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CHIRAL CORRELATORS IN LANDAU-GINSBURG THEORIES AND $N = 2$ SUPERCONFORMAL MODELS

P.S.Howe* and P.C.West*
CERN, Geneva, Switzerland

Abstract

Chiral correlation functions are computed in $N = 2$ Landau-Ginsburg models using the ϵ -expansion and the superconformal Ward identities for the Landau-Ginsburg effective action. They are also computed directly using superconformal model techniques. The same results are obtained yielding further confirmation of the identification of superconformal minimal models with Landau-Ginsburg models evaluated at their fixed points. The formulae for the chiral commutators that we compute are extremely simple when expressed in terms of effective actions.

* Permanent address: Dept. of Mathematics, King's College, London, U.K.

It has been conjectured that the minimal non-supersymmetric and $N = 1$ conformal models [1] in two dimensions are described by fixed points of Landau-Ginsburg (LG) Hamiltonians with ϕ^{2n} potentials [2]. This conjecture has been extended [3] to $N = 2$ superconformal models [4,5]. It is also understood within the context of conformal minimal models how to flow from one model to another [6]. However, these calculations of renormalisation group flows do not provide any direct evidence for the identification of the LG and conformal models. Moreover, the required fixed points lie outside the range of conventional $g = 0$ perturbation theory in two dimensions, so that other methods must be used.

For the $N = 2$ supersymmetric case it has been explicitly demonstrated [7] that LG models do possess non-trivial fixed points and that the conformal weights of chiral fields, their operator product expansions and the central charges c , evaluated at this point, are in exact agreement with those of $N = 2$ minimal models, thereby establishing the connection between LG models and ($N = 2$) minimal models. These results were found using the ϵ -expansion combined with superspace techniques, a method which guarantees the validity of the $N = 2$ non-renormalisation theorem.

In refs. [8,9,10] some interesting relations between $N = 2$ superpotentials and singularity theory were observed; in particular, it was noted that the singularities of the simplest such potentials fall into an ADE classification as does a class of $N = 2$ superconformal models. Alternative derivations of the central charge were also given [10,11], using $N = 2$ superconformal representation theory and assuming the existence of an infra-red stable fixed point in the LG model and a type of non-renormalisation theorem.

In this letter we compute chiral correlation functions starting from an $N = 2$ LG action.* This is first done using standard techniques of renormalised perturbation theory and the ϵ -expansion. We then use a less direct method. The results of ref.[7] guarantee the existence of a non-trivial infra-red stable fixed point in the LG model. This implies that the theory is dilation invariant at that point with the conformal weight of the field being the sum of its naive and anomalous dimensions. Correspondingly, the effective action will obey a dilation Ward identity which can be derived in the usual way from the path integral. Since dilation invariance implies conformal invariance † we can deduce from the ϵ -expansion the existence of an effective action (in two dimensions) obeying the (global) superconformal Ward identities. These allow us to determine chiral irreducible vertices uniquely. The Green functions are then constructed by attaching appropriate propagators onto the external lines. These propagators are simply the renormalisation group summed propagators evaluated at the fixed point, and are nothing other than the superconformal two-point functions. This method can be extended to include the other chiral operators in the theory.

Correlation functions of minimal models can also be calculated directly using group theoretic arguments alone [1]. We calculate the two- and three-point functions for minimal

* In this paper, as opposed to ref. [7], we use field theory terminology. The classical (effective) action in field theory corresponds to the Hamiltonian (free energy) in statistical mechanics.

† This is true for the models under consideration; subtleties can arise in certain two-dimensional non-linear sigma models [12].

$N = 2$ superconformal models. Using the differential equations corresponding to the particular null state that exists for a chiral primary state, together with the (global) superconformal Ward identities, we compute the Green functions for the chiral primary fields. These differential equations are none other than the requirement that the amplitudes be chiral.

The calculations within the context of conformal models are clearly related to the LG calculations of the same quantities, both making extensive use of the conformal Ward identities. In the LG models we start from an action and derive the Ward identities for the effective action using the usual path integral techniques. By contrast, the strategy for minimal models pioneered by BPZ [1] does not presuppose the existence of an action but relies on group theoretical methods which are applied directly to the Green functions. The results for the LG field theory and conformal model calculations are in agreement, as one would expect in view of the correspondence established in [7].

The renormalised action for an $N = 2$ LG model is given by*

$$S = - \int d^d x d^4 \theta \bar{\phi}_o \phi_o - (g_o \int d^d x d^2 \theta \frac{1}{n!} \phi_o^n + c.c.) \quad (1)$$

where $d = d_c - \epsilon$, $d_c (= 2 + \frac{2}{(n-2)})$ is the dimension in which g_o is a dimensionless coupling, and where the bare quantities g_o and ϕ_o are related to the renormalised ones in the usual way,

$$\phi_o = Z_\phi^{\frac{1}{2}} ; g_o = \mu^{\frac{(n-2)\epsilon}{2}} Z_g g \quad (2)$$

the Z 's being the renormalisation constants. Our conventions and a discussion of regularisation are given in ref. [7]. The $N = 2$ non-renormalisation theorem is valid in the ϵ -expansion and implies that

$$Z_g Z_\phi^{\frac{n}{2}} = 1 \quad (3)$$

from which the anomalous dimension of ϕ , γ_ϕ , can be expressed in terms of the β -function. There is an infra-red stable fixed point at $g^* = O(\epsilon)$ at which $\gamma_\phi = \frac{1}{n}$ [7].

The lowest order contribution to the chiral part of the effective potential is the tree-level contribution which is simply $\int d^d x d^2 \theta \frac{g_o}{n!} \phi_o^n$. The next order contribution to the n -point chiral irreducible vertex occurs at three loops for $n = 3$ and $n - 2$ loops for $n \geq 4$. The Feynman diagrams for the various cases are given in figs. 1, 2 and 3. We calculate the graph of figure 2 corresponding to n even. It yields a contribution to the effective action of the form

$$\int \prod_{i=1}^{\frac{n}{2}} \frac{d^d p_i}{(2\pi)^d} \frac{d^d q_i}{(2\pi)^d} \int d^4 \theta_1 d^4 \theta_2 d^4 \theta_3 \prod_{i=1}^{\frac{n}{2}} \phi(p_i, \theta_1) \phi(q_i, \theta_3) \delta(p + q) \times$$

$$\int \prod_{i=1}^{\frac{n}{2}} \frac{d^d k_i}{(2\pi)^d} \frac{d^d l_i}{(2\pi)^d} \frac{1}{(k_i)^2 (l_i)^2} \frac{1}{(k + p)^2 (l + q)^2} \delta_{12}^{(4)} (\bar{D}_1^2 D_2^2 \delta_{12}^{(4)})^{\frac{n}{2}-1} D_2^2 \delta_{23}^{(4)} (D_2^2 \bar{D}_3^2 \delta_{23}^{(4)})^{\frac{n}{2}-1} \quad (4)$$

* In this paper we consider only the A series of $N = 2$ models which are described by the action (1).

where $p = \sum_{i=1}^n p_i$ and $q = \sum_{i=1}^n q_i$, and similarly for k and l . The D -algebra can be easily evaluated yielding

$$\begin{aligned}
& \int \prod_{i=1}^n \frac{d^d p_i}{(2\pi)^d} \frac{d^d q_i}{(2\pi)^d} \int d^4 \theta D^2 \left\{ \prod_{i=1}^n \phi(p_i, \theta) \right\} \prod_{i=1}^n \phi(q_i, \theta) \times \\
& \quad \left\{ \int \prod_{i=1}^n \frac{d^d k_i}{(2\pi)^d} \frac{1}{(k_i)^2} \frac{1}{(k+p)^2} \right\}^2 \delta(p+q) \\
& \propto \int \prod_{i=1}^n \frac{d^2 p_i}{(2\pi)^2} \frac{d^2 q_i}{(2\pi)^2} \delta(p+q) \int d^4 \theta \frac{D^2}{p^2} \prod_{i=1}^n \phi(p_i, \theta) \phi(q_i, \theta) + O(\epsilon) \\
& \propto \int d^2 x d^2 \theta \phi^n + O(\epsilon)
\end{aligned} \tag{5}$$

Thus finite local quantum corrections to the effective potential can arise although one might expect that there would also be non-local contributions. The fact that contributions to the effective action have to be integrals over the full superspace is used to show that no divergences in the form of chiral integrals can arise because the divergences in a supersymmetric theory must be local on the full superspace [13]. Finite corrections having the same form as the potential term in the original action are allowed as they arise from non-local expressions in the full superspace, as we have seen.

Local contributions to the effective action of this type lead to corrections to the potential term in the classical action. In principle, there could also be non-local contributions from higher order graphs. However, as we shall see below, there are in fact no non-local contributions to the chiral n -point vertex at the fixed point in the ϕ^n LG model. (The chiral m -point vertices with $m < n$ vanish.) Thus the chiral n -point vertex has the same form as the tree-level n -point vertex but (probably) with a different coefficient. The Green function is obtained by attaching propagators to the external lines. The appropriate propagator is the renormalisation group summed propagator evaluated at the fixed point. In the ϕ^n theory it is given by

$$\langle T(\bar{\phi}(x_1, \theta_1) \phi(x_2, \theta_2)) \rangle = \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot (x_1 - x_2)} \frac{D_1^2(p) \bar{D}_2^2(-p) \delta_{12}^{(4)}}{(p^2)^{1-\gamma}} \tag{6}$$

where $\theta_{ij} = \theta_i - \theta_j$, $\delta_{12}^{(4)} (= \bar{\theta}_{12}^2 \theta_{12}^2)$ is the δ -function in superspace and the covariant derivative in momentum space is

$$D_\alpha(p) = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} (\gamma^a)_{\alpha\beta} \bar{\theta}^\beta p_a \tag{7}$$

The anomalous dimension of ϕ is denoted by γ and is equal to $\frac{1}{n}$ at the fixed point. Putting this altogether we find for the n -point chiral correlator

$$G(x_i, \theta_i) \equiv \langle T(\prod_{i=1}^n \phi(x_i, \theta_i)) \rangle$$

$$= \int \prod_{i=1}^n \left(\frac{d^2 p_i e^{i p_i x_i}}{(p_i^2)^{1-\gamma}} \right) \delta \left(\sum_i p_i \right) \prod_{i=1}^n \bar{D}_i^2(p_i) \{ D_1^2(-p_2) \delta_{12}^{(4)} \dots D_1^2(-p_n) \delta_{1n}^{(4)} \} \quad (8)$$

This expression factorizes, $G = G^L G^R$, where G^L depends only on the left-handed variables $Z_i = (z_i, \theta_i^+, \bar{\theta}_i^+)$, and G^R depends on the right-handed ones, $\bar{Z}_i = (\bar{z}_i, \theta_i^-, \bar{\theta}_i^-)$. It is also explicitly chiral on all legs because of the presence of the $\bar{D}^2$'s. Taking this into account we can recast G^L into the form

$$G^L(Z_1, \dots, Z_n) \propto \int \prod_{i=1}^n \left(\frac{d p_i e^{i p_i y_i}}{(p_i)^{1-\gamma}} \right) \delta \left(\sum_i p_i \right) \sum_i \theta_i^+ p_i \quad (9)$$

where $y_i = z_i + \frac{1}{2} \theta_i^+ \bar{\theta}_i^+$ is a chiral variable, $\bar{D}_+ y_i = 0$.

The conformal invariance of the theory at the fixed point is expressed by the conformal Ward identities for the effective action which read

$$\left(\int d^2 x d^2 \theta \delta_{F, \bar{F}} \phi(x, \theta) \frac{\delta}{\delta \phi(x, \theta)} + c.c. \right) \Gamma[\phi, \bar{\phi}] = 0 \quad (10)$$

where

$$\delta_{F, \bar{F}} \phi = \{ F \partial_z + \bar{D}_+ F D_+ + \gamma \bar{D}_+ D_+ F \} \phi + \{ \bar{F} \partial_{\bar{z}} + \bar{D}_- \bar{F} D_- + \gamma \bar{D}_- D_- \bar{F} \} \phi \quad (11)$$

The functions F , which are the vector components of the vector superfields which generate the superconformal diffeomorphisms, are given by

$$F = (1, z, z^2, \sqrt{2} \bar{\theta}^+, \sqrt{2} \theta^+, \sqrt{2} z \bar{\theta}^+, \sqrt{2} z \theta^+, \bar{\theta}^+ \theta^+) \quad (12)$$

and correspond to translations, rotations, scale transformations, special conformal transformations, supersymmetry and special supersymmetry transformations and U(1) transformations respectively. For the chiral n -point vertex function, Γ^n , in the ϕ^n model the Ward identities take the form

$$\left\{ \sum_i F_i \partial_i + \bar{D}_i^+ + (1 - \gamma) \bar{D}_i^+ D_i^+ F_i + \bar{F} \text{ terms} \right\} \Gamma^n = 0 \quad (13)$$

The unique solution to these equations is simply given by

$$\Gamma^n \propto \bar{D}_2^2 \delta_{12}^{(6)} \dots \bar{D}_n^2 \delta_{1n}^{(6)} \quad (14)$$

where $\delta^{(6)}$ denotes the δ -function on the full superspace. This corresponds to a contribution to the effective action of the form $\int d^2 x d^2 \theta \phi^n$ because the functional derivative for chiral fields has the basic property that

$$\frac{\delta}{\delta \phi(x_1, \theta_1)} \phi(x_2, \theta_2) = \bar{D}_2^2 \delta_{12}^{(6)} \quad (15)$$

The n -point chiral correlation function is found from Γ^n by sewing on external propagators as above; it is given in equation (8). However, in this way of looking at things, the propagator is determined by conformal invariance. It is still given by equation (6), of course.

This result can be generalised to include the set of chiral operators $\{\phi^2, \dots, \phi^{(n-2)}\}$. (Higher powers of ϕ are not independent operators due to the equation of motion.) To do this we first add to the action (1) source terms for the chiral composites,

$$S(K) = \int d^2x d^2\theta \sum_{p=2}^{(n-2)} K_p \frac{\phi^p}{p!} \quad (16)$$

Note that the non-renormalisation theorem also protects these terms so that (16) is renormalised as it stands. The generating functional for connected Green functions, $W(J, K_p)$, is given by

$$e^{W(J, K_p)} = \int [d\phi][d\bar{\phi}] \exp\{-S + ((S(K) + \int J\phi) + \text{c.c.})\} \quad (17)$$

where the source for ϕ is also chiral (as are the K 's.) We can define a generalised effective action, $\tilde{\Gamma}$, by Legendre transforming with respect to all the sources,

$$W(K_r) + \tilde{\Gamma}(\chi_r) = \sum_{r=1}^{(n-2)} \left(\int K_r \chi_r + \text{c.c.} \right) \quad (18)$$

The fields χ_r are the expectation values of the operators (with $\chi_1 \equiv \phi$). The superconformal Ward identities for $\tilde{\Gamma}$, evaluated at the fixed point, are

$$\left(\int d^2x d^2\theta \sum_r \delta_{F, \bar{F}} \chi_r(x, \theta) \frac{\delta}{\delta \chi_r(x, \theta)} + \text{c.c.} \right) \tilde{\Gamma} = 0 \quad (19)$$

where the variations of the fields are now

$$\delta_{F, \bar{F}} \chi_r = \{F \partial_z + \bar{D}_+ F D_+ + \gamma_r \bar{D}_+ D_+ F + \bar{F} \text{ terms}\} \chi_r \quad (20)$$

The anomalous dimensions are given by [7] $\gamma_r = \frac{r}{n}$ and the F 's are given in equation (12). The $U(1)$ charge of the field χ_r is also $\frac{r}{n}$. Consider now a chiral vertex $\tilde{\Gamma}^l$ where the sum of the $U(1)$ charges of the l fields is one. The Ward identities for this vertex are the same as in equation (13), but with the appropriate γ_r depending on the field involved. It is not difficult to verify that the solution to these Ward identities is again given by equation (14), with $l-1$ factors instead of $n-1$. This corresponds to a contribution to $\tilde{\Gamma}$ of the form $\int d^2x d^2\theta \chi_{r_1} \dots \chi_{r_l}$. To get the Green functions we again attach propagators which are the two-point functions for the composite operators evaluated at the fixed point,

$$\langle T(\bar{\phi}^r(x_1, \theta_1) \phi^s(x_2, \theta_2)) \rangle = \delta_{r,s} \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot (x_1 - x_2)} \frac{D_1^2(p) \bar{D}_2^2(-p) \delta_{12}^{(4)}}{(p^2)^{1-\gamma_r}}, \quad r, s = 1, \dots, (n-2) \quad (21)$$

The result for the Green function, $\tilde{G}$, is similar to equation (8) with appropriate anomalous dimensions. Such a Green function also factorises. The formula for the left-handed part is

$$\tilde{G}^L(Z_1, \dots, Z_l) \propto \int \prod_{i=1}^l \left(\frac{dp_i e^{ip_i y_i}}{(p_i)^{1-\gamma_i}} \right) \delta\left(\sum_i p_i\right) \sum_i \theta_i^+ p_i \quad (22)$$

We now turn to the superconformal model approach.* A primary field $\varphi(Z)$, (not necessarily chiral), with superconformal weights $(H, \bar{H})$ and chiral $U(1)$ charges $(T, \bar{T})$ transforms under a left-handed superconformal transformation by [4]

$$\delta_F \varphi = F \partial_z \varphi + \bar{D} F D \varphi + D F \bar{D} \varphi + H \partial_z F \varphi - \frac{T}{2} [D, \bar{D}] F \varphi \quad (23)$$

The functions F for the global transformations are given in (9); in general they are,

$$F = (z^{n+1}, \sqrt{2} z^{r+\frac{1}{2}} \bar{\theta}, \sqrt{2} z^{r+\frac{1}{2}} \theta, \theta \bar{\theta} z^n) \quad (24)$$

corresponding to the generators $(L_n, G_r, \bar{G}_r$ and $T_n)$ of the superconformal algebra. Substituting these F 's into the formula for the variation of ϕ we obtain differential operators which we shall denote by $\mathcal{L}_n, \mathcal{G}_r, \bar{\mathcal{G}}_r$ and $\mathcal{T}_n$. For example,

$$\mathcal{L}_0 = z \partial_z + \frac{1}{2} \theta \partial_\theta + \frac{1}{2} \bar{\theta} \partial_{\bar{\theta}} + H \quad (25)$$

Consider now an n -point correlation function; we denote its left-handed factor by $G(Z_1, \dots, Z_n)$. The invariance of the vacuum under the globally defined superconformal transformations implies that G satisfies the differential equations

$$\sum_{i=1}^n \Delta_i G(Z_1, \dots, Z_n) = 0 \quad (26)$$

where Δ is any one of the operators $\mathcal{L}_0, \mathcal{L}_{\pm 1}, \mathcal{G}_{\pm \frac{1}{2}}, \bar{\mathcal{G}}_{\pm \frac{1}{2}}$ or $\mathcal{T}_0$.

The two- and three-point functions in $N=0$ and $N=1$ conformal models are well-known to be determined by global (superconformal) invariance [14]. As expected this extends to the case $N=2$. For primary fields φ_1, φ_2 , one finds that the two-point function vanishes unless $H_1 = H_2$ and $T_1 = -T_2$, in which case it has the form

$$\frac{1}{(s_{12})^{H_1+H_2}} \left(1 - \frac{T_1}{2} \frac{\theta_{12} \bar{\theta}_{12}}{s_{12}} \right) \quad (27)$$

where

$$s_{ij} = z_i - z_j - \frac{1}{2} \theta_i \bar{\theta}_j - \frac{1}{2} \bar{\theta}_i \theta_j$$

* In this part of the paper we consider fields which depend only on the left-handed co-ordinates $Z = (z, \theta^+, \bar{\theta}^+)$. Since θ^- does not occur, we make the replacements $\theta^+ \rightarrow \theta, D_+ \rightarrow D, \bar{D}_+ \rightarrow \bar{D}$.

$$\theta_{ij} = \theta_i - \theta_j \quad ; \quad \bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j \quad (28)$$

The three point functions can be written in terms of the variables s_{ij} , θ_{ij} and $\bar{\theta}_{ij}$. The possible values of $T \equiv \sum_{i=1}^3 T_i$ are 0, ± 1 and ± 2 , because there are only two independent θ_{ij} 's and $\bar{\theta}_{ij}$'s. However, it is easy to verify that $T = \pm 2$ is not compatible with equations (13). When $T = 0$ the three point function has the form

$$\frac{1}{(s_{12})^{H_1+H_2-H_3}(s_{23})^{H_2+H_3-H_1}(s_{31})^{H_3+H_1-H_2}} \times \\ \left(1 + \frac{(T_2 - T_1)}{6} \frac{\theta_{12}\bar{\theta}_{12}}{s_{12}}\right) \left(1 + \frac{(T_3 - T_2)}{6} \frac{\theta_{23}\bar{\theta}_{23}}{s_{23}}\right) \left(1 + \frac{(T_1 - T_3)}{6} \frac{\theta_{31}\bar{\theta}_{31}}{s_{31}}\right) \quad (29)$$

When $T = +1$ we find the result

$$\left(s_{12}\theta_3 + s_{23}\theta_1 + s_{31}\theta_2 - \frac{1}{6}\theta_1(\theta_2 + \theta_3)\bar{\theta}_{23} - \frac{1}{6}\theta_2(\theta_3 + \theta_1)\bar{\theta}_{31} - \frac{1}{6}\theta_3(\theta_1 + \theta_2)\bar{\theta}_{12}\right) \times \\ \frac{1}{(s_{12})^{H_1+H_2-H_3+\frac{1}{2}}(s_{23})^{H_2+H_3-H_1+\frac{1}{2}}(s_{31})^{H_3+H_1-H_2+\frac{1}{2}}} \times \\ \left(\frac{(T_2 - T_1)}{6} \frac{\theta_{12}\bar{\theta}_{12}}{s_{12}} + \frac{(T_3 - T_2)}{6} \frac{\theta_{23}\bar{\theta}_{23}}{s_{23}} + \frac{(T_1 - T_3)}{6} \frac{\theta_{31}\bar{\theta}_{31}}{s_{31}}\right) \quad (30)$$

The higher point functions cannot in general be completely determined by purely group theoretic means. An exception is in rational conformal theories due to the existence of null states which lead to further differential equations [1]. An ansatz which solves these equations was given for the non-supersymmetric case in ref.[15] and discussed for some $N = 2$ correlators in ref.[4]. We now consider correlators for primary fields which have a given chirality. Such primary fields have $H = \pm \frac{T}{2}$ and define highest weight states. Applying either $G_{-\frac{1}{2}}$ or $\bar{G}_{-\frac{1}{2}}$ to these states we get null states. We concentrate on the chiral case ($H = \frac{T}{2}$). The corresponding highest weight state is $|H, T\rangle = |H, 2H\rangle$ such that $G_{-\frac{1}{2}}|H, 2H\rangle$ is null and hence vanishes in a unitary theory. The primary descendant field for this null state is

$$\Phi(Z) = G_{-\frac{1}{2}}(Z)\varphi_{H,2H}(Z) \quad (31)$$

where

$$G_{-\frac{1}{2}}(Z) = \oint_Z dZ' J(Z')(\theta' - \theta) \quad (32)$$

and $J(Z)$ is the supercurrent. We have

$$\langle \Phi(Z)\varphi_1(Z_1), \dots, \varphi_n(Z_n) \rangle = 0 \quad (33)$$

Using the definition of Φ and the *OPE* of J with the primary fields, φ_i [4], we find that chiral correlators must satisfy

$$\bar{D}_i G(Z_1, \dots, Z_n) = 0 \quad \forall i \quad (34)$$

as well as the differential equations (26). These are sufficient to determine these functions completely. Equation (34) implies that G is a function of the variables $y_i = z_i + \frac{i}{2}\theta_i\bar{\theta}_i$ and θ_i alone. The $\mathcal{L}_{-1}$ equation implies that $G = G(y_{ij}, \theta_i)$, where $y_{ij} = y_i - y_j$. In terms of these variables the remaining equations become

$$\begin{aligned}
\sum_i T_o^i G &= \sum_i (-\theta_i \frac{\partial}{\partial \theta_i} + T_i) G = 0 \\
\sum_i \mathcal{L}_o^i G &= \left\{ \sum_i y_i \frac{\partial}{\partial y_i} + \sum_i \left(\frac{1}{2} \theta_i \frac{\partial}{\partial \theta_i} + H_i \right) \right\} G = 0 \\
\sum_i \mathcal{L}_1^i G &= \left\{ \sum_i (y_i)^2 \frac{\partial}{\partial y_i} + \sum_i (y_i \theta_i \frac{\partial}{\partial \theta_i} + 2H_i y_i) \right\} G = 0 \\
\sum_i \mathcal{G}_{-\frac{1}{2}}^i G &= \sum_i \frac{\partial}{\partial \theta_i} G = 0 \\
\sum_i \bar{\mathcal{G}}_{-\frac{1}{2}}^i G &= - \sum_i \theta_i \frac{\partial}{\partial y_i} G = 0 \\
\sum_i \mathcal{G}_{+\frac{1}{2}}^i G &= \sum_i y_i \frac{\partial}{\partial \theta_i} G = 0 \\
\sum_i \bar{\mathcal{G}}_{+\frac{1}{2}}^i G &= - \sum_i (y_i \theta_i \frac{\partial}{\partial y_i} + 2H_i \theta_i) G = 0
\end{aligned} \tag{35}$$

where $\frac{\partial}{\partial \theta}$ is evaluated at constant y . The T_o equation implies that $\sum_i T_i \in \mathbb{Z}$. When the total charge is +1 G has the form

$$G = \sum_{i=1}^n \theta_i A_i \tag{36}$$

where the A_i 's are functions of the y_{ij} 's. The $\mathcal{L}_o$ equation then tells us that the A_i 's are homogeneous functions of degree -1. The $\mathcal{G}_{-\frac{1}{2}}$ and $\bar{\mathcal{G}}_{-\frac{1}{2}}$ equations then yield

$$\sum_{i=1}^n A_i = 0 \quad \text{and} \quad \partial_i A_j - \partial_j A_i = 0 \tag{37}$$

so that

$$A_i = \partial_i A \quad \text{and} \quad \sum_{i=1}^n \partial_i A = 0 \tag{38}$$

The $\mathcal{G}_{+\frac{1}{2}}$ equation tells us that A is a homogeneous function of degree zero while the $\bar{\mathcal{G}}_{+\frac{1}{2}}$ equation is

$$y_{ij} \partial_i \partial_j A = (H_j \partial_i - H_i \partial_j) A \tag{39}$$

The solution to (39) is most easily expressed as a Fourier transform from momentum space. It is

$$A = \int \left(\prod_{i=1}^n \frac{dp_i}{(p_i)^{1-2H_i}} \right) \delta\left(\sum_i p_i\right) e^{i \sum_i p_i y_i} \tag{41}$$

It is easy to see that this satisfies the remaining equation and is unique. The corresponding correlation function is

$$G(Z_1, \dots, Z_n) = \int \left(\prod_{i=1}^n \frac{dp_i}{(p_i)^{1-2H_i}} \right) \delta\left(\sum_i p_i\right) e^{ip_i y_i} \sum_i \theta_i p_i \quad (42)$$

This is the same as equation (22), with $\gamma_i \rightarrow 2H_i$.

In summary, we have calculated the general two- and three-point functions for $N = 2$ minimal models and chiral correlators for fields whose $U(1)$ charges sum to one. The LG field theory approach and the BPZ conformal model approach are similar, being related to one another by Legendre transformation. A feature of the LG approach is that it allows the construction of effective actions for conformal models. We have seen that momentum space techniques simplify the construction of the amplitudes and that the irreducible chiral vertices are extremely simple. We note that the calculation of the n -point chiral vertex in the ϕ^n model given here extends the result given in ref.[16] for the three-point vertex in the $d = 4$ Wess-Zumino model some time ago. However, three-point vertices can always be calculated in a conformal theory (in any dimension) whereas the calculation of the n -point vertices is rendered simple because of chirality. It should be possible to extend the results given here to other chiral vertices and to include non-chiral operators. One might hope in this way to be able to construct an effective action for all the primary fields of a minimal $N = 2$ superconformal model.

Note added

While we were writing up this work we received a preprint by Cecotti, Girardello and Pasquinucci [17] in which an alternative derivation of the link between the $N = 2$ models and LG field theory is given.

References

- [1] A.Belavin, A.Polyakov and A.Zamolodchikov, *Nucl.Phys.* **B241** (1984) 333.
- [2] A.Zamolodchikov, *Sov.J.Nucl.Phys.* **44** (1986) 529.
- [3] D.Kastor, E.Martinec and S.Shenker, "RG flow in $N = 1$ discrete series" EFI preprint 88-76 (1988)
- [4] P.DiVecchia, J.L.Petersen and H.B.Zheng, *Phys.Lett.* **B162** (1985) 327; P.DiVecchia, J.L.Petersen and M.Yu, *Phys.Lett.* **B162** (1986) 211; P.DiVecchia, J.L.Petersen, M.Yu and H.B.Zheng, *Phys.Lett.* (1986) 220.
- [5] M.Ademollo et al, *Phys.Lett.* **B94**; V.A.Fateev and A.B.Zamolodchikov, *Sov.Phys. JETP* **62** (1985) 215; W.Boucher, D.Friedan and A.Kent, *Phys.Lett.* **B162** (1986) 316.
- [6] J.Cardy and A.Ludwig, *Nucl.Phys.* **B285** (1987) 687; A.Zamolodchikov, *Sov.J.Nucl. Phys.* **46** (1987) 1090.
- [7] P.S.Howe and P.C.West, " $N = 2$ Superconformal Models, Landau-Ginsburg Hamiltonians and the ϵ -Expansion", CERN preprint TH.5318/89 (1989).
- [8] E.Martinec, "Algebraic Geometry and Effective Lagrangians", EFI preprint 88-76 (1988).
- [9] C.Vafa and N.Warner, "Catastrophes and the Classification of Conformal Theories", Harvard preprint HUTP-88/A047 (1988); B.Greene, C.Vafa and N.Warner, Harvard preprint HUTP-88/A047 (1988).
- [10] W.Lerche, C.Vafa and N.Warner, "Chiral Rings in $N = 2$ Superconformal Theories", Harvard preprint HUTP-89/A065 (1989).
- [11] E.Martinec, seminar at CERN (February 1989).
- [12] C.M.Hull and P.K.Townsend, *Nucl.Phys.* **B274** (1986) 349.
- [13] J.Wess and B.Zumino, *Phys.Lett.* **B49** (1974) 52; J.Iliopoulos and B.Zumino, *Nucl. Phys.* **B76** (1974) 310; D.M.Capper, *Nuovo Cimento* **25A** (1975) 646; B.Zumino, *Nucl.Phys.* **B89** (1975) 515; P.West, *Nucl.Phys.* **B106** (1976) 219; D.Capper and M.Ramon Medrano, *J.Phys A* (1976) 269. M.Grisaru, M.Roczek and W.Siegel, *Nucl. Phys.* **B159** (1979) 429.
- [14] A.M.Polyakov, *JETP Lett.* **12** (1970) 381; *Zh. JETP* **39** (1974) 10.
- [15] V.S.Dotsenko and V.A.Fateev, *Nucl.Phys.* **B240** (1984) 312.
- [16] S.Ferrara, J.Iliopoulos and B.Zumino, *Nucl.Phys.* **B77** (1974) 413.
- [17] S.Cecotti, L.Girardello and A.Pasquinucci, "Non-perturbative Aspects and Exact Results for the $N = 2$ Landau-Ginsburg Models.", Trieste/Milan preprint SISSA 51-89/EP, IFUM 364/FT (1989).

Figures

Fig. 1

Fig. 2

Fig. 3