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The Exclusive Exponentiation in the Monte Carlo - The Case of The Initial State Bremsstrahlung

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ABSTRACT

The structure of the QED infrared singularities is reviewed and the various types of the Monte Carlo approaches to deal with them are briefly described. It is shown how to exponentiate the lowest first and second-order QED calculation following the Yennie-Frautschi-Suura prescription and how to implement the corresponding multiphoton distributions in the Monte Carlo.

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1. Introduction

The problem of the exponentiation of the infrared divergences in the framework of the conventional Quantum Electrodynamics is discussed in a most complete way in the classical paper of Yennie, Frautschi and Suura [1] and the work presented in this talk relies heavily on the result obtained there. The casual reader of the above and other papers on exponentiation, see also ref. [2], may think, however, that these papers deal only with the inclusive integrated cross section. We shall spend, therefore, the first part of the talk on demonstrating what are the exclusive multiphoton distributions and we shall show that in fact they are well known in the framework of the ref. [1]. For pedagogical reasons we shall do it first on the simplified one-dimensional examples. The corresponding Monte Carlo generation algorithms will be discussed briefly. Then we shall show what are the main ingredients in the full scale *second order* exclusive exponentiation, i.e. what are the corresponding multiphoton distributions. A particular emphasis will be put on explaining what is the practical difference between the lowest, first and second order exponentiation. The talk will be concluded with some limited discussion on the second order Monte Carlo program (for the fermion production process in e^+e^- annihilation) and the corresponding numerical results. Let us note finally that more detailed information on the presented results and ideas can be found in refs. [3,4,5].

2. Infrared divergences - the Toy Model

One way of characterizing various Monte Carlo programs for QED calculations is to examine how they treat the virtual and real infrared divergences. In the following we shall describe in a simplified "pedestrian" way what is the structure of the infrared divergent poles in the QED distribution. We shall start with the first and second order (single and double bremsstrahlung) cases and then we shall go to the infinite order case (exponentiation). The corresponding Monte Carlo algorithms will be briefly discussed.

Let us start with the well-known $O(\alpha)$ case. The main purpose of this will be the introduction of the notation which will be used in more complicated second and infinite order cases. In the following discussion we shall limit ourselves to the leading infrared divergent term. The nonleading finite contributions will be included in the discussion later on. Let us consider the initial state bremsstrahlung in the e^+e^- annihilation. In such a simplified picture (leading infrared divergences only) the total cross section can be written as follows

$$\sigma(s) = \int_0^1 dK \rho_1(K) \sigma_{\text{Born}}(s(1-K)) \quad (2.1)$$

where

$$\rho_1(K) = \delta(K) + \gamma \left(\frac{1}{K} \right)_+ = \delta(K)(1 + \gamma \ln \epsilon) + \gamma \frac{1}{K} \theta(K - \epsilon) \quad (2.2)$$

is the photon energy $K = E\gamma/E_{\text{beam}}$ distribution and the "effective" perturbative expansion parameter is $\gamma \simeq 2\frac{\alpha}{\pi} \int_{m_e^2/s}^1 \frac{d\theta}{\theta} \simeq 2\frac{\alpha}{\pi} \ln \frac{s}{m_e^2} \simeq 0.10$. We keep track of the possible strong dependence of the Born cross section σ_{Born} on the center-of-mass energy $\sqrt{s}$. If this dependence is neglected then the integrated cross section in (2.1) will be equal to $\sigma_{\text{Born}}(s)$ due to $\int dK \left(\frac{1}{K} \right)_+ = 0$. The parameter ϵ was introduced here as an explicit regulator in the function $\left(\frac{1}{K} \right)_+$ i.e. it is the regulator of the infrared singularity. See also Fig. 1 for a pictorial representation. The negative contribution $\gamma \ln \epsilon$ in the cross section below ϵ threshold

$$\sigma(K < \epsilon) = \sigma_{\text{Born}}(s)(1 + \gamma \ln \epsilon) \quad (2.3)$$

is normally obtained by combining the virtual photon (vertex) contribution and the real soft photon contribution - both regularized with fictitious photon mass. The photon mass drops out and the ϵ remains as an effective infrared cut-off/regulator. In order to simplify the following discussion we denote the sum of these two contributions (vertex + virtual-soft) as the virtual one i.e. the term $\gamma \ln \epsilon$ we shall name as the virtual contribution.

In the Monte Carlo of the class of ref. [6] one uses the finite value of $\epsilon \simeq 10^{-2}$ and the distribution $\rho(K)\sigma_{\text{Born}}(s(1-K))$ is generated at the very beginning of the program. If $K < \epsilon$ then the photon multiplicity is assigned zero, $n = 0$, otherwise, for $K > \epsilon$, the photon multiplicity $n = 1$ is assigned. In the rest of the program the angular variables of the photon and fermions are generated. In order to keep the probability of $n = 0$ positive one cannot put ϵ too low. This restriction is relevant for the Monte Carlo - in the analytical calculation of the integrated cross section (for loose experimental cut-offs) the limit $\epsilon \rightarrow 0$ is always understood.

Let us come now to the *second order* cross section

$$\sigma = \int_{K_1+K_2 < 1} dK_1 dK_2 \sigma_{\text{Born}}(s(1-K_1-K_2))\rho(K_1, K_2) \quad (2.4)$$

where the double photon energy distribution (the leading infrared contribution as previously) can be written as follows

$$\begin{aligned} \rho_2(K_1, K_2) = & \delta(K_1)\delta(K_2)\left(1 + \gamma \ln \epsilon + \frac{1}{2}\gamma^2 \ln^2 \epsilon\right) \\ & + \frac{1}{2}\delta(K_1)\frac{\theta(K_2-\epsilon)}{K_2}\gamma(1 + \gamma \ln \epsilon) + \frac{1}{2}\delta(K_2)\frac{\theta(K_1-\epsilon)}{K_1}\gamma(1 + \gamma \ln \epsilon) \\ & + \frac{1}{2}\gamma^2\frac{\theta(K_1-\epsilon)\theta(K_2-\epsilon)}{K_1 K_2}, \end{aligned} \quad (2.5)$$

see Fig. 2 for a graphical representation. The above structure of the infrared divergences can be obtained by means of the explicit $O(\alpha^2)$ calculation, see ref. [7]. On the other hand the double photon energy distribution can be obtained by expanding the simple generating functional and keeping the terms up to $O(\gamma^2)$

$$1 \equiv \exp\left[\int_0^1 \left(\frac{1}{K}\right)_+\right] \xrightarrow{O(\gamma^2)} \int_0^1 dK_1 dK_2 \rho_2(K_1, K_2). \quad (2.6)$$

Of course, the same procedure provides $\rho_1(K)$ (by retaining terms of $O(\gamma)$) and the higher order multiphoton distributions. Before we go to the higher order case, let

us discuss the hypothetical *second order* Monte Carlo generator. In such a Monte Carlo in the beginning of the generation procedure one would generate the two-dimensional distribution $\rho_2(K_1, K_2)\sigma_{\text{Born}}(s(1-K_1-K_2))$ and assign the photon multiplicity $n = 0, 1, 2$ according to the number of the photons which go under the ϵ threshold. If σ_{Born} is strongly dependent on the energy $\sqrt{s}$ then it is more reasonable to rearrange the integral in the following way

$$\begin{aligned} \sigma &= \int_0^1 dK \sigma_{\text{Born}}(s(1-K))\rho_2(K) \\ \rho_2(K) &= \int dK_1 dK_2 \rho(K_1, K_2)\delta(K-K_1-K_2) \\ &= \delta(K)\left(1 + \gamma \ln \epsilon + \frac{1}{2}\gamma^2 \ln^2 \epsilon\right) + \frac{\theta(K-\epsilon)}{K}\gamma(1 + \gamma \ln \epsilon) \\ &\quad + \frac{1}{2}\gamma^2 \int dK_1 dK_2 \delta(K-K_1-K_2)\frac{\theta(K_1-\epsilon)}{K_1}\frac{\theta(K_2-\epsilon)}{K_2} \end{aligned} \quad (2.7)$$

In this case in the corresponding Monte Carlo one would first generate the one-dimensional distribution $\rho(K)\sigma_{\text{Born}}(s(1-K))$, assign $n = 0$ for $K < \epsilon$ and $n = 1, 2$ for $K > \epsilon$. The total energy K should be redistributed among two photon according to $\rho(K_1, K_2)$.

Let us note that in the above simple example one is already able to see the interesting phenomenon of the *competition of the soft photons for the total energy*. In spite of the fact that $\int_0^1 \int_0^1 dK_1 dK_2 \rho_2(K_1, K_2) \equiv 1$ the constrained distribution integrates

$$\int_0^1 dK \rho_2(K) = 1 - \gamma^2 \frac{\pi^2}{12} \quad (2.8)$$

to a value smaller than one. The inspection of the $\rho_2(K)$ distribution reveals that much of the negative contribution is located in the range $\epsilon < K < 2\epsilon$. This is related to the fact that the variable K in this range cannot be obtained by summing the two energies $K_1, K_2 > \epsilon$. A similar phenomenon shows up for an

infinite number of soft photons (see the remaining part of this Section). See also fig. 3 for an illustration.

Let us now go to the case of infinite order with an arbitrary number of photons in the game. Neglecting the $O(\gamma^3)$ and higher terms the second order formula can be rewritten as follows

$$\begin{aligned} \rho_2(K) = e^{\gamma \ln \epsilon} & \left\{ \delta(K) + \int dK_1 \delta(K - K_1) \frac{\theta(K_1 - \epsilon)}{K_1} \right. \\ & \left. + \frac{1}{2} \int dK_1 dK_2 \delta(K - K_1 - K_2) \frac{\theta(K_1 - \epsilon) \theta(K_2 - \epsilon)}{K_1 K_2} \right\}. \end{aligned} \quad (2.9)$$

The above expression suggests quite strongly an infinite order generalization of the following type

$$\rho(K) = e^{\gamma \ln \epsilon} \left\{ \delta(K) + \sum_{n=1}^{\infty} \prod_{i=1}^n \frac{dK_i}{K_i} \theta(K_i - \epsilon) \delta(K - \sum_{i=1}^n K_i) \right\} \quad (2.10)$$

Let us note immediately that the above formula is well suited for the Monte Carlo calculation since for a given photon multiplicity *the differential cross section is always positive*. There is no problem (except of computing time) to set the value of ϵ arbitrarily low. The Monte Carlo exercise based on this distribution was done in ref. [8]. In reality, the exclusive distribution (2.10) is not the result, however, of a pure guesswork based on the second order explicit calculation but it results from the careful analysis of the infrared virtual and real singularities in all orders of the perturbative QED which was done in the work of Yennie-Frautschi-Suura [1]. In fact the result of this reference coincides exactly with formula (2.10). For those readers who know the paper [1] the formula (2.10) may look unfamiliar, however. Let us elaborate on this rather important point. The main difference between (2.10) and the notation of ref. [1] and also other papers on the exponentiation, see for instance ref. [2], is that here we introduced the explicit regulator for the infrared singularity ϵ and we avoided the use of the Mellin transform which is unacceptable for the Monte Carlo. We can, however, rather easily translate (2.10) to a notation

typical for classical papers on the exponentiation :

$$\begin{aligned} \rho(K) = e^{\gamma \ln \epsilon} & \int \frac{dx}{2\pi} e^{iKx} \left\{ 1 + \int dK_1 \gamma \frac{\theta(K_1 - \epsilon)}{K_1} e^{-ixK_1} \right. \\ & \left. + \frac{1}{2!} \int dK_1 \gamma \frac{\theta(K_1 - \epsilon)}{K_1} e^{-ixK_1} \int dK_2 \gamma \frac{\theta(K_2 - \epsilon)}{K_2} e^{-ixK_2} + \dots \right\} \quad (2.11) \\ = & \int \frac{dx}{2\pi} e^{iKx} \exp \left\{ \gamma \ln \epsilon + \gamma \int dK' \frac{\theta(K' - \epsilon)}{K'} e^{-ixK'} \right\} \end{aligned}$$

The explicit dependence on the ϵ regulator may be removed by noticing that $\gamma \ln \epsilon = -\gamma \int_e^1 \frac{dK'}{K'}$ and $\int_0^e (e^{-iK'x} - 1) \rightarrow 0$ for $\epsilon \rightarrow 0$ and finally we arrive at a compact, manifestly infrared finite and regulator free formula

$$\rho(K) = \int \frac{dx}{2\pi} e^{iKx} \exp \left\{ \gamma \int_0^{\infty} \frac{dK'}{K'} \left(e^{-xK'} - 1 \right) \right\} = \gamma K^{\gamma-1} \frac{e^{\gamma C}}{\Gamma(1+\gamma)}, \quad (2.12)$$

which is precisely what can be found in ref. [1] ($C = 0.57721566\dots$). The above exercise can be done in the opposite direction i.e. the formula (2.10) can be obtained from (2.12) by introducing ϵ regularization, expanding the exponent and performing the x -integration. *The main lesson from this exercise is that although at first sight the formula (2.12) looks as an expression for the total cross section it contains, however, the full information on the differential cross sections with an arbitrary number of photons, as shown in eq. (2.10).* In other words the inclusive formula (2.12) is a *generating functional* for the exclusive formula (2.10). The two are completely equivalent numerically* and algebraically.

* In ref. [8] it was checked that the Monte Carlo calculation based on (2.10) gives precisely the same numerical answer as the analytical result shown in the right hand side of eq. (2.12).

3. Full scale exclusive exponentiation

Let us discuss now a realistic case of the second order exponentiated initial state bremsstrahlung for the fermion pair production in electron positron annihilation. The two main differences with the previous Section is the restoration of the full phase space and the introduction of the nonleading (in the infrared limit) contributions. The QED multi-differential distribution can be encoded again either in the Mellin-type formula analogous to (2.12) or in a manifestly exclusive manner as in eq. (2.10). The Mellin-type generating functional [1] reads

$$\begin{aligned} \sigma = & \int \frac{d^4 x}{(2\pi)^4} \int \frac{d^3 q_1}{q_1^0} \frac{d^3 q_2}{q_2^0} \frac{d^3 q_3}{q_3^0} e^{ix(p_1 + p_2 - q_1 - q_2 - q_3)} \exp(2\alpha \text{Re} B(p_1, p_2) + \int \frac{d^3 k}{k^0} \tilde{S}(p_1, p_2, k) e^{-ixk}) \\ & \left\{ \tilde{\beta}_0(p_1, q_j) + \int \frac{d^3 k_1}{k_1^0} e^{-ixk_1} \tilde{\beta}_1(p_1, q_j, k_1) \right. \\ & \left. + \frac{1}{2!} \int \frac{d^3 k_1}{k_1^0} \int \frac{d^3 k_2}{k_2^0} e^{-ixk_1 - ixk_2} \tilde{\beta}_2(p_1, q_j, k_1, k_2) \right\} \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} 2\alpha B(p_1, p_2) &= \frac{i\alpha}{4\pi^2} \int \frac{d^4 k}{k^2 - m_f^2} \left(\frac{2p_1 - k}{k^2 - 2p_1 \cdot k} + \frac{2p_2 + k}{k^2 + 2p_2 \cdot k} \right)^2, \\ \tilde{S}(p_1, p_2, k) &= -\frac{\alpha}{4\pi^2} \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2 \end{aligned} \quad (3.2)$$

and the exclusive Monte-Carlo-friendly formulation reads

$$\begin{aligned} \sigma = & \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^3 q_1}{q_1^0} \frac{d^3 q_2}{q_2^0} \left(\prod_{i=1}^n \frac{d^3 k_i}{k_i^0} \tilde{S}(p_1, p_2, k_i) \theta\left(\frac{2k_i^0}{\sqrt{s}} - \varepsilon\right) \delta^4(p_1 + p_2 - q_1 - q_2 - \sum_{i=1}^n k_i) \right) \\ & \exp\left(2\alpha \text{Re} B(p_1, p_2) + \int \frac{d^3 k}{k^0} \tilde{S}(p_1, p_2, k) \theta\left(\varepsilon - \frac{2k^0}{\sqrt{s}}\right)\right) \\ & \left\{ \tilde{\beta}_0(p_1, p_2, q_1, q_2) + \sum_{i=1}^n \frac{\tilde{\beta}_1(p_1, p_2, q_1, q_2, k_i)}{\tilde{S}(k_i)} + \sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{\tilde{\beta}_2(p_1, p_2, q_1, q_2, k_i, k_j)}{\tilde{S}(k_i) \tilde{S}(k_j)} \right\}. \end{aligned} \quad (3.3)$$

The above formulas coincide with the one-dimensional simplified distributions of the previous Section for $\tilde{\beta}_1 = \tilde{\beta}_2 \equiv 0$. To see it more clearly let us show how

to recover the characteristic Sudakov form factor $e^{\gamma \ln \varepsilon}$ in the above equations.

Introducing the usual infrared regulator $\varepsilon < 1$ we find

$$\exp\left(2\alpha \text{Re} B + \int \frac{d^3 k}{k^0} \tilde{S} e^{-ixk}\right) = \exp\left(R(\varepsilon) + \int_{2k^0/\sqrt{s} > \varepsilon} \frac{d^3 k}{k^0} \tilde{S} e^{-ixk}\right) \quad (3.4)$$

where

$$R(\varepsilon) = 2\alpha \text{Re} B + \int_{2k^0/\sqrt{s} < \varepsilon} \frac{d^3 k}{k^0} \tilde{S} = 2\frac{\alpha}{\pi} \left(\ln \frac{s}{m_e^2} - 1 \right) \ln \varepsilon + \frac{\alpha}{\pi} \left(\frac{1}{2} \ln \frac{s}{m_e^2} - 1 + \frac{1}{3} \pi^2 \right) \simeq \gamma \ln \varepsilon$$

is present in the exponent in eq. (3.3). The exclusive sum over photon multiplicity in (3.3) is obtained by means of expanding $\exp\left[\int \frac{d^3 k}{k^0} \tilde{S} e^{-ixk} \theta(2k^0/\sqrt{s} - \varepsilon)\right]$ and integrating over $d^4 x$. The integration over photon angles leads to the multi-photon energy distributions precisely as those in eq. (2.10). The function $\tilde{\beta}_0(p_1, p_2, q_1, q_2)$ is up to a normalization the Born differential cross section $d\sigma_{\text{Born}}/d\Omega$ at the reduced center of the mass energy $s' = (p_1 + p_2 - \sum k_i)^2$. The K variable is replaced here by $v = 1 - s'/s$.

The above case of $\tilde{\beta}_1 = \tilde{\beta}_2 = 0$ and the example discussed in the previous Section represent the *lowest order exponentiation*. As can be read from the eqs. (3.1) and (3.3) in this case the differential cross section for production of n photons is

$$d\sigma_n \sim \prod_{i=1}^n \frac{d^3 k_i}{k_i^0} \tilde{S}(k_i) \exp(2\alpha \text{Re} B(p_1, p_2)) |\mathcal{M}_{\text{Born}}|^2, \quad (3.5)$$

that is, up to a normalization constant, we have*

$$\tilde{\beta}_0 = |\mathcal{M}_{\text{Born}}|^2.$$

The factor $\exp(2\alpha \text{Re} B)$ stems from the sum over infinite number on the virtual photons and the distribution of the n real photons is good as long as $k_i^0 < \sqrt{s}/2$, i.e., there is no single hard photon.

* In the paper [1] it was proven in the arbitrary order of the perturbative QED that the most singular term in the differential cross section with n real photons looks like that in eq. (3.5) and that the residue coefficient $\tilde{\beta}_0$ is always the same, independently of n .

refs. [3,5]

$$\tilde{\beta}_1(k) = \rho_1(k) - \tilde{S}(k)\tilde{\beta}_0. \quad (3.7)$$

Again, as is the case of $\tilde{\beta}_0$, the definition of $\tilde{\beta}_1$ must *necessarily* include the reduction procedure which eliminates from the four-momentum balance all photons but one. It should be also clarified that in the above definition one should use the lowest order $\tilde{\beta}_0^{(0)} \sim |\mathcal{M}_{\text{Born}}|^2$ while in the first term of the eq. (3.6) one uses rather

$$\begin{aligned} \tilde{\beta}_0^{(1)} &= |e^{-2\alpha B} \mathcal{M}(\mathcal{O}(\alpha))|^2 = (1 - 2\alpha \text{Re} B)(1 + 2\text{Re} F_1) |\mathcal{M}_{\text{Born}}|^2 \\ &= (1 - 2\alpha \text{Re} B + 2\text{Re} F_1) |\mathcal{M}_{\text{Born}}|^2 = \left(1 + \frac{\alpha}{\pi} \left(\ln \frac{s}{m_e^2} - 1\right)\right) |\mathcal{M}_{\text{Born}}|^2 \end{aligned}$$

where $\mathcal{M}(\mathcal{O}(\alpha)) = (1 + F_1)\mathcal{M}_{\text{Born}}$ is again the first order (one loop) result obtained directly from the Feynman rules.

What happens if among n photons there is not one but rather *two hard photons*? In this case the distributions with $\tilde{\beta}_1$ only are not sufficiently good. One has to include $\tilde{\beta}_2$, update virtual corrections in $\tilde{\beta}_0$ and $\tilde{\beta}_1$ one order higher and generally this will be called the second order exponentiation. We refer the reader to ref. [3] for more details.

The recipe for the Yennie-Frautschi-Suura exponentiation can be summarized as follows:

1. Calculate in a traditional way, out of Feynman diagrams, the second order QED matrix element with zero one and two real photons, including the virtual corrections regularized by photon mass.* Extract ultraviolet divergences.
2. Eliminate virtual infrared contribution by factoring out the $\exp(2\alpha B)$ term. (Photon mass will disappear.)

* One should use, in principle, the exact $\mathcal{O}(\alpha^2)$ matrix elements as an input in the exponentiation procedure. In ref. [3] one uses for this purpose the leading and next-to-leading logarithmic approximations because (a) the relevant exact matrix elements are not available in the literature and (b) because such approximations are good enough to reach the necessary precision level for LEP/SLC experiments.

Let us note an important point about the definition of $\tilde{\beta}_0$: the Born amplitude $\mathcal{M}_{\text{Born}}$ is, strictly speaking, defined within the two-body phase space $p_1 + p_2 = q_1 + q_2$. The $\tilde{\beta}_0$ in eqs. (3.1), (3.3) and (3.5) are defined for momenta which do not obey this equation. It means that the definition of $\tilde{\beta}_0$ must *necessarily* embody a mapping procedure (so-called reduction procedure, see ref. [3]) $p_i, q_i \rightarrow \mathcal{R}p_i, \mathcal{R}q_i$ such that $\mathcal{R}p_1 + \mathcal{R}p_2 = \mathcal{R}q_1 + \mathcal{R}q_2$. This transformation should tend to identity when the sum of photon energies goes to zero.

The principal question: *what is the first and second order exponentiation* should be restated as follows: *what is the proper multi-distribution in the case when in addition to many soft real photons there are one or two hard photons*? The answer was found in the Yennie-Frautschi-Suura work [1] and is already built in eqs. (3.1) and (3.3). To see it more clearly let us look into the n -photon differential cross section in eq. (3.3) assuming that $\tilde{\beta}_2 = 0$ and $\tilde{\beta}_1 \neq 0$ (first order exponentiation)[†]

$$d\sigma_n \sim \tilde{\beta}_0 \prod_{i=1}^n \tilde{S}(k_i) + \sum_{l=1}^n \tilde{\beta}_l(k_l) \prod_{i \neq l} \tilde{S}(k_i). \quad (3.6)$$

If all photons are soft $k_i^0 \ll \sqrt{s}/2$ then the second sum is negligible because $\tilde{\beta}_1(k)$ is not singular in the infrared limit $k^0 \rightarrow 0$. If one photon, say $l = L$, is hard $k_L^0 \sim \sqrt{s}/2$ then in the second sum only the term $l = L$ is non-negligible and we can rewrite (3.6) as follows

$$\tilde{S}(k_1) \dots \tilde{S}(k_{L-1}) (\tilde{\beta}_0(k_L) \tilde{S}(k_L) + \tilde{\beta}_1(k_L)) \dots \tilde{S}(k_n) = \tilde{S}(k_1) \dots \tilde{S}(k_{L-1}) \rho_1(k_L) \dots \tilde{S}(k_n),$$

where the $\rho_1(k)$ is the conventional single bremsstrahlung matrix element calculated from the Feynman rules. The above implies the definition of the $\tilde{\beta}_1(k)$, see

[†] In fact, as is shown in ref. [1], the differential cross section for n real photons can be expanded exactly (without any approximation) into n terms which include $\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_n$. Here, we keep the two most singular terms. The point to be noted is that the residue coefficients $\tilde{\beta}_0$ and $\tilde{\beta}_1$ are precisely the same regardless the value of the photon multiplicity n .

3. Calculate the $\tilde{\beta}_2, \tilde{\beta}_1$ by extracting in a recursive way the singular factors $\tilde{S}(k)$, as in eq. (3.7) (do not forget to include the reduction procedure in the definition).
4. Construct the differential cross section for n real photons using $\tilde{\beta}_{0,1,2}$ according to eq. (3.3) and integrate over the phase space.
5. Check that you do not use the calculation for the region of the phase space which includes three or more hard photons (or evaluate that such a contribution is negligible).

4. The second order Monte Carlo

The Monte Carlo program for the initial state bremsstrahlung in the electron-positron annihilation with the second order exclusive exponentiation is described in a detail in ref. [3]. The Monte Carlo algorithm depicted in Fig. 3 consists of the following steps:

1. Choose $v = 1 - s'/s$ according to a distribution which is roughly $\gamma v^{\gamma-1} \sigma_{\text{Born}}(s(1-v))$.
2. If $v < \epsilon$ then choose photon multiplicity $n = 0$ otherwise generate $n - 1$ according to Poisson distribution with the average $\gamma \ln \frac{1}{\epsilon}$.
3. Generate photon four-momenta according to the density

$$\frac{d^3k}{k^0} \tilde{S}(k)$$

4. Construct four-momenta of all photons and fermions.
5. Reject events according to weight which adds the effects due to $\tilde{\beta}_1$ and $\tilde{\beta}_2$.

Let us finally present the example of a numerical results from the YFS2 Monte Carlo of ref. [3]. In Table 1 below we show the values of the integrated cross sections

in the R-units. The following input parameters were used: $M_Z = 92 \text{ GeV}$, $\Gamma_Z = 2.45346$, $\sin^2 \theta_W = 0.228818$. An upper limit on the photon phase space was $s'/s > 0.2$ where $\sqrt{s'}$ is an effective mass of the muon pair. The cross section σ_E is the best available non-Monte-Carlo result from ref. [7]; we show their "exponentiated" second order result, with the omission of the production of additional fermion pairs. No electroweak corrections were included. The two calculations agree to within 0.2%.

Table 1

$\sqrt{s} [\text{GeV}]$	σ Monte Carlo	$(\sigma_E - \sigma)/\sigma \cdot 10^3$
40	1.1026 ± 0.0006	-0.3 ± 0.50
90	33.65 ± 0.01	0.6 ± 0.4
91	73.33 ± 0.02	0.6 ± 0.3
92	131.53 ± 0.04	0.8 ± 0.3
93	93.38 ± 0.03	0.7 ± 0.3
94	58.38 ± 0.02	1.4 ± 0.4
100	13.21 ± 0.01	0.6 ± 0.7

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FIG. 1

FIG. 2

FIG. 3