha} \in SL(2, R)$). Had we chosen the space $SO(1, 2)/SO(2) \sim SL(2, R)/U(1) \sim SU(1, 1)/U(1)$, it would have been possible to define harmonic expansions, but the harmonics $u^{\pm\alpha} \in SU(1, 1)$ would have been complex. This point is crucial when introducing light-cone space-time variables x^{++}, x^{--}, x^{+-} (see the discussion below).

It is convenient to introduce the notation of composite vector harmonic variables (we consider MC first):

$$u_{\mu}^{\pm\pm} = (\sigma_{\mu})^{\alpha\beta} u_{\alpha}^{\pm} u_{\beta}^{\pm}, \quad (1.12)$$

where $(\sigma_{\mu})^{\alpha\beta} = (\sigma_{\mu})^{\beta\alpha}$ are the Minkowski γ -matrices in 3 dimensions, satisfying the identities:

$$\begin{aligned} (\sigma_{\mu})^{\alpha\beta} (\sigma^{\mu})_{\gamma\delta} &= -\delta_{\gamma}^{\alpha} \delta_{\delta}^{\beta} - \delta_{\delta}^{\alpha} \delta_{\gamma}^{\beta}; \\ (\sigma_{\mu})_{\alpha}^{\beta} (\sigma_{\nu})_{\beta}^{\gamma} + (\sigma_{\nu})_{\alpha}^{\beta} (\sigma_{\mu})_{\beta}^{\gamma} &= 2\eta_{\mu\nu} \delta_{\alpha}^{\gamma} = (-++). \end{aligned} \quad (1.13)$$

From (1.6), (1.12), (1.13) one can derive the following useful identities:

$$u_{\mu}^{++} u^{--\mu} = -2, \quad u_{\mu}^{+-} u^{+-\mu} = 1,$$

$$u_{\mu}^{++} u^{++\mu} = u_{\mu}^{--} u^{--\mu} = u_{\mu}^{++} u^{+-\mu} = u_{\mu}^{--} u^{+-\mu} = 0; \quad (1.14a)$$

$$u_{\mu}^{++} u_{\nu}^{--} u_{\lambda}^{+-} \epsilon^{\mu\nu\lambda} = 2; \quad (1.14b)$$

$$u_{\mu}^{++} u^{--\nu} + u_{\mu}^{--} u^{++\nu} - 2u_{\mu}^{+-} u^{+-\nu} = -2\delta_{\mu}^{\nu}. \quad (1.14c)$$

It is easy to see that $(\frac{1}{2}(u^{++\mu} + u^{--\mu}), \frac{1}{2}(u^{--\mu} - u^{++\mu}), u^{+-\mu})$ form a matrix belonging to $SO(1, 2)$. Moreover, $u^{++\mu}, u^{--\mu}, u^{+-\mu}$ act as Lorentz-covariant projectors onto a "moving light-cone frame" [2]. In other words, given the 3-vector x^{μ} , one can define its covariant light-cone projections:

$$x^{++} = u_{\mu}^{++} x^{\mu}, \quad x^{--} = u_{\mu}^{--} x^{\mu}, \quad x^{+-} = u_{\mu}^{+-} x^{\mu}, \quad (1.15a)$$

and vice versa,

$$x_{\mu} = -\frac{1}{2} u_{\mu}^{++} x^{--} - \frac{1}{2} u_{\mu}^{--} x^{++} + u_{\mu}^{+-} x^{+-}. \quad (1.15b)$$

Those projections transform as scalars of the Lorentz group $SO(1,2)$, and as non-trivial representations of the subgroup $SO(1,1)$ (indicated by their charges).

The definitions (1.12-15) were given in MC. We point out that due to our choice of the charge subgroup of $SO(1,2)$ ($SO(1,1)$ rather than $SO(2)$) the newly introduced variables (1.15a) are real:

$$MC : \quad (x^{\pm\pm})^* = x^{\pm\pm}. \quad (1.16)$$

One can easily write down the analogous formulas for EC. There the metric $(-++)$ is replaced by $(+++)$, (1.14b) becomes:

$$u_\mu^{++} u_\nu^{--} u_\lambda^{+-} \epsilon^{\mu\nu\lambda} = 2i; \quad (1.14b')$$

and one finds that:

$$EC : \quad (x^{++})^* = -x^{--}, \quad (x^{+-})^* = x^{+-}. \quad (1.16')$$

This point will be of crucial importance in the forthcoming discussion of light-cone analytic fields, and especially of their integrals.

Concluding this section, we shall define **light-cone harmonic space-time** as the space parametrized by:

$$\{x_\mu, u_\mu^{\pm\pm}\}. \quad (1.17)$$

In that space one can go to a new basis which will be called analytic (for reasons to become clear shortly):

$$\text{Analytic basis : } \{x^{++}, x^{+-}, x^{--}, u_\mu^{\pm\pm}\}. \quad (1.18)$$

If we define the analytic-basis space-time derivatives as follows:

$$\begin{aligned} \partial_x^{++} &\equiv \frac{\partial}{\partial x^{--}} = -\frac{1}{2} u^{++\mu} \partial_\mu, & \partial_x^{--} &\equiv \frac{\partial}{\partial x^{++}} = -\frac{1}{2} u^{--\mu} \partial_\mu, & \partial_x^{+-} &\equiv \frac{\partial}{\partial x^{+-}} = u^{+-\mu} \partial_\mu; \\ \partial_\mu &= u_\mu^{++} \partial_x^{--} + u_\mu^{--} \partial_x^{++} + u_\mu^{+-} \partial_x^{+-}, \end{aligned} \quad (1.19)$$

the harmonic derivatives (1.7) will have the following analytic-basis form (remember that $u^{\pm\pm}$ and $x^{\pm\pm}$ are now independent variables):

$$\begin{aligned} D^{++} &= \partial_u^{++} + x^{++} \partial_x^{+-} + 2x^{+-} \partial_x^{++}, \\ D^{--} &= \partial_u^{--} + x^{--} \partial_x^{+-} + 2x^{+-} \partial_x^{--}, \\ D^0 &= \partial_u^0 - 2x^{--} \partial_x^{++} + 2x^{++} \partial_x^{--}. \end{aligned} \quad (1.20)$$

2. Gauge field, bridge, analytic gauge field

In this section we shall introduce the fundamental concept of analytic fields, i.e. fields depending only on the variables x^{++}, x^{+-}, u (and not on x^{--}) in the analytic basis (1.18). Then we shall show how the gauge field $A_\mu(x)$ with its gauge invariance (1.2) can be equivalently replaced by an analytic gauge field $V^{++}(x^{++}, x^{+-}, u)$ with a new gauge invariance.

We begin with the observation that the $(++)$ -projection of $A_\mu(x)$ (see (1.15a)):

$$A^{++} = u^{++\mu} A_\mu(x) \quad (2.1)$$

can be presented in the form:

$$A^{++} = -\partial_x^{++} b(x, u) \quad (2.2)$$

for some function $b(x, u)$. Clearly, $b(x, u)$ undergoes gauge transformations with parameter $\tau(x)$ (see (1.2)):

$$\delta_\tau b(x, u) = -\tau(x). \quad (2.3)$$

Notice that the parameter $\tau(x)$ in (2.3) is a function only of x , so it cannot gauge away the field $b(x, u)$. At the same time, eq.(2.2) determines $b(x, u)$ up to the following gauge freedom:

$$\delta_\lambda b(x, u) = \lambda(x, u), \quad (2.4a)$$

$$\partial_x^{++} \lambda = 0. \quad (2.4b)$$

The new parameter $\lambda(x, u)$ is restricted by condition (2.4b), so it cannot gauge away $b(x, u)$ either. In fact, condition (2.4b) is very easy to solve if we go to the analytic basis (1.18):

$$\partial_x^{++} \lambda(x, u) = 0 \quad \Rightarrow \quad \lambda = \lambda(x^{++}, x^{+-}, u). \quad (2.5)$$

In EC, where x^{++} and x^{--} are complex conjugate (see(1.16b)), eq.(2.5) simply means that λ is an **analytic** function of the complex variable x^{++} . We shall call this property light-cone harmonic analyticity, and shall use the same name in MC, where x^{++} and x^{--} are two independent real variables (see (1.16a)).

Let us now go back to the gauge field $A_\mu(x)$. From (2.1), (2.2), (1.7), (1.19) we find:

$$D^{++} A^{++} = 0, \quad (2.6)$$

so

$$\partial_x^{++} D^{++} b = 0 \Rightarrow \partial_x^{++} V^{++} = 0, \quad (2.7)$$

where

$$V^{++} = D^{++} b(x, u). \quad (2.8)$$

The second equation in (2.7) means that

$$V^{++} = V^{++}(x^{++}, x^{+-}, u) \quad (2.9)$$

is also an analytic field. This field is inert under τ gauge transformations ($D^{++}\tau(x) = 0$), but changes under λ transformations:

$$\delta_\lambda V^{++} = D^{++}\lambda. \quad (2.10)$$

Conversely, given an arbitrary analytic gauge field V^{++} , one can try to solve (2.8) for $b(x, u)$, then construct an A^{++} from (2.2), which will automatically satisfy (2.6). Then the natural question arises: Can one conclude from (2.6) that A^{++} is always of the form (2.1), i.e. that (2.6) defines a u -independent gauge field $A_\mu(x)$? In EC the answer is affirmative, because there all functions of u have harmonic expansions (1.11), and then (2.6) obviously leads to (2.1). In MC the situation is more complicated, since there one cannot rely on harmonic-expansion arguments. Nevertheless, we shall demonstrate that the above conclusion is true even in that case. Moreover, we shall show that the chains:

$$\begin{aligned} A_\mu(x) &\rightarrow b(x, u) \rightarrow V^{++}(x^{++}, x^{+-}, u); \\ V^{++}(x^{++}, x^{+-}, u) &\rightarrow b(x, u) \rightarrow A_\mu(x) \end{aligned} \quad (2.11)$$

establish an equivalence between the gauge fields A_μ and V^{++} , up to gauge freedom. Note that the object $b(x, u)$ plays the role of an intermediary between A_μ and V^{++} , i.e. of a "bridge" between the τ and λ gauge frames [3]. This means that given a matter field $\phi(x)$ transforming under the τ gauge group ($\delta\phi = e^{i\tau}\phi$), one can convert it into a field $\Phi(x, u) = e^{ib(x, u)}\phi(x)$ transforming under the λ gauge group ($\delta\Phi = e^{i\lambda}\Phi$). In this new frame V^{++} plays the role of a gauge field for the covariant derivative D^{++} .

At first sight it may seem strange that a field like V^{++} , which depends only on 2 space-time variables, can have the same gauge-independent content as A_μ . In fact, V^{++}

depends on the harmonic variables as well, and they can substitute for the missing space-time one, as we shall see.

In order to study the relations (2.11) in detail, we shall construct explicit solutions for $b(x, u)$ from (2.2) and (2.8) for given A_μ and V^{++} , respectively. For this purpose we need Green's functions for the operators ∂_x^{++} and D^{++} . Let us start with EC. There, as we have already pointed out, x^{++} and x^{--} are complex conjugate, therefore one can claim that:

$$EC : \quad \partial_x^{++} \frac{1}{x^{++}} = 2\pi i \delta(x^{++}, x^{--}), \quad (2.12)$$

where $1/x^{++}$ is defined as a complex distribution (recall the relation $\partial/\partial\bar{z}(1/z) = -2\pi i \delta(z, \bar{z})$ [5]). Further, since in this case x^{+-} is a real variable (see (1.16b)), one can define the distribution $\delta(x^{+-})$. Combining both distributions one finds:

$$EC : \quad \partial_x^{++} \frac{\delta(x^{+-})}{x^{++}} = 2\pi i \delta(x^{++}, x^{--}) \delta(x^{+-}) = \pi \delta^3(x) \quad (2.13)$$

(the Jacobian of the change of variables from analytic to real basis is $2i$).

The situation with D^{++} is similar. In [6] the Green function for D^{++} was found to be $1/(u^+v^+)$, defined as follows:

$$(u^+v^+) \frac{1}{u^+v^+} = 1, \quad u^+v^+ \equiv u^{+\alpha}v_\alpha^+; \quad (2.14)$$

$$D_u^{++} \frac{1}{u^+v^+} = \delta^{(1,-1)}(u, v) = (u^+v^-) \delta^{(0,0)}(u, v).$$

Eq. (2.14) was derived in [6] with the help of the (formal) harmonic expansions of the distributions involved. In fact, one can give a direct proof of (2.14) by reducing it to the relation $\partial/\partial\bar{z}(1/z) = -2\pi i \delta(z, \bar{z})$. Indeed, instead of (1.9) one can use a complex parametrization of $SU(2)/U(1) \sim S^2$:

$$(u^{\pm\alpha}) = \frac{1}{\sqrt{1+z\bar{z}}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \rightarrow$$

$$u^{+\alpha}v_\alpha^+ = u^{+1}v^{+2} - u^{+2}v^{+1} = \frac{\xi - z}{\sqrt{(1+z\bar{z})(1+\xi\bar{\xi})}}. \quad (2.15)$$

Since $D^{++}(u^+v^+) = 0$, one sees that D^{++} is the analog of $\partial/\partial(\xi - z)$, so the meaning of (2.14) becomes clear. Relations (2.13) and (2.14), defining Green's functions for ∂_x^{++} and D^{++} , can be adapted to MC too. Indeed, let us define the distribution:

$$MC : \quad \frac{1}{x^{++}} = \mathcal{P} \frac{1}{x^{++}} - i\pi \epsilon(x^{--}) \delta(x^{++}) \rightarrow x^{++} \frac{1}{x^{++}} = 1. \quad (2.16)$$

Here $\mathcal{P}(1/x^{++})$ means the principal value of $1/x^{++}$ in the sense of Cauchy, and $\epsilon(x^{--})$ is the sign function of the real variable x^{--} . Using the relation $\epsilon'(x) = 2\delta(x)$ [5], one obtains:

$$MC : \quad \partial_x^{++} \frac{1}{x^{++}} = -2\pi i \delta(x^{--}) \delta(x^{++}), \quad (2.17)$$

since no other term in (2.16) depends on x^{--} . Then one proceeds as in (2.13):

$$MC : \quad \partial_x^{++} \frac{i\delta(x^{+-})}{x^{++}} = 2\pi \delta(x^{++}) \delta(x^{--}) \delta(x^{+-}) = \pi \delta^3(x) \quad (2.18)$$

The harmonic Green's function $1/(u^+v^+)$ can be defined in a similar manner, assuming that u^+v^+ is one of the two real variables parametrizing the coset $SL(2, R)/R$, and that D^{++} represents the derivative with respect to the other one ($D_u^{++}(u^+v^+) = 0$).

Now we are prepared to solve equations (2.2) and (2.8) for the bridge $b(x, u)$. Given an arbitrary gauge field $A_\mu(x)$, we find:

$$b(x, u) = -\frac{i}{\pi} \int d^3y \frac{\delta(x^{+-} - y^{+-})}{x^{++} - y^{++}} A^{++}(y) + \lambda(x^{++}, x^{+-}, u). \quad (2.19)$$

Here

$$x^{\pm\pm} - y^{\pm\pm} \equiv (x_\mu - y_\mu) u^{\pm\pm\mu}, \quad (2.20)$$

and we have used (2.13) or (2.18), according to the case. The analytic function λ is the freedom of the solution, as explained earlier. Having found $b(x, u)$, we can substitute it in (2.8) and obtain V^{++} :

$$\begin{aligned} V^{++} &= D^{++}b \\ &= -\frac{i}{\pi} \int d^3y [(x^{++} - y^{++}) \partial_x^{+-} + 2(x^{+-} - y^{+-}) \partial_x^{++}] \frac{\delta(x^{+-} - y^{+-})}{x^{++} - y^{++}} A^{++}(y) + D^{++}\lambda \\ &= -\frac{i}{\pi} \int d^3y \delta'(x^{+-} - y^{+-}) A^{++}(y) + D^{++}\lambda. \end{aligned} \quad (2.21)$$

Here we have used the facts that $D^{++}A^{++}(y) = 0$ and $(x^{+-} - y^{+-})\delta(x^{+-} - y^{+-}) = 0$. Clearly, V^{++} (2.21) does not depend on x^{--} , so it satisfies the analyticity condition (2.7). Moreover, the integral term in (2.21) does not depend on x^{++} as well, so we can conclude that it represents V^{++} in a special λ gauge (see Section 3).

Similarly, if we are given an arbitrary analytic V^{++} , we can solve (2.8) for $b(x, u)$ (see (2.14)):

$$b(x, u) = \int dv \frac{u^+v^-}{u^+v^+} V^{++}(x, v) + \tau(x), \quad (2.22)$$

where $\tau(x)$ is again the freedom of the solution. The integral over $v^{\pm\alpha}$ is understood as the invariant integral over the corresponding coset space. In EC, where all harmonic functions are assumed to have harmonic expansions, the u -integration can be reduced to the following simple rules [3]:

$$EC : \quad \int du \cdot 1 = 1, \quad \int du u_{(\alpha_1}^+ \dots u_{\alpha_m}^+ u_{\beta_1}^- \dots u_{\beta_n}^- = \delta_{0m} \delta_{0n}. \quad (2.23)$$

It is easy to see from (2.23) that the following relations take place:

$$\int du D^{++} f^{(-2)}(u) = 0, \quad \int du D^{--} f^{(+2)}(u) = 0, \quad (2.24)$$

allowing us to integrate by parts.

This time we are following the second chain in (2.11), so the next step is to obtain an expression for A^{++} from $b(x, u)$ (2.22):

$$A^{++} = -\partial_x^{++} b(x, u) = \frac{1}{2} u^{++\mu} \int dv \frac{u^+ v^-}{u^+ v^+} \partial_\mu V^{++}(x, v) - \partial_x^{++} \tau(x). \quad (2.25)$$

Using (1.14c) and the fact that $V^{++}(x, v)$ is analytic ($\partial_x^{++}(v) V^{++}(x, v) = 0$), one obtains:

$$\begin{aligned} A^{++} &= \frac{1}{2} u^{++\mu} \int dv \frac{u^+ v^-}{u^+ v^+} [v_\mu^{+-} \partial_x^{--}(v) + v_\mu^{+-} \partial_x^{+-}(v)] V^{++}(x, v) - \partial_x^{++} \tau \\ &= \int dv \frac{u^+ v^-}{u^+ v^+} [-(u^+ v^+)^2 \partial_x^{--}(v) - (u^+ v^+)(u^+ v^-) \partial_x^{+-}(v)] V^{++}(x, v) - \partial_x^{++} \tau \\ &= \int dv [-(u^+ v^+)(u^+ v^-) \partial_x^{--}(v) - (u^+ v^-)^2 \partial_x^{+-}(v)] V^{++}(x, v) - \partial_x^{++} \tau \\ &= u^{++\mu} \left[\frac{1}{2} \int dv (v_\mu^{+-} \partial_x^{--}(v) + v_\mu^{--} \partial_x^{+-}(v)) V^{++}(x, v) + \frac{1}{2} \partial_\mu \tau \right]. \end{aligned} \quad (2.26)$$

Here we have used the identities (see (1.12), (1.13)):

$$\begin{aligned} u^{++\mu} v_\mu^{++} &= -2(u^+ v^+)^2, \quad u^{++\mu} v_\mu^{--} = -2(u^+ v^-)^2, \\ u^{++\mu} v_\mu^{+-} &= -2(u^+ v^+)(u^+ v^-), \quad u^{+-\mu} v_\mu^{+-} = -(u^+ v^+)(u^- v^-) - (u^+ v^-)(u^- v^+). \end{aligned} \quad (2.27)$$

Clearly, A^{++} from (2.26) is a linear function of $u^{++\mu}$ (as in (2.1)), so one derives:

$$A_\mu(x) = \frac{1}{2} \int du [u_\mu^{+-} \partial_x^{--}(u) + u_\mu^{--} \partial_x^{+-}(u)] V^{++}(x, u) + \frac{1}{2} \partial_\mu \tau(x). \quad (2.28)$$

So far we have shown that from any $A_\mu(x)$ we can construct a V^{++} and, vice versa, from a given V^{++} we can find an $A_\mu(x)$. Thus both chains in (2.11) take place. The remaining question is: Can **any** V^{++} be obtained from some $A_\mu(x)$ and, vice versa, can **any** $A_\mu(x)$ be constructed from some V^{++} ? In other words, do relations (2.11) form a closed loop? We shall start answering this question from the V^{++} end. Take a V^{++} and construct a bridge $b(x, u)$, as in (2.22). Further, from $b(x, u)$ we can construct $A^{++} = -\partial_x^{++} b(x, u)$ and substitute it in (2.21). Thus we find a new V^{++} :

$$\begin{aligned}
\hat{V}^{++} &= \frac{i}{\pi} \int d^3 y \delta'(x^{+-} - y^{+-}) \partial_y^{++} b(y, u) \\
&= \frac{i}{\pi} D^{++} \int d^3 y \frac{\delta(x^{+-} - y^{+-})}{x^{++} - y^{++}} \partial_y^{++} b(y, u) \\
&= D^{++} \left[\int d^3 y \delta^3(x - y) b(y, u) + \lambda \right] \\
&= V^{++} + D^{++} \lambda.
\end{aligned} \tag{2.29}$$

Here we have applied the argument of (2.21) in the reversed order, then integrated ∂_y^{++} by parts and used (2.18). The λ term in (2.29) has the following origin. In MC the integration by parts of ∂_y^{++} can in principle produce a surface term, in which $|y^{--}| \rightarrow \infty$. Consequently, that term will not depend on y^{--} , i.e. it will be analytic. In EC one can use the relation:

$$\int \int dz d\bar{z} \frac{\partial}{\partial \bar{z}} w(z, \bar{z}) = - \oint dz w, \tag{2.30}$$

reducing the integral over some area to the integral along its boundary. When the area is enlarged to cover the whole plane, the contour integral will become an insignificant constant. The conclusion from (2.29) is that we have recovered the initial V^{++} up to a gauge transformation (note that the parameter λ can in principle be singular). In other words, we have found the A_μ which generates a given V^{++} . So, the loop $V^{++} \rightarrow b \rightarrow A_\mu \rightarrow b \rightarrow V^{++}$ closes.

Let us now close that loop from the A_μ end, i.e. find the V^{++} which can generate a given A_μ . First we construct the corresponding $b(x, u)$ (2.19), then calculate $V^{++} = D^{++} b$ and substitute it into (2.28). Thus we obtain a new A_μ :

$$\tilde{A}_\mu(x) = \frac{1}{2} \int du [u_\mu^{+-} \partial_x^{--} + u_\mu^{--} \partial_x^{+-}] D^{++} b. \tag{2.31}$$

Now we can repeat the steps of (2.26) in the reversed order:

$$\begin{aligned}
u^{++\mu} \tilde{A}_\mu(x) &= \frac{1}{2} u^{++\mu} \int dv \frac{u^+ v^-}{u^+ v^+} [\partial_\mu - v_\mu^{--} \partial_x^{++}(v)] D_v^{++} b(x, v) \\
&= \frac{1}{2} u^{++\mu} \int dv [\delta^{(0,0)}(u, v) - 1] \partial_\mu b(x, v) \\
&= -\partial_x^{++}(u) b(x, u) - \partial_x^{++}(u) \tau(x) \\
&= u^{++\mu} [A_\mu(x) + \frac{1}{2} \partial_\mu \tau(x)].
\end{aligned} \tag{2.32}$$

Removing $u^{++\mu}$, we see that the initial $A_\mu(x)$ has been recovered up to a (possibly singular) τ gauge transformation.

This concludes our proof of equivalence between the (A_μ, τ) and (V^{++}, λ) descriptions of the 3-dimensional gauge theory.

3. Gauge fixing

In the preceding section we showed that the gauge field $V^{++}(x^{++}, x^{+-}, u)$ has the same gauge-independent (modulo λ transformations) content as $A_\mu(x)$ (modulo τ transformations). Next we are going to find out how to fix the gauge for V^{++} . We recall the familiar gauge-fixing condition for $A_\mu(x)$:

$$\partial^\mu A_\mu = 0. \quad (3.1)$$

In the case of V^{++} we have a hint for a possible gauge fixing. Indeed, the integral term in expression (2.21) for V^{++} does not depend on x^{++} , so one can say that it satisfies the condition:

$$\partial_x^{--} V^{++} = 0, \quad (3.2)$$

in addition to the analyticity condition $\partial_x^{++} V^{++} = 0$. If we substitute this into the first term in expression (2.28) for A_μ , we see that (3.2) implies (3.1). A further argument in favor of this gauge fixing is as follows. Consider the spin Casimir operator of, e.g., $ISO(1, 2)$:

$$C = \partial_x^{(\alpha\beta)} M_{(\alpha\beta)}, \quad M_{(\alpha\beta)} = x_{(\alpha}^\gamma \partial_{\gamma\beta)}^x + u_{(\alpha}^+ \partial_{\beta)}^{u-} + u_{(\alpha}^- \partial_{\beta)}^{u+}. \quad (3.3)$$

Using (1.20) one can rewrite (3.3) as follows:

$$C = \partial_x^{--} D^{++} - \partial_x^{++} D^{--} - \frac{1}{2} \partial_x^{+-} D^0. \quad (3.4)$$

Let us apply C to an analytic field $\phi^{(q)}(\partial_x^{++} \phi = 0)$ with charge $q(D^0 \phi^{(q)} = q \phi^{(q)})$:

$$C \phi^{(q)} = (D^{++} \partial_x^{--} - \frac{1}{2} q \partial_x^{+-}) \phi^{(q)}. \quad (3.5)$$

In the case of $V^{++}(q = 2)$ with the gauge condition (3.2), eq.(3.5) becomes:

$$C V^{++} = -\partial_x^{+-} V^{++} \rightarrow C^2 V^{++} = (\partial_x^{+-})^2 V^{++} = \square V^{++}. \quad (3.6)$$

This can be regarded as a Poincaré irreducibility condition, similar to that on A_μ ($C A_\mu = \square A_\mu$ if $\partial^\mu A_\mu = 0$).

The proof that (3.2) is an appropriate gauge-fixing condition consists of two steps. First, one has to show that the condition on the gauge parameter λ following from (3.2),

$$\partial_x^{--} D^{++} \lambda = -\partial_x^{--} V^{++} \equiv J(x^{++}, x^{+-}, u), \quad (3.7)$$

has a solution, i.e. to find a Green's function for it. Second, using (3.2) one must be able to invert the operator in the action (1.1) (written down in terms of V^{++} instead of A_μ). In this section we shall go through the first part of the proof.

The Green function for equation (3.7) is defined as follows. The solution of (3.7) can be written down in the form:

$$\lambda(x^{++}, x^{+-}, u) = \int dv \int d^3y G^{(0,0)}(x, u|y, v) J(y^{++}, y^{+-}, v), \quad (3.8)$$

where the Green function $G^{(0,0)}$ ((0,0) refer to the u and v charges which are preserved separately) should satisfy the equations:

$$\partial_x^{++} G^{(0,0)}(x, u|y, v) = 0, \quad (3.9a)$$

$$\partial_x^{--} D_u^{++} G^{(0,0)}(x, u|y, v) = \delta^{(0,0)}(u, v) \delta^3(x - y). \quad (3.9b)$$

Eq.(3.9a) means that $G^{(0,0)}$ is analytic with respect to its first argument. In MC we can split the integral $\int d^3y = \int dy^{++} dy^{+-} dy^{--}$ and use the fact that J in (3.8) does not depend on y^{--} to combine $\int dy^{--}$ with $G^{(0,0)}$:

$$\begin{aligned} MC : \quad \lambda(x^{++}, x^{+-}, u) &= \int dv \int dy^{++}(v) dy^{+-}(v) G^{(0,-2)}(x, u|y, v) J(y^{++}, y^{+-}, v), \\ G^{(0,-2)}(x, u|y, v) &= \int_{-\infty}^{\infty} dy^{--}(v) G^{(0,0)}(x, u|y, v). \end{aligned} \quad (3.10)$$

In this case we obtain a Green's function which is analytic with respect to both arguments. Correspondingly, eq.(3.9b) now reads:

$$\partial_x^{--} D_u^{++} G^{(0,-2)}(x, u|y, v) = \int_{-\infty}^{\infty} dy^{--}(v) \delta^3(x - y) \delta^{(0,0)}(u, v) \equiv \delta_A^{(0,-2)}(x, u|y, v), \quad (3.11)$$

thus defining an analytic delta-function (one can show that $\partial_x^{++}(u)$ annihilates $\delta_A^{(0,-2)}$ by converting $\partial_x^{++}(u)$ into $\partial_y^{++}(v)$ with the help of the δ -functions, and then using $\int dy^{--}$).

Note that in EC the procedure of pulling dy^{--} out of d^3y and combining it with the Green function is not well-defined (such an integral would be contour-dependent). For reasons which will become clear in the next section, we prefer the manifestly analytic expression (3.10), so from now on we shall restrict ourselves to MC.

Now we shall show that one solution of eq.(3.11) is given by:

$$G^{(0,-2)}(x, u|y, v) = -\frac{1}{\pi^2} \frac{(u^+ v^-)^3}{u^+ v^+} \int_{-\infty}^{\infty} dy^{--}(v) \partial_x^{++}(v) \int d^3 \xi \frac{\delta(x^{+-} - \xi^{+-})}{x^{++} - \xi^{++}}(u) \frac{\delta(\xi^{+-} - y^{+-})}{\xi^{--} - y^{--}}(v). \quad (3.12)$$

Here $1/(\xi^{--} - y^{--})$ is defined by analogy with (2.16). First we have to check that $G^{(0,-2)}$ satisfies the analyticity condition (3.9a) (use (2.18)):

$$\begin{aligned} \partial_x^{++}(u) G^{(0,-2)} &= \frac{i}{\pi} \frac{(u^+ v^-)^3}{u^+ v^+} \int_{-\infty}^{\infty} dy^{--}(v) \partial_x^{++}(v) \int d^3 \xi \delta^3(x - \xi) \frac{\delta(\xi^{+-} - y^{+-})}{\xi^{--} - y^{--}}(v) \\ &= -\frac{i}{\pi} \frac{(u^+ v^-)^3}{u^+ v^+} \int_{-\infty}^{\infty} dy^{--}(v) \partial_y^{++}(v) \frac{\delta(x^{+-} - y^{+-})}{x^{--} - y^{--}}(v) \\ &= -\frac{i}{\pi} \frac{(u^+ v^-)^3}{u^+ v^+} \frac{\delta(x^{+-} - y^{+-})}{x^{--} - y^{--}}(v) \Big|_{y_v^{--}=-\infty}^{y_v^{--}=\infty} = 0. \end{aligned} \quad (3.13)$$

At first sight it might seem that the dy^{--} integral in (3.12) should also vanish, because one could change $\partial_x^{++}(v)$ into $\partial_y^{++}(v)$, using the transitivity of the $d^3 \xi$ integral. However, in that case $\int d^3 \xi (...) \neq 0$ when $y^{--} \rightarrow \pm\infty$ owing to the singularity of the harmonic factor in (3.12) when $u \rightarrow v$. Instead of demonstrating this directly, we shall show that there exist non-vanishing derivatives of $G^{(0,-2)}$.

Next we check that (3.12) solves (3.11). To this end we first calculate $D_u^{++} G^{(0,-2)}$. The harmonic factor is converted by D_u^{++} into a δ -function (see (2.14)), and D_u^{++} acts on $\delta(x^{+-} - \xi^{+-})/(x^{++} - \xi^{++})$ as in (2.21), so we get:

$$\begin{aligned} D_u^{++} G^{(0,-2)} &= -\frac{1}{\pi^2} \delta^{(4,-4)}(u, v) \int_{-\infty}^{\infty} dy^{--}(v) \partial_x^{++}(v) \int d^3 \xi \frac{\delta(x^{+-} - \xi^{+-})}{x^{++} - \xi^{++}}(u) \frac{\delta(\xi^{+-} - y^{+-})}{\xi^{--} - y^{--}}(v) \\ &\quad - \frac{1}{\pi^2} \frac{(u^+ v^-)^3}{u^+ v^+} \int_{-\infty}^{\infty} dy^{--}(v) \partial_x^{++}(v) \int d^3 \xi \delta'(x^{+-} - \xi^{+-})(u) \frac{\delta(\xi^{+-} - y^{+-})}{\xi^{--} - y^{--}}(v). \end{aligned} \quad (3.14)$$

In the first term one can use $\delta(u, v)$ to change $\partial_x^{++}(v)$ into $\partial_x^{++}(u)$, and then apply (2.18). In the second term $\partial_x^{++}(v)$ can be decomposed in the u basis with the help of (1.14c); only the term $(v^{++\mu} u_\mu^{+-}) \partial_x^{+-}(u)$ will contribute, because $\delta'(x^{+-} - \xi^{+-})(u)$ does not depend on x^{++} and x^{--} . Thus one finds:

$$\begin{aligned} D_u^{++} G^{(0,-2)} &= \frac{i}{\pi} \delta^{(2,-2)}(u, v) \int_{-\infty}^{\infty} dy^{--}(v) \frac{\delta(x^{+-} - y^{+-})}{x^{--} - y^{--}}(v) \\ &\quad - \frac{1}{\pi^2} (u^- v^+) (u^+ v^-)^3 \int_{-\infty}^{\infty} dy^{--}(v) \int d^3 \xi \delta''(x^{+-} - \xi^{+-})(u) \frac{\delta(\xi^{+-} - y^{+-})}{\xi^{--} - y^{--}}(v). \end{aligned} \quad (3.15)$$

Then it is very easy to see that (3.11) holds: in the first term $\partial_x^{--}(u)$ can be changed into $\partial_x^{--}(v)$, and then the x^{--} analog of (2.18) will give $\delta^3(x - y)$; the second term in (3.15) does not depend on $x^{++}(u)$.

4. Gauge-fixed action and propagators

In Section 3 we proved that the gauge (3.2) can be achieved by constructing a Green function for the gauge parameter λ . Next we have to show that condition (3.2) is sufficient for constructing a propagator for the gauge field V^{++} . To this end we should rewrite the action (1.1) in terms of V^{++} instead of A_μ , and add a gauge fixing term. Replacing the field variable A_μ by V^{++} is easy, one just makes use of relation (2.28). In order to implement the gauge condition (3.2) in the action, we need a Lagrange multiplier (recall that this is the case with the original action (1.1) and gauge condition (3.1), too). Thus we find for the gauge-fixed action:

$$\begin{aligned}
S_{g.f.} = & \frac{1}{8} \int d^3x \epsilon^{\mu\nu\lambda} \int du (u_\mu^{+-} \partial_x^{--} + u_\mu^{--} \partial_x^{+-}) V^{++}(x, u) \partial_\nu \int dv (v_\lambda^{+-} \partial_x^{--} + v_\lambda^{--} \partial_x^{+-}) V^{++}(x, v) \\
& + \int dudx^{++} dx^{+-} L^{(-2)} \partial_x^{--} V^{++}.
\end{aligned} \tag{4.1}$$

The Lagrange multiplier $L^{(-2)}$ is analytic, $\partial_x^{++} L^{(-2)} = 0$.

In order to derive equations of motion from the action, we have to vary the fields in it. The variation of V^{++} in the first term deserves special attention. After some obvious steps we arrive at an expression of the type:

$$\int dud^3x \delta V^{++}(x^{++}, x^{+-}, u) F^{(-2)}(x, u), \tag{4.2}$$

where $F^{(-2)}$ is some non-analytic expression. In MC we can use the fact that δV^{++} does not depend on x^{--} to rewrite (4.2) in the form:

$$\int dudx^{++} dx^{+-} \delta V^{++}(x^{++}, x^{+-}, u) \left(\int_{-\infty}^{\infty} dx^{--} F^{(-2)} \right). \tag{4.3}$$

Then δV^{++} is an arbitrary function of the integration variables, so one can derive the equation of motion. Note that the latter will be manifestly analytic due to $\int dx^{--}$. In EC, where x^{--} is complex, it is hard to make sense of a procedure like this, and it is not clear how one can derive equations of motion for V^{++} . **This is the main reason why we have to discriminate EC in favor of MC.** On the other hand, in EC the harmonic functions

have harmonic expansions, which make manipulations with them easier. In MC one should avoid relying on arguments related to harmonic expansions. e

Let us now go back to the question of finding propagators for the action (4.1). It is clear that we need two of them, $\langle V^{++}V^{++} \rangle$ and $\langle V^{++}L^{(-2)} \rangle$. They should satisfy the following equations:

$$\partial_x^{--} \langle V^{++}L^{(-2)} \rangle = \delta_A^{(0,-2)}, \quad (4.4a)$$

$$\partial_x^{--} \langle V^{++}V^{++} \rangle = 0, \quad (4.4b)$$

obtained by varying $L^{(-2)}$, and

$$\int_{-\infty}^{\infty} dx^{--}(u)(u_{\mu}^{+-}\partial_x^{--} + u_{\mu}^{--}\partial_x^{+-})\epsilon^{\mu\nu\lambda}\partial_{\nu} \int dv(v_{\lambda}^{+-}\partial_x^{--} + v_{\lambda}^{--}\partial_x^{+-})\langle V^{++}L^{(-2)} \rangle = 0, \quad (4.4c)$$

$$\begin{aligned} & \frac{1}{4} \int_{-\infty}^{\infty} dx^{--}(u)(u_{\mu}^{+-}\partial_x^{--} + u_{\mu}^{--}\partial_x^{+-})\epsilon^{\mu\nu\lambda}\partial_{\nu} \int dv(v_{\lambda}^{+-}\partial_x^{--} + v_{\lambda}^{--}\partial_x^{+-})\langle V^{++}V^{++} \rangle \\ & -\partial_x^{--} \langle L^{(-2)}V^{++} \rangle = \delta_A^{(-4,2)}, \end{aligned} \quad (4.4d)$$

obtained by varying V^{++} , in addition to the usual analyticity conditions.

First we shall construct $\langle V^{++}L^{(-2)} \rangle$. Not surprisingly, it is related to the Green function $G^{(0,-2)}$ (3.12) for the gauge-fixed parameter λ :

$$\langle V^{++}L^{(-2)} \rangle = D_u^{++}G^{(0,-2)} \quad (4.5)$$

(see (3.15)). Indeed, (4.4a) is just (3.11), and (4.4c) can be checked by repeating the steps (2.31-32), using the analyticity of $G^{(0,-2)}$:

$$\int dv(v_{\lambda}^{+-}\partial_x^{--} + v_{\lambda}^{--}\partial_x^{+-})\langle V^{++}L^{(-2)} \rangle = -\partial_{\lambda} \int dv G^{(0,-2)} \quad (4.6)$$

which is annihilated by $\epsilon^{\mu\nu\lambda}\partial_{\nu}$ in (4.4c).

The propagator $\langle L^{(-2)}V^{++} \rangle = D_w^{++}G^{(-2,0)}(x,u|y,w)$ makes the following contribution to eq. (4.4d):

$$\begin{aligned} -\partial_x^{--} \langle L^{(-2)}V^{++} \rangle &= \delta^{(-2,2)}(u,w) \int_{-\infty}^{\infty} dx^{--}(u)\delta^3(x-y) \\ &+ \frac{i}{\pi}(u^+w^-)(u^-w^+)^3 \int_{-\infty}^{\infty} dx^{--}(u)\delta''(x^{+-}-y^{+-})(w). \end{aligned} \quad (4.7)$$

The first term in (4.7) is just $\delta_A^{(-4,2)}$, so we have to find a propagator $\langle V^{++}V^{++} \rangle$ which will cancel the second term in (4.7) after substitution in (4.4d). This turns out to be:

$$\langle V^{++}V^{++} \rangle = -\frac{1}{\pi^2} \int d^3\xi \delta(x^{+-} - \xi^{+-})(u) u^{++\mu} v^{++\nu} \epsilon_{\mu\nu\lambda} \partial^\lambda \delta(\xi^{+-} - y^{+-})(v). \quad (4.8)$$

It depends only on $x^{+-}(u)$ and $y^{+-}(v)$, so analyticity and (4.4b) are obvious. Note also the symmetry of expression (4.8), as was to be expected. Eq. (4.8) can be rewritten in a different form by decomposing ∂^λ in the u basis, pulling it out of the integral and keeping only the $\partial_x^{+-}(u)$ term (see (1.14) and (2.27)):

$$\langle V^{++}V^{++} \rangle = -\frac{2}{\pi^2} (u^+ v^+)^2 \int d^3\xi \delta'(x^{+-} - \xi^{+-})(u) \delta(\xi^{+-} - y^{+-})(v). \quad (4.9)$$

Further, we can pull out a D_u^{++} from (4.9) in the same way as we did it in (2.29):

$$\langle V^{++}V^{++} \rangle = -\frac{2}{\pi^2} D_u^{++} \left[(u^+ v^+)^2 \int d^3\xi \frac{\delta(x^{+-} - \xi^{+-})}{x^{++} - \xi^{++}}(u) \delta(\xi^{+-} - y^{+-})(v) \right]. \quad (4.10)$$

Next, as in (2.31-32), we find for the integral $\int dv$ in (4.4d) the following (we use an intermediate harmonic variable $t^{++\lambda}$):

$$\begin{aligned} & t^{++\lambda} \int dv (v_\lambda^{+-} \partial_x^{--} + v_\lambda^{--} \partial_x^{+-}) \langle V^{++}(x, v) V^{++}(y, w) \rangle \\ &= \int dv t^{++\lambda} \frac{t^+ v^-}{t^+ v^+} \partial_\lambda \langle V^{++}(x, v) V^{++}(y, w) \rangle \\ &= -\frac{2}{\pi^2} \int dv t^{++\lambda} [\delta^{(0,0)}(t, v) - 1] (v^+ w^+)^2 \partial_\lambda \int d^3\xi \frac{\delta(x^{+-} - \xi^{+-})}{x^{++} - \xi^{++}}(v) \delta(\xi^{+-} - y^{+-})(w) \\ &= \frac{-4i}{\pi^2} (t^+ w^+)^2 \delta(x^{+-} - y^{+-})(w) - \partial^{++} \tau(x - y, w) \\ &= t^{++\lambda} \left[\frac{2i}{\pi} w_\lambda^{++} \delta(x^{+-} - y^{+-})(w) + \frac{1}{2} \partial_\lambda \tau \right]. \end{aligned} \quad (4.11)$$

Now we can remove the auxiliary variable $t^{++\lambda}$ and obtain:

$$\begin{aligned} & \epsilon^{\mu\nu\lambda} \partial_\nu \int dv (v_\lambda^{+-} \partial_x^{--} + v_\lambda^{--} \partial_x^{+-}) \langle V^{++}(x, v) V^{++}(y, w) \rangle \\ &= \frac{2i}{\pi} \epsilon^{\mu\nu\lambda} \partial_\nu [w_\lambda^{++} \delta(x^{+-} - y^{+-})(w)] \\ &= -\frac{2i}{\pi} \delta'(x^{+-} - y^{+-})(w) w^{++\mu}. \end{aligned} \quad (4.12)$$

Finally, inserting (4.12) into (4.4d), decomposing $\partial_x(u)$ in the w basis and using the fact that only $x^{+-}(w)$ is present in (4.11), we find that the first term in (4.4d) gives:

$$-\frac{i}{2\pi} \int_{-\infty}^{\infty} dx^{--}(u) [(u^{+-\mu} w_{\mu}^{++})(u^{--\nu} w_{\nu}^{+-}) + (u^{--\mu} w_{\mu}^{++})(u^{+-\nu} w_{\nu}^{+-})] \delta''(x^{+-} - y^{+-})(w). \quad (4.13)$$

Using (2.27), we see that (4.12) does indeed cancel the second term in (4.7).

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