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DUALITY AND QUANTUM GROUPS

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ABSTRACT

We show that the duality properties of Rational Conformal Field Theories follow from the defining relations and the representation theory of quantum groups. The fusion and braiding matrices are q -analogues of the $6j$ -symbols and the modular transformation matrices are obtained from the properties of the co-multiplication. We study in detail the Wess-Zumino-Witten models and the rational gaussian models as examples, but carry out the arguments in general. We point out the connections with the Chern-Simons approach. We give general arguments of why the general solution to the polynomial equations of Moore and Seiberg describing the duality properties of Rational Conformal Field Theories defines a Quantum Group acting on the space of conformal blocks. A direct connection between Rational Theories and knot invariants is also presented along the lines of Jones' original work.

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1. INTRODUCTION

Rational Conformal Field Theories (RCFT) [1] are those conformally invariant field theories whose physical Hilbert space $\mathcal{H}$ splits into a finite sum of irreducible representations of the chiral algebra $\mathcal{A}_L \otimes \mathcal{A}_R$; $\mathcal{H} = \oplus \mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}$ and $\mathcal{A}_L, \mathcal{A}_R$ contain at least the identity operator and the left and right Virasoro algebras. Well-known examples are provided by the minimal series of Virasoro representations [2], the unitary series [3] and the Wess-Zumino-Witten models (WZW) , [4, 5, 6, 7, 8] whose chiral algebras are Kac-Moody algebras. The classification of RCFT is a simplified version of the more difficult problem of classifying all two dimensional conformally invariant quantum field theories of interest in the study of classical solutions to String Theory and in the critical limits of statistical mechanical models.

There are some general constraints satisfied by any RCFT embodied in a series of polynomial equations ([9, 10, 11] and references therein). The basic ingredients in these equations are the duality (braiding and fusing) matrices of the conformal blocks and the matrices expressing the modular properties of the characters of the irreducible representations of the chiral algebra appearing in the conformal theory. If one translates the notions of locality, duality and modular invariance (conformal bootstrap [2]) to the language of blocks and characters, it is not difficult to derive the polynomial equations which were used in [9] to prove Verlinde's conjecture [12] stating that the fusion rules are diagonalized by the modular transformation properties of the characters. In spite of the simplicity of the polynomial equations (they only involve genus zero and genus one information) , Moore and Seiberg were able to prove that they provide sufficient conditions to have modular invariance for any genus Riemann surface with a number of distinguished points and local parameters about them. Hence given any RCFT all we have to do to check modular invariance is to make sure that the braiding and genus one modular transformations satisfy these equations. They can be interpreted as a representation in the language of conformal field theory of the set of generators and relations of

the mapping class group for any Riemann surface. It is, however, not yet clear to what extent a RCFT can be reconstructed from a solution to these equations. One has for example the problem of determining the chiral algebra. The dimensions of the primary fields are only determined by the equations up to integers, and there is no a priori trace or scalar product that would allow some reconstruction of the physical Hilbert space.

It has been clear for some time that there is a close connection between Statistical Mechanics and link invariants (see [13] and references therein). Recently [14] E. Witten has shown how to construct and understand geometrically link invariants in terms of topological Chern-Simons theory in $2 + 1$ dimensions. A consequence of this construction is that the space of conformal blocks and their duality properties can be understood from the principle of general covariance in three dimensions. One can envisage two general approaches to the classification of RCFT :

- 1). Use Witten's three dimensional point of view where the classification problem is related to the classification of three dimensional topological theories with compact phase spaces. The clarification and exploration of this approach has been pursued in [15, 16, 17, 18, 19, 20].
- 2). Exploit the duality properties of RCFT and try to characterize the general algebraic solution to the problem (see for example [9, 10, 11, 13, 21, 22, 26, 24, 25, 23]).

In [24, 25] it was first pointed out explicitly the connection between the duality properties of RCFT and Quantum Groups. In this paper we continue our study [24, 25] of the connection between RCFT and Quasi-Triangular Yang-Baxter algebras (Quantum Groups for short).

It is believed that any RCFT can be represented as a coset model in analogy with the GKO construction [27] of the minimal unitary series of Virasoro and super-Virasoro theories. At least any known RCFT can be expressed in this form. It is one of the main goals of both approaches to settle whether this conjecture is

true. In [11] it was shown that the duality properties of those RCFT admitting a classical limit can be described in terms of the representation theory of ordinary groups (finite or continuous). More precisely, the theories with a classical limit fall in sequences of theories with a parameter playing the role of Planck's constant. For example, in WZW theories we consider a group G and level k for the corresponding Kac-Moody algebra, and the classical limits is obtained by taking $k \mapsto \infty$. For many of the known sequences, the duality properties of all the elements in these sequences can be obtained by quantizing the group appearing in the classical limit à la Drinfel'd Jimbo, i.e. as a Quantum Group [28, 29, 30, 31, 32]. This paper is intended in part to provide evidence supporting this fact. For theories which do not admit a classical limit, it is not clear which of the two approaches will be more powerful in finding and describing "exceptional" RCFT. It seems plausible that with some extra physical assumptions, any solution to the polynomial equations can be described as a Quantum Group with a rational representation ring (only a finite number of representations appear; this will be explained in detail in subsequent sections). In this paper we set up and explore the quantum group formulation of RCFT in order to carry out this program. Eventually we believe that if we make an analogy with topology, the Chern-Simons approach will be the analogue of differential topology while the quantum group approach is closer to combinatorial topology.

The organization of this paper is as follows: In Section 2 we briefly review the general algebraic set up common to any conformal field theory. We present the polynomial equations and analyze some of the general properties of the fusion algebra. We then motivate the defining properties of a quantum group from the solutions to the polynomial equations to show why quantum groups capture the duality properties of RCFTs. In sec. 3 we treat more formally the theory of quantum groups and study their representations when the deformation parameter is a root of unity. The fusion rules are shown to follow from the decomposition of tensor products of representations of quantum groups. Furthermore the general structure of the blocks and their braiding properties can be read off from the

representation theory of quantum groups. We close this section by showing that the Gauss law constraint in the Chern-Simons approach [14] can be interpreted nicely in terms of the comultiplication of the quantum group. In sec. 4 we study in more detail how one extracts duality information from the quantum groups and reproduce the duality properties of WZW models and also of rational gaussian models. In sec. 5 we compute the modular transformation properties purely in terms of the quantum group and give a proof in this context of the Verlinde conjecture. Here we make contact again with the Chern-Simons theory. From the construction of the modular transformation S (corresponding to the behavior of the character under the modular transformation $\tau \mapsto -1/\tau$ it is obvious that S is a link invariant, which in the language of [14] corresponds to the expectation value of two linked Wilson lines on the three sphere. The Markov property of the traces appearing in quantum groups and RCFT are the basic reason why there is such a close connection between RCFT and three dimensional Topological Field Theories. Furthermore, the modular transformation S can be interpreted in terms of complex conjugation of quantum group characters computed as the trace of a particular element that exists in any Quasi-Triangular Yang-Baxter algebra and which can be interpreted as the remnant in the quantum group of the Virasoro algebra. In sec. 6 we use the previous results to show with generality that to any set of fusion rules given by a RCFT one can associate a collection of link invariants. The argument is close to the original algebraic approach of V. Jones [33, 34], and it shows clearly that the necessary ingredients to construct such invariants are essentially contained in the fusion algebra. Finally in Section 7 we present our conclusions and speculations.

Our approach here to quantum groups is complementary to the approaches followed in [24, 35, 36] where a direct action of the quantum group on the conformal blocks is defined. Nevertheless there is as yet no natural construction of operators within a given conformal theory which satisfy the commutation relations for a quantum group. We believe that at this moment it is more natural to consider the quantum groups as the mathematical objects whose representation theory

characterizes the topological content of conformal theories, or at least RCFT.

2. DUALITY PROPERTIES FUSION RULES AND QUANTUM GROUPS

Apart from invariance under the Virasoro or a bigger chiral algebra, the distinguishing features of conformally invariant theories with respect to other field theories are their duality properties [2]. The main aim of this paper is to show that the basic duality properties of any RCFT embodied in the polynomial equations [9] and some physical conditions are captured by the algebraic structure of a quantum group and its representation theory for particular values of the deformation parameters (roots of unity). As outlined in the introduction, this section is intended to summarize the duality properties of RCFT in a convenient form, to analyze some of the general properties of the fusion algebra, and to provide the first motivation for why the quantum groups are of relevance in the study of RCFT.

The physical correlation functions are constructed in terms of monodromy invariant combinations of holomorphic and antiholomorphic functions [2] : the conformal blocks. The n -point blocks are multivalued functions of its arguments with singularities described by the conformal dimensions of the fields appearing in the block. We can think of them diagrammatically as skeletons with labels describing the quantum numbers of the external and internal lines, and with three point vertices describing the couplings between three fields. An example is drawn in fig. 1. For a RCFT there will be n fields $\phi_0, \dots, \phi_{n-1}$, and for every field ϕ_i we can always find another one $\phi_{\bar{i}}$ such that the operator product expansion (OPE) of $\phi_i \times \phi_{\bar{i}}$ contains the identity operator. The mapping $C : \phi_i \mapsto \phi_{\bar{i}}$ defines the charge conjugation matrix of the theory $C^2 = 1$. The number of possible three point couplings between i, j, k is N_{ij}^k (fig. 2). The associativity of the OPE implies that the matrices $(N_{ij}^k)_{j,k}$ commute pairwise, and it is also clear that $N_{ij}^k = N_{\bar{j}\bar{i}}^{\bar{k}}$. If we define abstractly an algebra with n generators x_i $i = 0, \dots, n-1$, then the

fusion algebra can be written as

$$x_i x_j = N_{ij}^k x_k \quad (2.1)$$

and the previous properties imply that the algebra is commutative and associative. Hence it has n one-dimensional irreducible representations. Not all the possible examples of (2.1) give fusion rules for conformal field theories. It is easy to find counterexamples. Nevertheless, (2.1) guarantees that the dimension of the space of blocks with fixed external legs is independent of the way in which we choose to draw the skeleton diagrams.

Geometrically the blocks can be described in terms of sections of a (projectively) flat vector bundle. At genus zero for instance they are sections of a vector bundle over the moduli space of the sphere with n distinguished points $S^2 - \{P_1, \dots, P_n\}$. This moduli space can be described by $n-3$ copies of S^2 (this corresponds to fixing three points to be $0, 1, \infty$ by SL_2 invariance) and removing the "diagonal" subsets where two or more points coincide*. The group $Diff(S^2 - \{P_1, \dots, P_n\})$ (including diffeomorphisms that permute the points) acts on the space of blocks. If $Diff^+(S^2 - \{P_1, \dots, P_n\})$ is the subset of diffeomorphisms connected with the identity the duality properties are related to the representations of the quotient group $Diff/Diff^+$ furnished by the conformal blocks. Since the bundle is flat, we can use the flat connection (naturally given by the energy-momentum tensor) to identify all the fibers with one of them and study the action of the quotient group on this fiber. Since a RCFT has a finite number of irreducible representations of the chiral algebra, and the properties of the descendants under duality are determined by those of the primary fields it is clear that the duality properties of a RCFT can be given in terms of finite dimensional quantities. Formally we only need to look at $\mathcal{H}/\mathcal{A}$ where $\mathcal{H}$ is the Hilbert space of the theory and $\mathcal{A}$ is the chiral algebra. Since $\mathcal{H} = \oplus \mathcal{H}_i$ $\mathcal{H}_{\bar{i}}$ is a finite sum, we

* The geometrical aspects of these bundles and their importance in Conformal Field Theory have been emphasized by Friedan and Shenker [1].

only need to keep the highest weight subspaces $V_i \subset \mathcal{H}_i$ generated by the states in the module with the same L_0 eigenvalue as the highest weight vector generating the representation. For WZW theories V_i is specified by an irreducible representation of an ordinary Lie group. A basis in the space of blocks with $\phi_1, \dots, \phi_N$ on the external legs is obtained by describing a way to decompose (according to the fusion rules) the product $V_1 \otimes \dots \otimes V_N$. The associativity of (2.1) guarantees that at least the dimension of the bundle obtained does not change for different description to draw skeleton representing the same space of blocks. This is reminiscent of group theory albeit not classical group theory. To clarify this statement, consider the three point block in fig. 2. It can be thought of as describing an intertwiner from the representations i, j to the representation k (similar but not the same as the Clebsch-Gordan coefficients). The difference however is that if we exchange (braid) the i, j legs, in the case of classical groups this will amount to a sign counting whether the representation k appears symmetrically in the tensor product. In conformal theories this exchange which braids the i, j legs (which in terms of blocks is obtained by analytic continuation) gives a phase depending on the dimensions of the fields i, j, k . If σ_{ij} represents this braiding operation (the Ω operation in [9, 11]) and W_k^{ij} is the block in fig. 3, then

$$\sigma_{ij} W_k^{ij} = \epsilon e^{\pi i(h_k - h_i - h_j)} W_k^{ji} \quad (2.2)$$

where h_i, h_j, h_k are respectively the conformal dimensions of the i, j, k fields and ϵ is a sign with the same interpretation as for classical groups. The duality properties of conformal theories imply that blocks related by braiding or fusing two external legs are linearly related (fig. 3), and the matrices B and F providing the basic moves in the space of blocks satisfy the hexagon and pentagon identities [9]. The first one follows from the defining relations of the braid group. The braid group on n strands B_n has $n - 1$ generators (fig. 4 shows the i -th generator) σ_i $i = 1, \dots, n - 1$ satisfying the relations (see for example [37]) :

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| \geq 2 \quad (2.3)$$

the matrix B gives a representation of the braid group acting on the conformal blocks, and the hexagon identity expresses the first equation in (2.3) in terms of B . The pentagon identity is a consequence of the associativity of the OPE, and it only involves F . It is obtained by starting with a general block as in fig. 1, selecting three contiguous external legs, and then requiring associativity of the operation implied by the second move in fig. 3 (see [9] for details). The matrices F, B are not independent and they are related by the moves shown in fig. 5 where σ_{ab} represents braiding of the a and b legs. In the cases considered in fig. 5 the braiding produces only a phase, and we obtain

$$B_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} = \epsilon_{kl}^p \epsilon_{p'k}^i e^{i\pi(h_p + h_{p'} - h_i - h_l)} F_{pp'} \begin{bmatrix} j & l \\ i & k \end{bmatrix} \quad (2.4)$$

where the ϵ_{kl}^p factors determine whether p appear symmetrically in the product of k, l . The hexagon and pentagon identities follow from tree level considerations only. The genus 1 conditions are formulated in terms of characters. If $\chi_i(\tau)$ is the character of the irreducible representation of the chiral algebra generated by ϕ_i :

$$\chi_i(\tau) = \text{tr}_i q^{L_0 - c/24} \quad q = e^{2\pi i \tau} \quad (2.5)$$

then the modular transformations act on $\chi_i(\tau)$ according to

$$T : \chi_i(\tau + 1) = e^{2\pi i(h_i - c/24)} \chi_i(\tau)$$

$$S : \chi_i\left(\frac{-1}{\tau}\right) = S_i^j \chi_j(\tau) \quad (2.6)$$

What Verlinde conjectured [12] and Moore and Seiberg proved in general [9] was that the matrix S diagonalizes the fusion algebra. Since the N 's are mutually

commuting they can all be brought to diagonal form simultaneously

$$N_i \sim \text{diag}(\lambda_i^{(0)}, \lambda_i^{(1)}, \dots, \lambda_i^{(n-1)})$$

$$\lambda_i^{(k)} \lambda_j^{(k)} = N_{ij}^k \lambda_i^{(k)} \lambda_j^{(k)} \quad (2.7)$$

and the matrix implementing this diagonalization is S_i^j . The charge conjugation matrix C can be written as $C_{ij} = N_{ij}^0$, and if we raise and lower indices with C we find that $N_{ij;k}$ is totally symmetric. There are a number of properties satisfied by the matrix S that we now list :

$$\lambda_i^{(j)} = \frac{S_i^j}{S_0^j}$$

$$S^* = SC = S^{-1}$$

$$S^2 = C = (ST)^3$$

$$S^t = S$$

$$N_{ij}^k = \sum_m \frac{S_i^m S_j^m S_m^k}{S_0^m} \quad (2.8)$$

These conditions on S provide strong constraints on the possible fusion algebras if they are to come from a conformal field theory. Furthermore, for unitary theories we find that S_{i0}/S_{00} is positive. This number defines the “quantum dimension” for the field ϕ_i in the following sense [38] :

$$\lim_{\tau \rightarrow 0} \frac{\chi_i(\tau)}{\chi_0(\tau)} = \frac{S_{i0}}{S_{00}} \quad (2.9)$$

reminiscent of the Murray-Von Neumann theory of dimensions in $*$ -algebras.

It is sometimes useful to express the fusion rules in terms of Bratteli diagrams (in particular in the connection between RCFT and link invariants explained in the appendix). To make the ideas clear, the simplest blocks to consider are those with $N + 1$ external fields, N of which are equal to the field ϕ and the other j is a “spectator” field (fig. 6). The braid group in our construction will only act on the ϕ legs. Using the fusion rules, the dimension of the space of blocks with the external legs as in fig. 6 is given by

$$d_j^N = (N_\phi^{N-1})_j^\phi \quad (2.10)$$

to obtain the dimensions of the blocks with $N + 1$ external ϕ fields and some spectator j , we use N_ϕ :

$$d_i^{N+1} = \sum_j d_j^N (N_\phi)_j^i \quad (2.11)$$

this can be best expressed in terms of a graph (the Bratteli diagram). In the row at level N (corresponding to N external ϕ fields) we have the dimensions d_i^N for all possible spectators. To obtain the dimensions at the next level we draw $(N_\phi)_j^i$ lines from d_i^N to d_j^{N+1} (fig. 7), then the tree obtained contains as much information as (2.11) but it is easier to visualize. The different paths in this tree from the top to a given element j in the N -th row give the different internal states in the block with N ϕ 's and j as spectator. This is illustrated in fig. 8 for the case of a theory containing only two fields 1 and ϕ and fusion rules $\phi x \phi = 1 + \phi$ (which generate a Fibonacci tree). At each level in the Bratteli diagram in fig. 8 we have drawn schematically the conformal blocks that will appear and the dimension of their spaces. Another example is provided by the $SU(2)_k$ WZW model with large k . This is shown in fig. 9. If ϕ is the primary field of spin $1/2$ the first few levels of the diagram agree with the Clebsch-Gordan decomposition of the tensor products $V_{1/2}^{\otimes n}$ for ordinary $SU(2)$. For a fixed spectator the blocks at each level generate irreducible representations of the braid group acting on the ϕ legs. Instead of using the Bratteli diagrams, we can picture the fusion matrix

$(N_\phi)_i^j$ as a Dynkin-like diagram. For instance in $SU(2)_k$ the primary fields have spins $j = 0, 1/2, \dots, k/2$. If the field $\phi_{1/2}$ is the spin $1/2$ field, we can represent the operator products between $\phi_{1/2}$ and ϕ_j as in fig. 10. Each dot in the diagram stands for one of the primary fields, and we draw $(N_\phi)_i^j$ lines joining position i to position j in the graph. For $SU(2)_k$ this leads always to the Dynkin diagram A_{k+1} . Following the rules in this diagram it is easy to reconstruct the Bratelli diagram in fig. 9. In cases where the operator algebra is more complicated we will get in general multi-dimensional Dynkin-like diagrams (for instance, in the minimal models [2] we have two generators of the operator algebra $\psi_{(2,1)}, \psi_{(1,2)}$ leading to a two-dimensional diagram) and in this case we obtain “Bratelli towers”.

The fusion algebra (2.1) can be studied in parallel with the theory of representations of finite groups [39]. First, the existence of a conjugate field ϕ_i^* for each field ϕ_i implies that (2.1) is not only commutative and associative, but also semi-simple. Hence the regular representation $\phi_i \mapsto N_i$ is both reducible and fully reducible. All representations are one dimensional and one can write orthogonality properties for the characters. If there are n distinct generators in (2.1), the n one-dimensional representations are given in terms of the eigenvalues (2.7)

$$\lambda^{(i)}(x_i) = \lambda_i^{(j)} = \lambda_j(i) \quad (2.12)$$

Given the matrix $\Lambda, \Lambda_{ij} = \lambda^{(i)}(x_j)$, let $N, N_{ij} = n_i(j) = n_j(i)$ be its inverse. Then one can derive the following orthogonality properties [39] :

$$\begin{aligned} \sum_k \lambda_i(k) \bar{\lambda}_j(k) &= \frac{1}{n_0(i)} \delta_{ij} \\ \sum_k n_0(k) \lambda_k(i) \bar{\lambda}_k(j) &= \delta_{ij} \\ \bar{\lambda}_j(k) &= \lambda_j(\bar{k}) \end{aligned} \quad (2.13)$$

and the matrices N_i can be simultaneously diagonalized by the matrix

$$S_{ij} = \sqrt{n_0(j)} \lambda_j(i) \quad (2.14)$$

If S is symmetric (as required by (2.8)) (2.13) implies easily that $S^2 = C$. In section 5 we will show that for fusion rules coming from quantum groups S is necessarily symmetric and therefore one of the basic modular properties of RCFT is automatically satisfied.

We can interpret the B, F operations (fig. 3) in (quantum) group theoretic language. Although the precise definitions of quantum groups and quasi-triangular Yang-Baxter algebras are introduced in the next section, we want to motivate these definitions from the point of view of duality properties of a RCFT. In ordinary group theory, the intertwiner can be used to define composition of angular momentum. In general we have the situation in fig. 11. If K_k^{ij} is the intertwiner and the algebra A acts on the representation space V_k via the matrices $\varrho^k(a)$, $a \in A$ the diagram in fig. 11 can be used to define the action of A on the tensor product. Technically this is the same as defining a co-multiplication operation in the algebra $\Delta : A \mapsto A \otimes A$ and $\varrho^i \otimes \varrho^j(\Delta(a)) = \Delta^{ij}(a)$ gives the action of the algebra on $V_i \otimes V_j$. If the algebra is associative and the tensor product of representations is also associative, then the co-multiplication is co-associative. The algebra A that will appear from the duality properties is not the zero mode subset of the chiral algebra $\mathcal{A}$ but a deformation of it. This is because the intertwiner has non-trivial monodromy. At this stage we do not yet know what this algebra might be. We will proceed as though such an algebra exists and try to extract its defining properties. At the end of the section we will look at the $SU(2)_k$ WZW model to identify in detail the algebra A with the quantum group $SU(2, q)$ for $q = e^{2\pi i/k+2}$. Although this procedure may seem a bit backwards, it exhibits more directly why duality in RCFT leads to quantum groups.

Recall that in $\mathcal{H}/\mathcal{A}$ we are identifying blocks with different prescriptions of

decomposing the tensor product $V_{i_1} \otimes \dots \otimes V_{i_N}$

$$V_{i_1} \otimes \dots \otimes V_{i_N} = \oplus_j F_j^{i_1 i_2 \dots i_N} \otimes V^j \quad (2.15)$$

where the spaces $F_j^{i_1 i_2 \dots i_N}$ are generated by the blocks $F_j^{i_1 i_2 \dots i_N}(p_1, \dots, p_{N_2})$ and the multiplicity of V^j is given by the dimension of $F_j^{i_1 i_2 \dots i_N}$ i.e. by the number of non-vanishing conformal blocks counted according to the fusion rules. Comparing with fig. 1, $i_{(N+1)}$ plays the role of the spectator field j . There are some properties of (2.15) which follow from the general structure of RCFT. In the decomposition (2.15) we have assumed the representation on the right hand side to be not only reducible but also fully reducible. This is not the case for general algebras but it holds in Conformal Field Theory. The space of conformal blocks can be divided into subsets each providing an irreducible representation of the braid group acting on its external legs. The action of A on the tensor product is obtained by iterating the co-multiplication. From general arguments in representation theory, the blocks $F_j^{i_1 \dots i_N}$ provide representations of the centralizer of A acting on the tensor product. By definition, if A, B are two algebras acting on the same vector space V B is the centralizer of A if the set of operations commuting with the action of A in V generate the algebra B . If A and B are semisimple and finite dimensional then if B is the centralizer of A then A is the centralizer of B . This is just a very general way of stating Schur's lemma. Consider first the simple case when all the representation $V_{i_1}, \dots, V_{i_N}$ are identical corresponding to some field ϕ . If we knew the algebra A we could in principle find its centralizer* whose representations would completely characterize the decomposition of the tensor product V_ϕ^n . The complete reducibility of the braid representations provided by the conformal blocks implies that the centralizer algebra must be semisimple. As we will see later in the more interesting cases (quantum groups whose deformation parameters are roots

* For classical groups, for example $GL(n)$, and the fundamental representation, the centralizer algebra of the action of $GL(n)$ on the tensor product of N copies of the fundamental is generated by the permutations exchanging the factors together with the constraint that the $n+1$ -row antisymmetrizer vanishes, i.e. only tableaux with at most n rows appear as representations of the group

of unity) some restrictions have to be imposed on their representation theory in order to obtain effectively a semisimple centralizer. This is equivalent to dividing the centralizer by some abelian ideals and working with the semisimple quotient) Without going into these complications now, let us assume the centralizer to be semisimple. Since the braid group acts on the blocks (2.15) via some representation, we can identify the centralizer of A with the algebra representing B_n on the blocks. Reading (2.15) backwards and using the braid representations on the blocks we can construct a matrix $R^{\phi\phi}$ acting on contiguous spaces in the tensor product (the generators of the braid group only exchange contiguous labels), this matrix commutes with the co-multiplication operation i.e. it commutes with $\Delta(a)$, $a \in A$ by the definition of centralizer

$$R^{\phi\phi} \Delta^{\phi\phi} = \Delta^{\phi\phi} R^{\phi\phi}$$

if the representations in the external legs are different the analogous equation is

$$R^{ij} \Delta^{ij} = \Delta^{ij} R^{ij} \quad (2.16)$$

where

$$R^{ij} : V_i \otimes V_j \mapsto V_j \otimes V_i$$

if $P^{ij} : V_i \otimes V_j \mapsto V_j \otimes V_i$ is the permutation map $P^{ij}(x \otimes y) = y \otimes x$ we can define $\mathcal{R}^{ij} = P^{ji} R^{ij} : V_i \otimes V_j \mapsto V_i \otimes V_j$, $P^{ji} P^{ji} = 1$, and (2.16) can be interpreted in terms of the algebra as follows : We can introduce two co-multiplications, Δ and $\Delta' = \sigma \cdot \Delta$, $\sigma : A \otimes A \mapsto A \otimes A$, $\sigma(a \otimes b) = b \otimes a$. Equation (2.16) implies the existence of a matrix $\mathcal{R} \in A \otimes A$ such that Δ and $\Delta' = \sigma \cdot \Delta$ are equivalent :

$$\Delta'(a) = \sigma \cdot \Delta(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1} \quad (2.17)$$

using that

$$P^{ij} \Delta^{ji} P^{ji} = \Delta^{ij}$$

it is easy to see that (2.17) leads to (2.16). Two extra equations on the co-multiplication are obtained from the fact that in conformal theories fusing and

braiding are compatible. This is illustrated in fig. 12. As will be explained in detail in sec. 3 (2.17) and the compatibility between fusing and braiding imply all but one of the defining relations for a Quasi-Triangular Yang-Baxter algebra. The missing axiom can be explained in terms of the relation between the braiding and fusing properties of a set of fields and those of the conjugate fields. A simple consequence of these axioms are the Yang-Baxter equation and the hexagon and pentagon identities (see the next two sections for details). Furthermore, we will show in sec. 5 that the modular matrix S with the properties (2.8) can be obtained from the properties of the co-multiplication.

Although in general we do not yet know whether it is possible to associate a quantum group to any RCFT (or to a solution to the polynomial equations), in some cases it is not difficult to derive the quantum group whose representation theory reproduces the duality properties of the RCFT. Ideally one would like to find an explicit construction of the quantum algebra $\mathcal{A}$ in terms of the generators of the chiral algebra $\mathcal{A}$. There are several ways of constructing quantum group actions on the space of blocks for RCFT [24, 35, 36], however it is not known at present how to relate them to $\mathcal{A}$. The previous arguments are helpful in understanding why such a construction may not be so simple. The duality properties coded by the quantum groups correspond to rather non-local operations in terms of the actual blocks (as sections of a vector bundle over some moduli space), for example one has to look at analytic continuation from one region to another, etc. Hence the quantum group generators should be non-local functionals of the generators of $\mathcal{A}$. We find it more appealing and easier to find the quantum group as a centralizer of the half monodromies, and then use its representation theory to express the B, F, S, T matrices characterizing a solution to the polynomial equations.

As an example consider the $SU(2)_k$ WZW theory, and take $\phi = \phi_{1/2}$ for simplicity. This field generates the operator algebra in the sense that any other primary field $\phi_j, j \leq k/2$ can be obtained in terms of the OPE of a sufficient number of ϕ fields. Consider the blocks in fig. 6. The dimension of the field ϕ_j is

$h_j = j(j+1)/k+2$. From the $SU(2)$ fusion rules we know that

$$\begin{aligned} \phi_{1/2} \times \phi_{1/2} &= \phi_0 + \phi_1 \\ \phi_{1/2} \times \phi_j &= \phi_{j-1/2} + \phi_{j+1/2} \quad \text{if } j \leq \frac{k-1}{2} \\ \phi_{1/2} \times \phi_{k/2} &= \phi_{(k-1)/2} \end{aligned} \quad (2.18)$$

Then the Bratteli diagram of fig. 9 will not keep on expanding indefinitely and it will truncate as shown in fig. 13. We can characterize quite accurately the representations of the braid group furnished by the blocks $F_{\phi_i}^{(N)}$ from the information we have about the fusion rules (2.18) and the dimensions h_j of the fields ϕ_j . First in the operation (2.2) we have

$$\begin{aligned} \sigma_{12}(W_{j_3}^{i_1 j_2}) &= (-1)^{j_3-j_1-j_2} q^{(c_{j_3}-c_{j_1}-c_{j_2})/2} W_{j_3}^{i_2 j_1} \\ q &= e^{\frac{2\pi i}{k+2}} \quad c_l = l(l+1) \end{aligned}$$

Using the F move (fig. 3) we can diagonalize the matrix ϱ_i braiding the legs $i, i+1$. From (2.18) we learn that all the ϱ_i have the same eigenvalues. By a phase redefinition these eigenvalues can be taken to be 1, q . Therefore the matrices ϱ_i satisfy a quadratic equation. This together with the braid relations (2.3) defines the structure of a Hecke algebra of type $A_N : H_N(q)$:

$$\begin{aligned} \varrho_i^2 &= (1-q)\varrho_i + q \\ \varrho_i \varrho_{i+1} \varrho_i &= \varrho_{i+1} \varrho_i \varrho_{i+1} \\ \varrho_i \varrho_j &= \varrho_j \varrho_i \quad |i-j| \geq 2 \end{aligned} \quad (2.19)$$

For generic values of q the representations of $H_N(q)$ are in one-to-one correspondence with those of the permutation group S_N which satisfies the same defining

relation as the Hecke algebra except that $q = 1$. When q is a root of unity the algebra is not semisimple and to obtain a semisimple representation theory we have to consider quotients of the algebra by some ideals. Wenzl [40] has studied this question in detail and has classified the necessary restrictions in the representations of $H_N(q)$. From the general properties of conformal theories we know that the algebra representing the action of the braid group must be effectively semisimple because the blocks provide fully reducible representations of B_N . For the blocks considered in this example the semisimple quotient of the Hecke algebra provides the representation of the braid group on the blocks. This can be seen from the fusion rules or the associated Brattelli diagram. The representations of $H_N(q)$ are labelled by the same Young tableau as those of S_N . In the case of $SU(2)$ we can replace the spin labels in fig. 13 by Young tableaux. Spin j is represented by a tableau with at most two rows of λ_1, λ_2 boxes each such that $\lambda_1 - \lambda_2 = 2j$. Hence the representation of $SU(2)$ appearing in the WZW model are those satisfying $\lambda_1 - \lambda_2 = 2k$. These are the diagrams of type $(2, k)$ in Wenzl notation and as shown in [40] they provide the $*$ -representations of $H_N(q)$ when $q = e^{2\pi i/k+2}$ (the $*$ -representations are always semisimple algebras). The absence of tableaux with more than two rows can be written as an extra relation in the Hecke algebra demanding the three row antisymmetrizer to vanish

$$1 - \varrho_1 - \varrho_2 + \varrho_1\varrho_2 + \varrho_2\varrho_1 + \varrho_1\varrho_2\varrho_1 = 0 \quad (2.20)$$

(since we are working with $SU(2)$ this was expected). Equations (2.19) and (2.20) are equivalent to the Temperley-Lieb-Jones algebra [34]. Define

$$e_i = \frac{1 - \varrho_i}{1 + \varrho_i} \quad (2.21)$$

one can then easily show that

$$e_i e_{i+1} e_i = \beta^{-1} e_i, \quad \beta = 2 + q + q^{-1}$$

$$e_i^2 = e_i$$

$$e_i e_j = e_j e_i \quad |i - j| \geq 2 \quad (2.22)$$

These algebras are known as $A_{\beta,N}$ and they have been extensively studied by Jones [33, 34] where he has classified the subfactor when q is a root of unity and analyzed the $*$ -representations. It is remarkable that many of the properties of the semisimple quotients of these algebras can be obtained directly from the theory of the WZW model whose chiral algebra is a Kac-Moody algebra. We have therefore learned that the representations of the braid group provided by the blocks F_{ϕ}^N in the $SU(2)_k$ WZW theory are in one-to-one correspondence with the $*$ -representations of the algebra $A_{\beta,2}$. Since we know this algebra, we can ask for its centralizer in order to find the algebra A . As we will show in the next section, the centralizer of this algebra is the quantum group $SU(2, q)$, $q = e^{2\pi i/k+2}$ (with some restrictions on its representation theory so that it also becomes a semisimple algebra). This information can now be used to construct explicitly the duality matrices B, F in terms of the q -analogues of the $6j$ -symbols. The F move (fig. 3) corresponds to the change of basis implementing the associativity law for the composition of (quantum) angular momentum ($V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$). For generic values of q the $6j$ -symbols were computed in [31] and it was shown in [25] that they coincide when q is a root of unity with the duality matrices for the $SU(2)$ level k WZW model. The arguments in [25] were essentially the same as those appearing here. Using Wenzl's analysis of the $*$ -representation of the braid representations of this conformal field theory and then finding the quantum group through the centralizer of the Temperley-Lieb-Jones algebras. The duality matrices were obtained in [7] by solving the Knizhnik-Zamolodchikov equation [5] corresponding to the decoupling of null vectors of the combined Virasoro and Kac-Moody algebras. The results of both computations (via the quantum group or via the differential equation) agree up to conjugation by a diagonal matrix. This arbitrariness is used to normalize the blocks according to one's favorite conventions. For instance the prescription in [2].

To summarize this section, we have collected the basic duality properties of a RCFT given by the genus zero polynomial equations and the properties of the fusion algebra. They contain the purely topological information characterizing a conformal field theory. We then argued that when we have some information about the highest weight subspaces V_i the braiding and fusing properties of the conformal blocks can be translated into the defining properties of a Quasi-Triangular Yang-Baxter algebra which can be obtained as the centralizer of the action of the braid group on the blocks. With so little information one can actually go quite far in the explicit computation of the duality matrices in terms of the representation theory of the associated quantum group although in the more standard approach one obtains this information after solving the Knizhnik-Zamolodchikov differential equation. From the arguments above it is clear that the two computations differ only in the convention chosen to normalize the blocks.

The reconstruction problem in RCFT is to find the extra conditions that determine when a particular representation space of the duality transformations (a solution of the polynomial equations can be interpreted as the space of blocks of a RCFT. The quantum group approach is a first step in this direction. As we have seen we get the quantum group directly from the duality properties of the conformal theory. What we see at this point is that RCFT are always associated with C^* -irreducible representations of the duality transformations. If these C^* -representations are II_1 -factors i.e. they admit a normalized finite trace then we will be able to identify the space of conformal blocks with the Hilbert space obtained by completion of the duality C^* -algebra with respect to the scalar product defined by the trace ($\langle x|y \rangle = \text{tr}(x^*y)$). The extra physical requirement that naturally appears in the quantum group approach is that the conformal theories are in one-to-one correspondence with solutions of the polynomial equations such that: 1) They define irreducible C^* -representations of the duality transformations, and 2) They are equipped with a normalized finite trace satisfying the Markov property (see sec. 3,7). This Markov property which holds under very general circumstances is at the root of the connection between Conformal Field

Theory and three-dimensional Topological Field Theories. In general if one begins with a solution to the polynomial equations some of the steps in the previous paragraphs establishing the relation with quantum groups are missing. For instance, in general we will not get the spaces V_i themselves or even their real dimensions, but only their q -dimensions (2.9). In this case more powerful techniques are required. It is reasonable to expect that an extension of the results of Deligne quoted in [11] concerning the reconstruction of classical groups may yield ultimately that to any solution of the polynomial equations with some extra condition about the possible q -dimensions can be given in terms of quantum groups. In the next sections we will look in more detail into the structure of quantum groups and show that those groups coming from classical groups do indeed satisfy the polynomial equations and look also into the general situation.

3. QUANTUM GROUPS AND THEIR REPRESENTATIONS

In this section we look in some detail at the definition and properties of quantum groups and their representations with particular emphasis on the representations when q is a root of unity. Let us begin with the definitions : Given an associative algebra A with unit, we say that A is a Hopf algebra, if we can define three operations $\Delta, \gamma, \epsilon : A \mapsto A \otimes A$ is the comultiplication, $\gamma : A \mapsto A$ is the antipode, and $\epsilon : A \mapsto C$ is the co-unit. Δ and ϵ are homomorphisms of algebras ; $\Delta(ab) = \Delta(a)\Delta(b)$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$ while γ is an antihomomorphism $\gamma(ab) = \gamma(b)\gamma(a)$. The three operations must satisfy the following axioms :

$$a, b \in A \quad (id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a)$$

$$m(id \otimes \gamma)\Delta(a) = m(\gamma \otimes id)\Delta(a) = \epsilon(a)1$$

$$(\epsilon \otimes id)\Delta(a) = (id \otimes \epsilon)\Delta(a) = a \quad (3.1)$$

where m is the multiplication in the algebra. In words, the first condition is the associativity of the co-multiplication, the second is the definition of the antipode

(the analogue of an inverse), and the last one is the condition satisfied by the co-unit. If $\sigma : A \otimes A \mapsto A \otimes A$ is the permutation map $\sigma : (x \otimes y) = y \otimes x$, it is easy to check that $\Delta' = \sigma \cdot \Delta$ is another co-multiplication in the same algebra with antipode $\gamma' = \gamma^{-1}$. A Hopf algebra is a quasi-triangular Yang-Baxter algebra if the co-multiplications Δ, Δ' are related by conjugation :

$$\sigma \cdot \Delta(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1} \quad \mathcal{R} \in A \otimes A \quad (3.2)$$

and the following conditions are satisfied :

$$(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}$$

$$(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}$$

$$(\gamma \otimes id)(\mathcal{R}) = \mathcal{R}^{-1} \quad (3.3)$$

since $(id \otimes \Delta)(\mathcal{R}) \in A \otimes A \otimes A$, $\mathcal{R}_{13}$ acts as the identity in the second factor and as $\mathcal{R}$ in the first and third factors in $A \otimes A \otimes A$. The motivation for these axioms came originally from the theory of integrable models (see [28] and references therein). Later in this section we will give a motivation from the point of view of conformal theories. An immediate consequence of these axioms is the Yang-Baxter equation for $\mathcal{R}$: Computing $[(\sigma \cdot \Delta) \otimes id] \mathcal{R}$ using (3.2) of the second equation in (3.3) we obtain

$$\mathcal{R}_{13} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \quad (3.4)$$

the matrix $\mathcal{R} : A \otimes A \mapsto A \otimes A$ is known as the universal $\mathcal{R}$ matrix for the algebra A .

As an example consider any Cartan matrix (a_{ij}) of an ordinary Lie group of affine Kac-Moody algebra. Then we can associate to it a deformation of the

classical algebra [29, 28]. Working in a Chevalley basis, we can associate to every simple root three generators X_i^+, X_i^-, H_i satisfying the following relations

$$[H_i, H_j] = 0$$

$$[H_j, X_i^\pm] = \pm \langle \alpha_i, \alpha_j \rangle X_i^\pm \quad \langle \alpha, b \rangle = 2a \cdot b / b^2$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{H_i/2} - q^{-H_i/2}}{q^{1/2} - q^{-1/2}}$$

$$[X_i^\pm, X_j^\pm] = 0 \quad \text{if } a_{ij} = 0$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q_i} (X_i^\pm)^{1-a_{ij}-\nu} X_j^\pm (X_i^\pm)^\nu q_i^{-\nu(1-a_{ij}-\nu)/2} = 0 \quad i \neq j, q_i = q^{\alpha_i^2/2} \quad (3.5)$$

the last condition is the deformation of the Serre relations. From now on we use the following conventions for q -numbers :

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$$

$$[n]! = [n][n-1] \dots [2][1] \quad (3.6)$$

the operators Δ, γ, ϵ are defined as :

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i$$

$$\Delta(X_i^\pm) = X_i^\pm \otimes q^{H_i/4} + q^{-H_i/4} \otimes X_i^\pm$$

$$\epsilon(1) = 1 \quad \epsilon(X_i^\pm) = \epsilon(H_i) = 0$$

$$\gamma(H_i) = -H_i$$

$$\gamma(X_i^\pm) = -q^{e/2} X_i^\pm q^{-e/2}$$

$$\varrho = \frac{1}{2} \sum_{\alpha > 0} H_\alpha \quad (3.7)$$

it is an easy exercise to check that (3.7) satisfy the axioms (3.1). Moreover, (3.5) also have the structure of a quantum group (3.2), (3.3)[28]. To keep the discussion simple, consider the $SU(2)$ case. Now we do not have the Serre relations and the defining properties are

$$[X^+, X^-] = [H]$$

$$[H, X^\pm] = \pm 2X^\pm$$

$$\epsilon(H) = \epsilon(X^\pm) = 0, \quad \epsilon(1) = 1$$

$$\gamma(H) = -H \quad \gamma(X^\pm) = -q^{\pm 1/2} X^\pm \quad (3.8)$$

the universal $\mathcal{R}$ -matrix is given by [31] :

$$\mathcal{R} = q^{H \otimes H/4} \sum_{n \geq 0} \frac{(1 - q^{-1})^n}{[n]!} q^{-n(n-1)/4} q^{nH/4} (X^+)^n \otimes q^{-nH/4} (X^-)^n \quad (3.9)$$

and it satisfies (3.2) and (3.3). As long as q is not a root of unity, the representation theory of quantum groups runs parallel with the classical theory. The representations are also labelled by the same highest weights and most of the standard expressions valid for classical algebras hold for quantum groups where numbers are replaced by q -numbers. From the point of view of representation theory the

co-multiplication can be interpreted as the rule of addition of angular momentum. Geometrically we can understand for example the representation theory of $SU(2, q)$ as a deformation of the Borel-Weil-Bott construction. The fundamental representations of $SU(2)$ has two states : spin up and spin down. These two states can be represented in terms of two variables u, v respectively. If we consider the space of polynomials in these variables, the irreducible representation of spin j is given by the homogeneous polynomials of degree $2j$. A basis is given by

$$\frac{u^{j+m} v^{j-m}}{\sqrt{(j+m)!(j-m)!}} \quad m = -j, -j+1, \dots, j \quad (3.10)$$

and the operators X^+, X^-, H are represented by the differential operators

$$X^+ = u \frac{\partial}{\partial v} \quad X^- = v \frac{\partial}{\partial u} \quad H = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \quad (3.11)$$

geometrically the vectors (3.10) are the holomorphic sections of the holomorphic line bundle of $-j$ differentials over the sphere, and the variables u, v are the homogeneous coordinates on the sphere considered as a complex projective space. Applying the Riemann-Roch theorem one finds that this line bundle has $2j+1$ holomorphic sections. The metric in the space of polynomials is

$$\langle f|g \rangle = f\left(\frac{d}{du}, \frac{d}{dv}\right) g(u, v)|_{u=v=0} \quad (3.12)$$

the q -deformation of this construction is obtained when we use the same polynomials in u, v but the generators are instead represented in terms of q -derivatives

$$D_u f(u) = \frac{f(q^{1/2}u) - f(q^{-1/2}u)}{(q^{1/2} - q^{-1/2})u}$$

$$D_v f(v) = \frac{f(q^{1/2}v) - f(q^{-1/2}v)}{(q^{1/2} - q^{-1/2})v} \quad (3.13)$$

To compute the commutation relations of these q derivatives we only need to compute them on the basic monomials $u^a v^b$. It is then straightforward to show

that the operators

$$X^+ = uD_u \quad X^- = vD_v \quad H = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \quad (3.14)$$

satisfy the commutation relations of $SU(2, q)$. The metric analogous to (3.12) is

$$\langle f|g \rangle = f(D_u, D_v)g(u, v)|_{u=v=0} \quad (3.15)$$

with respect to this scalar product an orthonormal basis for the spin j representation is given by

$$|j, m\rangle = \frac{u^{j+m} v^{j-m}}{\sqrt{[j+m]![j-m]!}} \quad m = -j, -j+1, \dots, j \quad (3.16)$$

$$[0]! = 1$$

and it is now easy to derive formulae like

$$(X^\pm)^a |j, m\rangle = \sqrt{\frac{[j \mp m]![j \pm m + a]!}{[j \mp m - a]![j \pm m]!}} |j, m \pm a\rangle \quad (3.17)$$

finally the analogue of the Casimir operator is

$$C = X^- X^+ + \left[\frac{H+1}{2} \right]^2 \quad (3.18)$$

To make contact with the remarks made at the end of the last section about the centralizer algebras for quantum groups, we first consider the example of $SU(N, q)$. After this example is worked out, the general approach for any group G should be clear. As in the $SU(2)$ case, the fundamental representations of $SU(N)$ and $SU(N, q)$ coincide. Suppose we want to decompose $V_N \otimes \dots \otimes V_N$ into irreducible components. As in the classical case we have to look for the algebra

commuting with the action of $SU(N, q)$. As shown in (2.16), if we define the action of $\Delta(a)$, $\mathcal{R}$ and R in the tensor product as

$$\Delta^{ij}(a) = q^i \otimes q^j(\Delta(a)) : V_i \otimes V_j \mapsto V_i \otimes V_j$$

$$\mathcal{R}^{ij} = q^i \otimes q^j(\mathcal{R})$$

$$R^{ij} = P^{ij} \mathcal{R}^{ij} : V_i \otimes V_j \mapsto V_j \otimes V_i \quad (3.19)$$

it follows from (3.2) and (3.3) that

$$R^{ij} \Delta^{ij}(a) = \Delta^{ji}(a) R^{ij} \quad (3.20)$$

in our example i, j are the fundamental representation and in order to simplify the notation we will remove the labels in R^{ij}, Δ_{ij} . Then (3.20) becomes $R\Delta(a) = \Delta(a)R$ and the generators of the centralizer of $SU(N, q)$ in $V_N^{\otimes n}$ are :

$$\varrho_i = 1 \otimes \dots \otimes R_{i,i+1} \otimes \dots \otimes 1 \quad (3.21)$$

where the subindices in R indicate the positions of the spaces in the tensor products where it acts non-trivially. The matrix R can be computed explicitly from our knowledge of the co-multiplication, the fact that the fundamental representation of the quantum group and the classical group coincide and the relation (3.20). If e_{ij} is the $N \times N$ matrix whose only non-vanishing entry is 1 and it appears in the position i, j it is easy to show that

$$R = \sum_{i \neq j} e_{ij} \otimes e_{ji} + q^{1/2} \sum_i e_{ii} \otimes e_{ii} + (q^{1/2} - q^{-1/2}) \sum_{i < j} e_{ij} \otimes e_{ji} \quad (3.22)$$

and a straightforward computation shows that

$$R^2 = (q^{1/2} - q^{-1/2})R + 1 \quad (3.23)$$

to agree with the conventions of the previous section we define the generators of

the Hecke algebra to be

$$\begin{aligned}\varrho_i &= -q^{-1/2} R \\ \varrho_i^2 &= (1-q)\varrho_i + q\end{aligned}\tag{3.24}$$

The Yang-Baxter relation written as acting on $V_i \otimes V_j \otimes V_k$ becomes

$$\mathcal{R}^{ij}\mathcal{R}^{ik}\mathcal{R}^{jk} = \mathcal{R}^{jk}\mathcal{R}^{ik}\mathcal{R}^{ij}\tag{3.25}$$

Writing it in components and using the definition (3.19) we obtain the corresponding relation for R^{ij} :

$$(R^{jk} \otimes 1)(1 \otimes R^{ik})(R^{ij} \otimes 1) = (1 \otimes R^{ij})(R^{ik} \otimes 1)(1 \otimes R^{jk})\tag{3.26}$$

from (3.26) we derive :

$$\begin{aligned}\varrho_i \varrho_{i+1} \varrho_i &= \varrho_{i+1} \varrho_i \varrho_{i+1} \\ \varrho_i \varrho_j &= \varrho_j \varrho_i \quad |i-j| \geq 2\end{aligned}$$

this together with (3.24) shows that the centralizer of $SU(N, q)$ in $V_N^{\otimes n}$ satisfies the defining relations of the Hecke algebra $H_n(q)$. Moreover for each representation μ we can define a homomorphism π^μ from the braid group B_n into C_μ^n

$$\pi^\mu(\sigma_i) = 1 \otimes \dots \otimes R_{i,i+1}^{\mu\mu} \dots \otimes 1\tag{3.27}$$

As in the case of $SU(2, q)$ there is one extra relation [30], since for generic q the representations of $SU(N, q)$ are labelled by the same Young tableaux as those of $SU(N)$ (now however the symmetrizers and antisymmetrizers are constructed out of the generators ϱ_i), the $N+1$ -row anti-symmetrizer must vanish. Only tableaux with at most N rows appear in the representation ring of $SU(N, q)$. We denote the centralizer of $SU(N, q)$ in $V_N^{\otimes n}$ by C_N^n . This gives a sequence of algebra embeddings

$\dots \subset C_N^n \subset C_N^{n+1} \dots$ whose properties will be useful in section 7. For q generic the algebra C_N^n is semisimple. This implies that any reducible representation of $SU(N, q)$ is fully reducible, and from (3.20) we see that the centralizer of C_N^n defines the comultiplication operation in $SU(N, q)$. $SU(N, q)$ is the centralizer of C_N^n . This reciprocity relation was used in the previous section to relate the duality properties of the $SU(2)_k$ WZW model with $SU(2, q)$.

Before dealing with the case of q a root of unity, we present one more property of the general theory [30] that will be useful in sec. 7. For any representation μ of a quantum group we can consider the chain of algebras $\dots C_\mu^n \subset C_\mu^{n+1} \subset \dots$ of the formal union $C_\mu^\infty = \bigcup_n C_\mu^n$ describing the braiding operations in $V_\mu^{\otimes n}$ for any n . If $R^{\mu\mu}$ is the matrix R^{ij} for this representation and $q = \frac{1}{2} \sum_{\alpha > 0} H_\alpha$ then (3.20) implies

$$R^{\mu\mu} q^\varrho \otimes q^\varrho = q^\varrho \otimes q^\varrho R^{\mu\mu}\tag{3.28}$$

and we can use R and q^ϱ to define a trace on the algebra C_μ^∞ with the following properties

$$\begin{aligned}tr(1) &= 1 \\ tr(ab) &= tr(ba) \\ tr \alpha \varrho_n^{\pm 1} &= z_\pm tr \alpha \quad \alpha \in C_\mu^n\end{aligned}\tag{3.29}$$

where the generators ϱ_i are defined by (3.21), and

$$z_\pm = \frac{q^{c(\mu)}}{tr_\mu q^\varrho}\tag{3.30}$$

and $c(\mu)$ is the second Casimir for the classical group of the representation labelled by the highest weight vector μ . The trace is defined by

$$tr \alpha = \frac{1}{tr_\mu q^\varrho} tr(q^\varrho \otimes \dots \otimes q^\varrho \alpha)\tag{3.31}$$

for a proof see [30]. Using the homomorphism (3.27) we get from (3.31) a Markov trace on the braid group.

This trace suggests the definition of a quantum dimension for a representation of a quantum group. If we remove the normalization factor from (3.31) and take the braided element to be the identity we can define the q -dimension of a representation of highest weight μ to be

$$D_q(\mu) = \text{tr} q^e \quad (3.32)$$

with tr the standard matrix trace. This definition will be shown to agree with the definition (2.9) and it plays a very important role in the representation theory of quantum groups when q is a root of unity.

We finally come to analyze representations for quantum groups when the deformation parameter is a root of unity. Now the centralizer algebra and the quantum group are not semisimple and we want to find out under what conditions the representation theory can be restricted to a regular set of representations (modular representations of quantum groups appear in [32]). A more physical treatment can be found in [41]). Let us begin with $SU(2, q)$. The basic problem when $q = e^{2\pi i/p}$ is that $(X^\pm)^p = 0$ and this generates null vectors in some representations. Many of the representations appearing in the decomposition of tensor products of irreducible representations will be reducible but not fully reducible. Instead of using the basis (3.16), we construct a different basis for the representation with highest weight j as

$$\begin{aligned} X^+ |j\rangle &= 0 \\ |m\rangle &= \frac{(X^-)^{j-m}}{[j-m]!} |j\rangle \end{aligned} \quad (3.33)$$

then

$$\begin{aligned} X^- |jm\rangle &= [j-m+1] |j, m-1\rangle \\ X^+ |jm\rangle &= [j+m+1] |j, m+1\rangle \\ \frac{(X^\pm)^a}{[a]!} |jm\rangle &= \frac{[j \pm m + a]!}{[a]! [j+m]!} |j, m \pm a\rangle \end{aligned} \quad (3.34)$$

when $q^p = 1$, $[p] = [2p] = \dots = [kp] = 0$, however the operators $(X^\pm)^p/[p]!$ are still well defined as can be seen by taking $a = p$ in (3.34) for generic q and then taking the limit $q \mapsto e^{2\pi i/p}$. It is also clear from the previous equations that $(X^\pm)^p = 0$ and moreover that this property is preserved by the co-multiplication as one should expect :

$$\Delta(X^\pm)^N = \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix} q^{-(N-j)H/4} (X^+)^j \otimes q^{jH/4} (X^+)^{N-j}$$

when $N = p$ the first and the last terms in the right hand side contain $(X^\pm)^p = 0$ and they vanish, and any of the intermediate terms vanish because they are all proportional to $[p]$. Hence for large enough j the representation will have null vectors. Following [41] we begin to see the source of the problem if we calculate the eigenvalues of the Casimir operator (3.18) acting on highest weight vectors

$$C|j\rangle = [j + 1/2]^2 |j\rangle$$

for q generic the eigenvalues of C distinguish between all the different values of j . When $q = e^{2\pi i/p}$, it is easy to see that $j, j + kp, p - 1 - j + kp$ for any integer k have the same Casimir and if we compute the q -dimensions we find

$$D_q(j) = D_q(j + kp) = -D_q(p - 1 - j + kp) \quad (3.35)$$

this suggests that if we try to decompose the tensor product $V_{1/2}^{\otimes n}$ for n high enough into irreducible representations, odd things begin to happen. $V_{1/2}^{\otimes n}$ contains in its decomposition reducible but not fully reducible representations which pair up representations that would be distinct irreducible ones for q generic adding up their q -dimensions to zero. For example if $q^3 = 1$ we can try to decompose $V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$. For generic values of q this tensor product decomposes into $V_{3/2} \oplus 2V_{1/2}$. When $q^3 = 1$ the weight states of the spin $3/2$ and one of the spin $1/2$ representations mix into a reducible but not fully reducible representation

due to the presence of null vectors. It is not difficult to check that the state $|\alpha\rangle = X^-[1/2] \otimes |1/2\rangle \otimes |1/2\rangle$ with $m = 1/2$ is annihilated by X^+ . Since this state is null, it is orthogonal to itself with respect to the metric (3.15) (notice that this scalar product is the standard one in terms of bras and kets with the prescription that in constructing bras from kets one should not complex conjugate q), and out of the other two states in $V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$ only one of them is orthogonal to $|\alpha\rangle$. The other state $|\beta\rangle$ is not orthogonal to $|\alpha\rangle$ and therefore $X^+|\beta\rangle \propto |1/2\rangle \otimes |1/2\rangle \otimes |1/2\rangle$. The q -dimensions are $D_q(3/2) = [4] = [3-4] = -[1] = -1$ and $D_q(1/2) = [2] = [3-2] = [1] = 1$, where we have used $[p] = 0, [x] = [p-x], [-x] = -[x]$. Thus $D_q(3/2) + D_q(1/2) = 0$. Looking at other tensor product $V_{1/2}^{\otimes n}$ it is possible to prove that this situation is quite general [41]. If we want to have a regular representation theory (so that tensor products of irreducible representations are fully reducible) we have to impose some conditions on what representations can be considered “integrable” and modify the definition of the tensor product of representations. To find the integrable or regular representations when $p \geq 0$ we first notice that $V_{1/2}$ has positive q -dimension. Since $[x]$ is a ratio of sine functions, we can solve the equation

$$D_q(j) = [2j+1] = 0 \quad (3.36)$$

to find that the q -dimension vanishes whenever $j = \frac{p-1}{2} + kp$. As we keep on taking tensor products $V_{1/2} \otimes \dots \otimes V_{1/2}$ we eventually obtain the representation with $j = \frac{p-1}{2}$ with vanishing q -dimension and other with positive q -dimensions. Tensoring another copy of $V_{1/2}$ begins the pairing of representations. We can restrict the representations of $SU(2, q)$ to those with the smallest possible spin and positive q -dimension. This is achieved by requiring that in the tensor product of the fundamental representations we only keep those highest weight vectors annihilated by X^+ and at the same time not in the image of $(X^+)^{p-1}$. This restricts the representations to those with spin smaller or equal to $(p-2)/2$, defining $p = k+2$ we obtain the same restriction as in the $SU(2)$ level k WZW model. In this way we find an alcove in the space of weights where the q -dimension is strictly positive

with the lowest possible value of j . Using the scalar product (3.15)

$$\langle j| \frac{(X^+)^n}{[n!]} \frac{(X^-)^n}{[n!]} |j\rangle = \begin{bmatrix} j \\ n \end{bmatrix}$$

it is easy to check the unitarity and positive definiteness of the representation. The representations with $j < (p-1)/2$ are characterized by :1). They are highest weight representations. 2) The highest weight $|j\rangle$ cannot be written as $(X^+)^{p-1}|\text{anything}\rangle$. The highest weights are in the space [41]

$$|j\rangle \in \text{Ker } X^+ / \text{Im}(X^+)^{p-1}$$

Next we can define a restricted tensor product. We have the collection of regular representations $\mathcal{V}_{reg} = \{V_j, j \leq (p-2)/2\}$. Given any two regular representations in $\mathcal{V}_{reg}$ we construct the standard tensor product $V_{j_1} \otimes V_{j_2}$. If $j_1 + j_2 \leq (p-2)/2$ the standard tensor product rule for $SU(2)$ holds. As soon as the value of $j_1 + j_2$ is surpassed, the highest weight vector with the largest value of $m_1 + m_2$ will belong to $\text{Im}(X^+)^{p-1}$, if $j = (p-1)/2$ the q -dimension vanishes. For $j > (p-1)/2$ the representation pairs with some other states into a reducible but not fully reducible representations. If we consider the set of highest weight vector $(V_{j_1} \otimes V_{j_2})^+ \in V_{j_1} \otimes V_{j_2}$, we can restrict to those which are not in $\text{Im}(X^+)^{p-1}$ and to the representation they generate. This will be our definition of the restricted tensor product

$$(V_{j_1} \otimes V_{j_2})' \quad (3.37)$$

where the prime stands for the operation of eliminating those representations whose highest weight $\in \text{Im}(X^+)^{p-1}$, in other words eliminating those representations with vanishing q -dimension. Now the scalar product in (3.37) is positive definite and the representation it provides is reducible and fully reducible. Furthermore, the co-associativity of the co-multiplication guarantees this tensor product

to be associative

$$((V_{j_1} \otimes V_{j_2})' \otimes V_{j_3})' \cong (V_{j_1} \otimes (V_{j_2} \otimes V_{j_3}))' \quad (3.38)$$

From the pairing argument described one easily gets the decomposition of (3.37) into irreducible regular representations

$$(V_{j_1} \otimes V_{j_2})' = \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} V_j \quad k = p-2$$

Since the Bratelli decomposition of an algebra and that of its centralizer are the same, we see that the definition (3.37) automatically leads to *-representations of the Temperley-Lieb-Jones algebra if $j_1 = j_2 = \dots j_n = 1/2$ or the more complicated centralizer algebras for higher spins in the external legs.

Next we wish to consider a general group G whose quantized commutation relations appear in sec.3. When q is a root of unity we want to find the integrable representations and the conditions on the restricted tensor product. To find the regular alcove in the space of weight we use Weyl's character formula. For a representation with highest weight λ we obtain for the q -dimension the expression

$$D_q(\lambda) = \prod_{\alpha > 0} \frac{[(\lambda + \varrho, \alpha)]}{(\varrho, \alpha)} \quad \varrho = \frac{1}{2} \sum_{\alpha > 0} \alpha \quad (3.39)$$

for $q^p = 1$ we first determine when two representations will have equal or opposite q -dimension. If w_i is the Weyl reflection with respect to the simple root α_i then

$$\prod_{\alpha > 0} \frac{[(w_i(\lambda) + w_i(\varrho), \alpha)]}{(\varrho, \alpha)} = - \prod_{\alpha > 0} \frac{[(\lambda + \varrho, \alpha)]}{(\varrho, \alpha)}$$

and for any element in the Weyl group :

$$\prod_{\alpha > 0} \frac{[(w(\lambda + \varrho), \alpha)]}{(\varrho, \alpha)} = \epsilon(w) \prod_{\alpha > 0} \frac{[(\lambda + \varrho, \alpha)]}{(\varrho, \alpha)}$$

where ϵ is the parity of w . Then we obtain

$$D_q(w(\lambda + \varrho) - \varrho + p \sum_{i=1}^r n_i \alpha_i) = \epsilon(w) D_q(\lambda), \quad q^p = 1 \quad (3.40)$$

the $\alpha_i, i = 1, \dots, r$ are the simple roots of the classical algebra G . Every positive root can be written as $\alpha = \sum n_i \alpha_i, n_i \geq 0$, and $\sum_i n_i \equiv \text{level}(\alpha)$. For the highest root θ , the level plus one is the dual Coxeter number of the algebra $g = (\theta, \theta + 2\varrho)/\theta^2$. We will normalize the highest root to have length 2. For any weight λ , the largest value of $(\lambda + \varrho, \alpha), \alpha > 0$ is obtained for the highest root $\theta, (\varrho, \theta) = g - 1$. The denominator of (3.39) can be written as :

$$\prod_{\alpha > 0} [(\varrho, \alpha)] = \prod_{l(\alpha)=1}^{g-1} [l(\alpha)]^{Nl(\alpha)}$$

$l(\alpha)$ is the level of α and $Nl(\alpha)$ is the number of positive roots with the same level. For $q^p = 1, p > g$ the q -dimension of the generating representations of G are positive and if we consider representations in increasing values of (λ, θ) we find that the q -dimension stays positive until $(\lambda, \theta) = p - g$, for $(\lambda, \theta) = p - g + 1$ the q -dimension vanishes, and beyond this value it can be positive, negative or zero, and we begin to have null vectors, reducible but not fully reducible representations appear, etc. In analogy with $SU(2)$ let $p = k + g$, the integrable (regular) irreducible representations are such that their highest weights are not in

$$Im(X_\theta^+)^{p-g+1} \quad (3.41)$$

the first one with vanishing q -dimension appears when $(\lambda, \theta) = k + 1$ and it is such that $|\lambda\rangle = (X_\theta^+)^{k+1}|\alpha\rangle$. Hence the condition $|\lambda\rangle \notin (X_\theta^+)^{k+1}$ restricts the representations to those with $(\lambda, \theta) \leq k$. Moreover, since $(\lambda, \theta) \geq (\lambda, \alpha), \alpha > 0, (X_\alpha^-)^{k+1}|\lambda\rangle = 0, \forall \alpha$. This is the same condition of integrability as for Kac-Moody algebras. Proceeding as with $SU(2)$ one can derive the "depth" rule [6] giving

the fusion rules for WZW theories with group G and level k . If $\mathcal{V}_{reg}$ is the set of regular representations we can define the restricted tensor product

$$(V_1 \otimes V_2)'$$

as for $SU(2)$ and the co-associativity of the co-multiplication implies the associativity of the restricted tensor product. The representation $(V_1 \otimes V_2)'$ is reducible and fully reducible. If we analyze the decomposition of $((V_1 \otimes V_2 \otimes \dots V_n)')$ for regular representations, we can also read off the regular representations and the Bratelli diagram of the various centralizer algebras characterizing the braid representations generated by the conformal blocks associated to the decomposition of the tensor product (2.15). In the case of $SU(N)$ and with all but one of the external legs in the fundamental representation we can check that one recovers the results of Wenzl [40]. For other groups or other representations the analysis of Wenzl has not been carried out yet and the restricted tensor product constructed here provides a rather simple way of characterizing the regular representations of all the centralizer algebras that appear in the decomposition of representations of quantum groups when q is a root of unity. For $SU(N)$ the representations are labels in terms of Young tableaux $[\lambda_1, \lambda_2, \dots, \lambda_N]$, $\lambda_1 \geq \lambda_2 \geq \dots$ where λ_1 is the number of blocks in the first row, λ_2 is the number of boxes in the second row etc. Then $\langle \lambda, \theta \rangle = \lambda_1 - \lambda_N \leq p - g = k$ in Wenzl's notation, only diagrams of type (N, k) appear (at most N rows and the previous condition on the number of columns). The Bratelli diagram of the fundamental representation is easy to work out. Consider a Young tableau Y as in fig. 15 with $\lambda_1 < k$. If we take the tensor product with one more copy of the fundamental representation we proceed as with standard $SU(N)$ until $\lambda_1 - \lambda_N = k$. In this case the restricted tensor product eliminates the representation which could be generated by adding one more box to the first row (or any row with the same length as the first one). The argument is easier to explain with one example. In fig. 16 we present the Bratelli diagram for the fundamental representation of $SU(3, q)$ with $q^5 = 1$ i.e. level $k = 2$.

To conclude this section we want to show that the quantum dimension (2.9) for level k WZW models with group G are given by (3.39). Instead of taking the characters $\chi_\lambda(\tau)$ we can consider the characters with arbitrary angles in the Cartan subalgebra

$$\chi_\mu(z\tau) = t r_\mu q^{L_0 - c/24} e^{iz \cdot H}$$

z is an r -dimensional vector (r is the rank of G). The modular properties of $\chi_\mu(z\tau)$ are the same as those of $\chi_\mu(\tau)$ [42]. If we take $z = \varrho/(k+g)$ we can use the Weyl-Kac character formula to obtain

$$\frac{\chi_\mu(\frac{\varrho}{k+g}|\tau)}{\chi_0(\frac{\varrho}{k+g}|\tau)} = D_q(\mu) \frac{\sum_{\alpha \in M} e^{i\pi\tau((k+g)\alpha + \mu + \varrho)^2/(k+g)}}{\sum_{\alpha \in M} e^{i\pi\tau((k+g)\alpha + \mu + \varrho)^2/(k+g)}} \quad (3.42)$$

where M is the lattice generated by the long roots. We can take the $\tau \mapsto 0$ limit by first applying the Poisson resummation formula to the numerator and denominator of the right-hand side in (3.42) to obtain :

$$\lim_{\tau \mapsto 0} \frac{\chi_\mu(\frac{\varrho}{k+g}|\tau)}{\chi_0(\frac{\varrho}{k+g}|\tau)} = D_q(\mu) \quad q^{k+g} = 1$$

hence

$$D_q(\mu) = \frac{S_{\mu 0}}{S_{00}} \quad (3.43)$$

from (2.8) we conclude that the q -dimensions are the eigenvalues of the fusion algebra for WZW models. This agrees also with the derivation of the fusion rules based on the restricted tensor products.

It is intriguing that after the representation theory of the quantum group is restricted as indicated above, the quantum group looks closer to a finite group than to a classical or Kac-Moody group.

Before finishing this section we would like to make a few comments in relation with Witten's three dimensional construction of conformal field theories. The main

point in that construction is the identification of the space of blocks of the WZW theory with the physical Hilbert space of the Chern-Simons theory. Considering an S^2 section of the three-dimensional manifold and working in the temporal gauge, the physical Hilbert space for n external charges in representations R_i of the gauge group is given by

$$\mathcal{H}_{ph}^{R_1 R_2 \dots R_n} = \otimes_{i=1}^n \mathcal{H}^{R_i} / (\text{Gauss' law}) = \text{Inv} \left\{ \otimes_{i=1}^n \mathcal{H}^{R_i} \right\} \quad (3.44)$$

where the $\mathcal{H}^{R_i}$ are the representation spaces for the irreducible representations R_i and Gauss' law is defined by the constraint :

$$\frac{k}{8\pi} \epsilon^{ij} F_{ij}^a(x) = \sum_{k=1}^n \delta^{(2)}(x - P_i) T_{(k)}^a \quad (3.45)$$

for the fundamental representation, the spaces $\mathcal{H}_{ph}^{R_1 R_2 \dots R_n}$ define irreducible representations of the Hecke algebra $H_n(q)$, $q = e^{2\pi i/k+2}$. This follows from the identification of the space of conformal blocks with $\mathcal{H}_{ph}$. We can use the physical Hilbert space defined above to write down a "branching rule" decomposition :

$$\otimes_1^n \mathcal{H}^R = \bigoplus_j \left\{ \mathcal{H}^{R_1 \dots R_n} \right\} \otimes \mathcal{H}^{R_j}$$

which is equivalent to equation (2.15). From the last equation and the identification of $\mathcal{H}_{ph}$ with the space of blocks we learn that the commutant of the Hecke algebra (i.e. the quantum group) acts on $\otimes^n \mathcal{H}^R$. From the previous arguments we see that this group parametrizes the inequivalent solution to Gauss' law constraint. This connection was already foreseen in [14].

An alternative way to get the same result is to consider invariants in the quantum group. If we have a tensor product of n representations V^i , the space of invariants is given by

$$\text{Inv} \left(\otimes_{i=1}^n V^i \right) = \left\{ v \in \otimes_{i=1}^n V^i, \varrho_1 \otimes \varrho_2 \dots \varrho_n (\Delta^{n-1}(a))v = a \quad \forall a \in A \right\}$$

this set is non-empty only when invariant tensors can be constructed out of the representations appearing in the tensor product, and they define interaction ver-

tices according to the fusion rules. For instance, if we work with $SU(2, q)$, $q^3 = 1$ then $\text{Inv}(V^{1/2} \otimes V^{1/2} \otimes V^1)$ is empty. Coming back to (3.44), we know that the space of invariant tensors determined by Gauss' law are the ones allowed by the fusion rules of the $SU(2)$ level k WZW model. In this chapter we have seen that this is precisely what happens for $SU(2, q)$, $q = e^{2\pi i/k+2}$. Thus we can formally identify the generators of (3.45) for n charges in the representations R_i with $\otimes_{i=1}^n \varrho^{R_i}(\Delta^{n-1}(a))$ with a a generator of quantum $SU(2)$.

4. QUANTUM GROUPS AND DUALITY. EXAMPLES

In sec. 2 we argued why the quantum group underlies the duality properties of a RCFT. The last section was devoted to a systematic study of Quasi-Triangular Yang-Baxter algebras and their representations specially when q is a root of unity. In this section we would like to outline how one recovers the genus zero duality information from the quantum group. Since we have already done most of the work for WZW theories, we will be brief in our treatment of current algebras. We present also another example which illustrates the use of quantum groups : the rational gaussian models [43]. In the next section we will construct the matrix S (2.6) in the context of quantum groups and give a proof of Verlinde's conjecture [12] based only on the properties of the co-multiplication (3.2), (3.3).

When $q^p = 1$ the set of regular representations $\mathcal{V}_{reg}$ contains only those representations with highest weights satisfying $\langle \lambda, \theta \rangle \leq p - g \equiv k$. In quantum groups we can define the intertwiner

$$K_k^{ij} : (V^i \otimes V^j)' \mapsto V^k$$

and the universal $\mathcal{R}$ matrix yields the operation (2.2). As a general consequence of the properties of quantum groups [30] we have

$$K_k^{ij} R^{ji} = \epsilon q^{(c_k - c_i - c_j)/2} K_k^{ij} \quad (4.1)$$

where c_i is the quadratic Casimir of the classical algebra for the same represen-

tation. Hence, modulo integers we can identify the conformal dimensions of the field generating V^i as

$$h_i = \frac{c_i}{k+g} \quad (4.2)$$

Notice that in (4.1) the intertwiner is defined with respect to the restricted tensor product. In this case we have a positive definite scalar product and we can define the adjoint

$$K_{ij}^k : V^k \mapsto (V^i \otimes V^j)'$$

so that the orthogonality relations now become

$$\sum_k K_{ij}^k K_k^{lm} = \delta_i^l \delta_j^m (1_i \otimes 1_j)'$$

with the sum extended only over the representations appearing in the decomposition of the restricted tensor product $(V^i \otimes V^j)'$.

The hexagon identity is a simple consequence of (3.26) which is represented graphically in fig. 17. The explicit formulae for the braiding and fusing matrices are given in terms of the q -analogues of the $6j$ -symbols for $SU(2)$ or their generalizations for other groups. They determine explicitly the isomorphism

$$((V_1 \otimes V_2)' \otimes V_3)' \cong (V_1 \otimes (V_2 \otimes V_3))' \quad (4.3)$$

the pentagon identity is a consequence of the associativity of the restricted tensor product which in turn follows from the co-associativity of the co-multiplication.

As argued in sec. 2, the matrices of this change of basis give the F -move in the space of blocks and the N -move is obtained from it using (2.4). For theories where the representation ring is generated by one representation (in Conformal Field Theory this means that there is an operator out of which all other primaries are obtained by decomposing its OPEs) the braiding and fusion matrices of more

interest are those of the generating field with arbitrary spectactors. Using associativity and the fusion rules one can in principle determine the fusion and braiding matrices of other representations. In the case of $SU(N)$ the F, B matrices are obtained up to conjugation by a diagonal matrix using rather simple arguments and they are shown to be the same as those obtained in WZW theories by solving the Knizhnik-Zamolodchikov equation [5]. For level $k > 1$ there are two irreducible representations appearing in the tensor product of two fundamental representations $V \otimes V$, $V \otimes V = V_a + V_s$ (the subscripts s, a stand for q -symmetric and q -antisymmetric respectively). Using the F move we diagonalize the braiding matrix of two contiguous representations and find that they have only two distinct eigenvalues. Choosing the common phase of all the braiding matrices conveniently we can make these two eigenvalues equal to 1, $q = e^{2\pi i/k+g}$ and therefore the braid generators ϱ_i satisfy a Hecke algebra relation (2.19) together with the condition that the $N+1$ row antisymmetrizer vanishes. Both the quantum group and the WZW theory provide unitary representations of this algebra labelled by the same tableaux. This is a consequence of the relation between the Bratelli diagrams of the quantum group and its centralizer, and in the WZW theory it follows from the fusion rules. Since the basis we have chosen for the blocks have the same quantum numbers as those for the WZW theory, the matrices of these two representations must be the same up to diagonal conjugation [25, 44]. In WZW one can determine the explicit form of the representation matrix by solving the Knizhnik-Zamolodchikov equation. In the quantum group we know that the action of the braid group on tensor products is given by

$$\varrho(\sigma_i) = \varrho_i = 1 \otimes \dots \otimes R_{i,i+1} \otimes \dots \otimes 1$$

and after the tensor product is decomposed, we have an action on the blocks (see fig. 18) :

$$\varrho_i F_j(\dots p_{i-1}, p_i, p_{i+1} \dots) = \sum_{p'_i} B_{p_i, p'_i} \begin{bmatrix} \phi & \phi \\ p_{i-1} & p_{i+1} \end{bmatrix} F_j(\dots p_{i-1}, p'_i, p_{i+1} \dots)$$

in the $SU(2)$ case $\phi = \phi_{1/2}$. Using (2.4) and the definition of the $6j$ -symbols we

obtain [25] :

$$F_{jj'} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} = \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{Bmatrix}$$

$$B_{jj'} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} = (-1)^{j+j'-j_1-j_4} q^{(c_{j_1}+c_{j_4}-c_{j'}-c_{j'})/2} \begin{Bmatrix} j_2 & j_1 & j \\ j_3 & j_4 & j' \end{Bmatrix} \quad (4.4)$$

where

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \Delta(j_1, j_2, j_{12}) \Delta(j_3, j, j_{12}) \Delta(j_1, j, j_{23}) \Delta(j_3, j_2, j_{23}) \times$$

$$\sum_{z \geq 0} (-1)^z [z+1]! \frac{1}{z-j_1-j_2-j_{12}} \frac{1}{[z-j_3-j-j_{12}][z-j_1-j-j_{23}][z-j_3-j_2-j_{23}]} \quad (4.5)$$

$$\frac{1}{[j_1+j_2+j_3+j-j-z]![j_1+j_3+j_{12}+j_{23}-z]![j_2+j+j_{12}+j_{23}-z]!}$$

$$\Delta(a, b, c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}}$$

implementing the B, F moves for $SU(2)$ WZW theories.

Another example is provided by the rational theories at $c=1$ consisting of a single scalar field ϕ taking values on a circle of radius $R = \sqrt{r/s}$ (where s is odd). The chiral algebra is generated by the two fields $\partial\phi(z)$ and $\exp(\pm 2i\sqrt{rs}\phi(z))$. For details see [43]. The primary fields are vertex operators $\phi_p = e^{ip\phi(z)}$ where the momentum is quantized as

$$p = \frac{k}{2\sqrt{rs}} \quad k \in Z/4rsZ = Z_{4rs}$$

ϕ_p has conformal dimension $h_p = p^2 = k^2/4rs$ and the fusion rules are

$$\phi_k \times \phi_{k'} = \phi_{k+k'} \quad k, k', k+k' \in Z_{4rs} \quad (4.6)$$

The quantum group description of this RCFT is generated by the operators $p, t_{\pm}$ in correspondence with $\partial\phi(z)$ and $\exp(\pm 2i\sqrt{rs}\phi(z))$. They satisfy the commutation

relations

$$[p, t_{\pm}] = \pm M t_{\pm}$$

$$[t_+, t_-] = 0$$

We want to first show how one endows this algebra with the structure of a quantum group and then show that the duality properties of the rational gaussian model are recovered from its representation theory. The Hopf algebra axioms are satisfied by the following definitions

$$\Delta p = p \otimes 1 + 1 \otimes p \quad \epsilon(p) = 0 \quad \gamma(p) = -p$$

$$\Delta(t_{\pm}) = t_{\pm} \otimes q^{\pm} + q^{\mp} \otimes t_{\pm} \quad \epsilon(t_{\pm}) = 0 \quad \gamma(t_{\pm}) = -q^{\pm M} t_{\pm} \quad (4.7)$$

so far the two parameters q, M are independent. The next step is to find under what conditions this Hopf algebra $Q(q, M)$ becomes a Quasi-Triangular Yang-Baxter algebra (3.2), (3.3). Take as the universal $\mathcal{R}$ -matrix the ansatz

$$\mathcal{R} = q^{ap \otimes p} \quad (4.8)$$

then (3.2) and (3.3) imply the conditions

$$q^{4p} = q^{2Map} = 1 \quad (4.9)$$

It is interesting to see that quasi-triangularity imposes important restrictions on the Hopf algebra Q . Taking q as a phase we obtain the quantization conditions on the spectrum of p . We normalize q^a to $e^{2\pi i}$, which is equivalent to choosing conventions so that $h_p = p^2$. Then (4.9) lead to

$$p = \frac{a}{4} k \quad M = \frac{2\nu_0}{a} \quad k, \nu_0 \in Z \quad (4.10)$$

since t_+ changes the eigenvalue of p by M units to obtain a representation of the

algebra we must have that $p + M$ also takes the form $\frac{a}{4}k'$, hence

$$a = 2\sqrt{\frac{2\nu_0}{N}} \quad N \in \mathbb{Z}$$

all this together yields the quantization conditions

$$p = \frac{k}{2\sqrt{rs}} \quad N = 2\nu_0 rs \quad (4.11)$$

to obtain a rational representation theory note first that the irreducible representations of Q are labelled by the eigenstates of $p, |p\rangle$ with $p = (k + mN)/2\sqrt{rs}$, $m \in \mathbb{Z}$ obtained from $|k/2\sqrt{rs}\rangle$ by repeated action of $t_{\pm}$. Since the braiding properties of two elements in the same representation of the chiral algebra are the same we find that for a state in $\{|p_1\rangle\}$ and a state in $\{|p_2\rangle\}$ the braiding is

$$e^{2\pi i p_1 p_2} = e^{2\pi i k_1 k_2 / 4rs} e^{i\pi(m_1 k_2 + m_2 k_1)\nu_0}$$

implying that to have a consistent representation theory, ν_0 must be an even integer. In fact there is no loss of generality if we choose $\nu_0 = 2$. If $\nu_0 \geq 2$ we can always construct other operators $t'_{\pm}$ which are “roots” of $t_{\pm}$ and for them $\nu_0 = 2$. This implies that $k \in \mathbb{Z}_{4rs}$. When we enlarge the representations of p as indicated in order to label them only with integers mod. $4rs$ we have to make sure that $t_{\pm}$ is still compatible with the modding and the similarly for the co-multiplication. The fusion rules are very easy in this case. Assuming the Clebsch-Gordan rule

$$|p_1\rangle|p_2\rangle = |p_1 + p_2\rangle$$

we have

$$t_{+}|p\rangle = N_p|p + M\rangle$$

from the co-multiplication we can derive the compatibility condition

$$N_{p_1} q^{p_2} + N_{p_2} q^{-p_1} = N_{p_1 + p_2}$$

solved by

$$N_p = \sin \frac{k\pi}{2} \quad p = \frac{a}{4}k \quad N_{p+N} = N_p$$

and in the reduced theory $t_{\pm}$ does not act as the identity

$$t_{\pm}|k\rangle = \sin \frac{k\pi}{2}|k\rangle$$

t cannot act as the identity because $\Delta(1) = 1 \otimes 1$ and it would not be compatible with the reduction. Therefore the rational gaussian models can be recovered (or rather their duality properties) by looking at the quantum group Q and imposing a rationality condition on its representation ring (to have only a finite number on representations).

5. MODULAR PROPERTIES

We now construct the matrix S (2.6) using only quantum group information. Using the relation between quantum groups, RCFT and link invariants detailed in sec. 7 we will show that our representation of S is as expected from the topological Chern-Simons theory given by the expectation value of two linked Wilson lines on S^3 .

From the work of Moore and Seiberg [9, 10, 11] we know we can represent the Verlinde operators [12] in terms of the duality matrices. In particular, the action of the operators associated to the a -cycle on the characters is diagonal and it is given by the eigenvalues of the fusion matrices $\lambda_k^{(j)}$. The precise relation with the duality matrices is

$$\lambda_k^{(j)} = \frac{1}{F_k} \sum_p B_{op} \begin{bmatrix} k & \bar{j} \\ k & j \end{bmatrix} (+) B_{p0} \begin{bmatrix} \bar{j} & k \\ k & j \end{bmatrix} (+)$$

$$F_k = \epsilon_{k\bar{k}}^0 F_{00} \begin{bmatrix} k & \bar{k} \\ k & k \end{bmatrix} \quad (5.1)$$

where the line over an index refers to the conjugate field. (The $+$ argument in the braiding matrices indicates that we are braiding in the positive sense, the $i + 1$ -st

strand goes over the i -th strand). That this corresponds to the Verlinde operator associated to the a -cycle is easily seen from fig. 20. The Verlinde operator acts on characters according to

$$\phi_k(a)\chi_j = \lambda_k^{(j)} \chi_j \quad (5.2)$$

and it can be described as inserting the identity operator factorized as $k, \bar{k}$ on the block in fig. 20, then we braid the field k around j . These operations result in (5.1). Since the B and F matrices are related (2.4), we can write (5.1) instead as

$$\lambda_k^{(j)} = \frac{1}{F_k} \sum_p \frac{e_j^0 e_k^k e_j^p}{j j \bar{p} j} e^{2\pi i(h_p - h_j - h_k)} F_{op} \left[\begin{matrix} k & j \\ k & j \end{matrix} \right] F_{p0} \left[\begin{matrix} \bar{j} & j \\ k & k \end{matrix} \right] \quad (5.3)$$

it is easy to show as a consequence of the pentagon identity that

$$\frac{S_{jk}}{S_{00}} = \sum_p e^{2\pi i(h_p - h_j - h_k)} \frac{S_{0p} N_{jk}^p}{S_{00}} \quad (5.4)$$

In the case of $SU(2)$ we have

$$S_{ji} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(2j+1)(2l+1)}{k+2}$$

and (5.4) implies the following identity between q -numbers

$$[(2j+1)(2k+1)] = \sum_p q^{p-c_j-c_k} N_{jk}^p [2p+1] \quad c_j = j(j+1)$$

this relation is certainly different from the addition formula of q -dimensions,

$$[(2j+1)][(2k+1)] = \sum_p N_{jk}^p [2p+1] \quad (5.5)$$

which is a particular case of the Verlinde conjecture (2.8).

In the quantum group there is a useful relation between the trace of the comultiplication $\Delta(a)$ in the tensor product representation $j \otimes k$ and the trace of a in the representations appearing in the decomposition of the tensor product :

$$tr_{j \otimes k} \Delta(a) = \sum_p N_{jk}^p tr_p(a) \quad (5.6)$$

where

$$tr_{j \otimes k} \Delta(a) = tr q^j \otimes q^k (\Delta(a))$$

and q^j is the map from the quantum group A to the representation V^j . Using the diagram in fig. 11 one easily derives (5.6) using the orthogonality properties of the quantum Clebsch-Gordan coefficients. As a trivial example from (5.6) and the comultiplication

$$\Delta(q^{H/2}) = q^{H/2} \otimes q^{H/2}$$

we obtain (5.5). In any Quasi-Triangular Yang-Baxter algebra there is always a distinguished element u [30] defined in terms of the $\mathcal{R}$ -matrix and the antipode by

$$u = m(\gamma \circ id)(\sigma(\mathcal{R}))$$

$$u^{-1} = m(id \circ \gamma^2)(\sigma(\mathcal{R})) \quad (5.7)$$

where m is the multiplication in A and we have used the notation in sec. 3. The element u has the properties

$$uau^{-1} = \gamma^2(a) \quad \forall a \in A \quad (5.8)$$

when $A = SU(2, q)$, u and u^{-1} are given in the representation j by

$$q^j(u) = q^{-c_j} q^{H/2} \quad q^j(u^{-1}) = q^{c_j} q^{-H/2} \quad (5.9)$$

so that

$$tr_j u = q^{-c_j} [2j+1] \quad tr_j u^{-1} = q^{c_j} [2j+1] \quad (5.10)$$

since the construction of $\lambda_k^{(j)}$ follows from the polynomial equations directly, the

relation (5.4) has a very nice interpretation in the quantum group (for the time being we consider $SU(2)$):

$$\frac{S_{jk}}{S_{00}} = q^{-\epsilon_j - \epsilon_k} t r_j \otimes_k \Delta(u^{-1}) = q^{\epsilon_j + \epsilon_k} t r_j \otimes_k \Delta(u) \quad (5.11)$$

the element u can be identified with the remnant of the Virasoro algebra. The comultiplication rule for the element u is [31]

$$\begin{aligned} \Delta(u) &= \mathcal{R}^{-1} \sigma(\mathcal{R}^{-1})(u \otimes u) \\ \Delta(u^{-1}) &= \sigma(\mathcal{R}) \mathcal{R}(u^{-1} \otimes u^{-1}) \end{aligned} \quad (5.12)$$

finally we find the complete form of the matrix S in the quantum group^{*}

$$\begin{aligned} \frac{S_{jk}}{S_{00}} &= q^{-\epsilon_j - \epsilon_k} t r_j \otimes_k \sigma(\mathcal{R}) \mathcal{R}(u^{-1} \otimes u^{-1}) \\ &= q^{\epsilon_j + \epsilon_k} t r_j \otimes_k \mathcal{R}^{-1} \sigma(\mathcal{R}^{-1})(u \otimes u) \end{aligned} \quad (5.13)$$

We can make contact with the Chern-Simons theory [14]. We know from (3.25-3.27) that the quantum group comes endowed with a trace with the Markov property (3.27) which can be used to define link invariants [V. G. Turaev : Invent. Math. 92(1988)527.] (see section 7 for more details). As in $SU(2)$, u is proportional in general to q^ℓ , hence (3.25)

$$(u \otimes u)R = R(u \otimes u)$$

to understand what link we relate with S by (5.13), we recall that every link or knot can be obtained by closing a braid (see sec. 7). Since we have a representation

* In the limit $q \mapsto 1$, $\sigma(\mathcal{R})\mathcal{R} \mapsto 1 \otimes 1$, this is the condition for a quasi-triangular Yang-Baxter algebra to be also triangular [29]. Hence the classical limit of a quantum group is a triangular Yang-Baxter algebra.

of the braid group whose action is defined in terms of the matrix R , we see that the operation $\sigma(R)R$ used in the definition of S is the braid in fig. 21a. Taking traces is like closing the braid (identifying the end points of the braid in fig. 21a to get fig. 21b) and we prove directly that the modular matrix S is a link invariant associated to fig. 21. In the Chern-Simons theory this conclusion is obtained by doing surgery on the trivial link (two unknotted unlinked circles on $S^2 \times S^1$) according to the operation S . In other words, in the Chern-Simons theory S_{jk} is the expectation value in S^3 of two linked Wilson lines

$$S_{jk} = Z(S^3, L(R_j, R_k))$$

(see [14] for more details). In the quantum group this interpretation is a consequence of the co-multiplication rule for the distinguished element u .

We can now proceed along the argument presented in [14] to prove the Verlinde conjecture. Since here we do not have a functional integral interpretation (only duality properties) the proof will rely on the defining properties of a quantum group. Consider the link in S^3 in fig. 22. It can be represented by the braid in fig. 23 which as a representation in terms of traces as

$$\begin{aligned} W_{ijk} &= t r_i \otimes_j \otimes_k \left(\sigma(R_{12}) \sigma(R_{13}) R_{13} R_{12} q^{-H/2} \otimes q^{-H/2} \otimes q^{-H/2} \right) \\ &\equiv t r_i^M \otimes_j \otimes_k (\sigma(R_{12}) \sigma(R_{13}) R_{13} R_{12}) \end{aligned} \quad (5.14)$$

where in the last step we use the definition of the Markov trace (3.26). For simplicity we are presenting the argument in detail for $SU(2)$. The generalization to more complicated cases is straightforward. To prove Verlinde's conjecture we compute (5.14) in two ways. First we use the defining properties of a quantum group :

$$\begin{aligned} (1 \otimes \Delta) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{12} \\ (1 \otimes \Delta) \sigma(\mathcal{R}) &= \sigma(\mathcal{R})_{12} \sigma(\mathcal{R})_{13} \end{aligned}$$

to write

$$W_{ijk} = tr_{i \otimes j \otimes k} (1 \otimes \Delta) \sigma(R) R(q^{-H/2} \otimes q^{-H/2})$$

using (5.6) we and (5.13) we obtain

$$W_{ijk} = \sum_m N_{jk}^m tr_{i \otimes m} \sigma(R) R(q^{-H/2} \otimes q^{-H/2}) = \sum_m N_{jk}^m \frac{S_{mi}}{S_{00}} \quad (5.15)$$

on the other hand the trace (5.14) has the Markov property ((3.26) and next section) and for the braiding we perform in fig. 23 we have

$$W_{ijk} = \frac{(tr_{i \otimes j}^M \sigma(R) R) (tr_{i \otimes k}^M \sigma(R) R)}{S_{0i}/S_{00}} = \frac{S_{ij} S_{ik}}{S_{00} S_{0i}} \quad (5.16)$$

from (5.15) and (5.16) we obtain the proof of Verlinde's conjecture :

$$\frac{S_{ij} S_{ik}}{S_{0i}} = \sum_m N_{jk}^m S_{mi} \quad (5.17)$$

Since S is obviously symmetric and it diagonalizes the fusion rules, it follows from the general arguments of sec. 2 that $S^2 = C$ as expected. This now leads to all the formula in (2.8). Although for simplicity we have carried out the argument as though the representations were self-conjugate, the formulae derived hold in the general case. Using (2.8), in particular $S^2 = C, S^* = S^{-1} = SC = CS$ we can rewrite (5.4) as

$$S_i^j = \sum_k N_{ki}^j e^{2\pi i(h_i + h_j - h_k)} S_0^k \quad (5.18)$$

With respect to the other defining relation of the modular group $(ST)^3 = C$, it can be rewritten as

$$TSTST = S \quad (5.19)$$

although this equation seems to involve many conditions on the RCFT, it is equivalent to a single equation. Using (2.8)(5.4) and (5.18) one can show that (5.19) is

equivalent to

$$e^{2\pi i h_i} S_0^j = e^{i\pi c/4} \sum_k e^{2\pi i h_k} S_0^k S_0^j \quad (5.20)$$

this suggest the definition of a quantum character as

$$\chi_i(q) = e^{2\pi i h_i - i\pi c/8} \frac{S_0^i}{S_0}$$

and (5.20) expresses the relation between the character with argument q^{-1} and the characters with argument q . More generally this is a property of the quantum characters under charge conjugation, and it should be related to the completeness of the set of rational representations of the quantum group. For quantum groups coming from deformations of classical groups (5.20) holds as one can see from explicit computations, it is also quite easy to verify this relation for gaussian models. In general, however, an argument similar to the one for S in terms of u should ensure this transformation property of quantum characters, although at present this general argument is missing. It seems likely that the Markov property of the trace and its relation to knot and link invariants should provide again the steps to complete the argument in general.

This concludes the proof that quantum groups provide solutions to the polynomial equations. It should be stressed again that the quantum group comes endowed with a Markov trace, and that this is an important property with respect to the relation with knot invariants and also with respect to the partial reconstruction of the space of conformal blocks. For a general Quasi-Triangular Yang-Baxter algebra, the only missing step at this time is the general argument guaranteeing the relation (5.20) although we hope to come back to this point in the future.

6. KNOT INVARIANTS AND CONFORMAL FIELD THEORY

There are several ways to relate RCFT to link invariants. In [13] one uses the R -matrices coming from statistical mechanics of field theory leading to Turaev's construction [45] of the Jones polynomials and its generalizations. In the last part of this section we will make the connection along the lines of Jones' original approach [33, 34]. For this we need very little information : the fusion algebra and its diagonalization by the matrix S , and that in any unitary RCFT its blocks provide unitary representations of the braid group (if the theory is not unitary but all the dimensions of the primary field are given by real numbers , the representations of the braid group we will study are still unitary).

An important property of knots and links is that they can be obtained by closing braids. Some examples are shown in fig. 24. The problem is that there are many ways to represent the same knot in terms of braids. If one has representations of the braid groups $\dots B_n \subset B_{n+1} \dots$ constructing a knot invariant is equivalent to finding traces on these groups satisfying the properties

$$\begin{aligned} \alpha, g \in B_n \quad \text{tr} \alpha g \alpha^{-1} &= \text{tr} g \\ \alpha \in B_n, \sigma_n^{\pm 1} \in B_{n+1} \quad \text{tr} \alpha \sigma_n^{\pm 1} &= \text{tr} \alpha \end{aligned} \quad (6.1)$$

these are the Markov properties of the trace that correspond to the invariance of the trace under two Markov moves on braids, and a non-trivial theorem in knot theory [37] implies that these two properties alone are enough to guarantee that the traces (6.1) are invariant under the Reidemeister moves. The first Markov move in (6.1) is a consequence of the cyclicity of the trace. The second move is illustrated in fig. 25 and it is clear that if we close both braids, they generate links that can be deformed to each other. We now show that to any RCFT we can associate a trace with the properties (6.1). At each level in the Brattelli diagram we can

construct a matrix algebra :

$$A_\phi^{(n)} = \bigoplus_i M_{d_{\phi,i}^{(n)}}(C) \quad (6.2)$$

To be more precise, each of the spaces of square matrices $M_{d_{\phi,i}^{(n)}}(C)$ should be thought of as the algebra generated by the irreducible representation of the braid group provided by the blocks $F_{\phi,i}^{(n)}$ (n ϕ fields and one spectator). At different levels we have $A_\phi^{(n)}, A_\phi^{(n+1)} \dots$ and the dimensions of the matrix subalgebras are related by the fusion rules (2.11). In a more general setting, we have two finite matrix algebras A, B and $B \subset A$. Then

$$\begin{aligned} A &= \bigoplus_i M_{n_i}(C) \\ B &= \bigoplus_\alpha M_{m_\alpha}(C) \end{aligned} \quad (6.3)$$

and the inclusion is completely specified by the inclusion matrix $\Lambda_{i\alpha}$ whose entries are non-negative integers relating the dimensions of the irreducible representations of A, B :

$$n_i = \sum_\alpha \Lambda_{i\alpha} m_\alpha \quad (6.4)$$

the inclusion matrix in our case is $(N_\phi)_i^j$. Next we want to construct traces in these algebras with the Markov properties. Since each factor in (6.2) has the ordinary trace for square matrices, the traces in $A_\phi^{(n)}$ are given by the relative normalizations

$$\text{tr}^{(n)}(x) = \sum_i t_i^n \text{tr} x_i \quad (6.5)$$

where $x \in A_\phi^{(n)}$ and $x_i \in M_{n_i}(C)$. The trace is positive iff $t_i^n \geq 0$. Since $B \subset A$ (and in our case we have a chain of inclusions) we want the traces on A and B to

agree on the elements of B . This implies in general

$$\sum_i t_i^A \Lambda_{i\alpha} = t_\alpha^B \quad (6.6)$$

in the conformal field theory this leads to

$$\sum_j (N_\phi)_i^j t_{\phi,j}^{n+1} = t_{\phi,i}^n \quad (6.7)$$

note that (6.7) is dual to (2.11). Using (2.8) we can find a solution to (6.7):

$$t_{\phi,j}^n = \frac{S_{j0}}{S_{00}} \left(\frac{S_{00}}{S_\phi} \right)^n \quad (6.8)$$

since $S_{j0}/S_{00} \geq 0$, we obtain a positive trace on the whole family $A_\phi = \oplus A_\phi^n$. To complete the argument we can proceed in two ways. If we think about those theories whose duality properties can be represented in terms of quantum groups, we can use the Markov property of the trace to obtain the desired result. If one does not want to rely on quantum groups, we need to invoke a result in [33] which guarantees under very general circumstances that for chains of embeddings of algebras $\dots \subset A_n \subset A_{n+1} \dots$ the trace defined here has the Markov property which translates into the statement :

$$tr^{(n+1)} \alpha \sigma_n^{\pm 1} = z_\pm tr^{(n)} \alpha \quad \alpha \in B_n \quad (6.9)$$

to compute the value of $z_\pm$ it suffices to consider the algebra $A^{(2)}$. Since

$$tr^{(n)}(1) = 1$$

as a consequence of (2.8); we find

$$z_\pm = tr^{(2)} \sigma_1^{\pm 1} = \frac{e^{\mp 2\pi i h_\phi}}{(S_{\phi 0}/S_{00})} \quad (6.10)$$

Now it is easy to construct a trace satisfying (6.1). If $g \in B_n$, define

$$T(g) = \left(\frac{S_{\phi 0}}{S_{00}} \right)^n e^{2\pi i h_\phi w(g)} tr^{(n)}(B) \quad (6.11)$$

where B is a representation of $g \in A_\phi^{(n)}$ made out of a direct sum of matrices B^j ,

the irreducible representations of the braid group on the blocks with spectator j . The coefficient $w(g)$ is the wraith of the braid g . Since g can be constructed in terms of the generators σ_i , $w(g)$ is uniquely determined by

$$w(\sigma_i) = 1 = -w(\sigma_i^{-1})$$

$$w(ab) = w(a) + w(b) \quad (6.12)$$

it is clear that (6.11) satisfies the first condition in (6.1) and we only need to check the second one :

$$\begin{aligned} T(g\sigma_n) &= \left(\frac{S_{\phi 0}}{S_{00}} \right)^{n+1} e^{2\pi i h_\phi w(g\sigma_n)} tr^{(n+1)}(B\sigma_n) \\ &= \left(\frac{S_{\phi 0}}{S_{00}} \right)^{n+1} e^{2\pi i h_\phi w(g)} e^{2\pi i h_\phi} \left(\frac{S_{00}}{S_\phi} \right) tr^{(n)}(B) = T(g) \end{aligned}$$

and similarly

$$T(g\sigma_n^{-1}) = T(g)$$

As an example, consider n unlinked, unknotted circles (fig. 26). T in this case gives

$$T = \left(\frac{S_{\phi 0}}{S_{00}} \right)^n$$

In RCFT, the dimensions of all the primary fields are rational numbers [46, 47] and we can always write the dimensions as $h_j = p_j/N \bmod 1$. This implies that :

$$T(\sigma_1^{2N}) = e^{2\pi i h_\phi(2N)} \sum_j \frac{S_{j0}}{S_{00}} N_\phi^j e^{i\pi 2N(2h_\phi - h_j)} = \sum_j \frac{S_{j0}}{S_{00}} N_\phi^j = \left(\frac{S_{\phi 0}}{S_{00}} \right)^2$$

by the properties of S (2.8). Graphically this is interpreted by saying that the link shown in fig. 27a is seen in this theory as two unlinked, unknotted circles. In

terms of braids we obtain fig. 27b. Since all the braid group generators are related to σ_1 by conjugation, this implies that

$$T(\sigma_i^{2N}) = \left(\frac{S_{\emptyset 0}}{S_{00}}\right)^2$$

giving some constraints on the level of complexity of knots and links that can be distinguished by a given RCFT. As a final application we consider the $SU(2)_k$ WZW theory. In this case the fusion algebra is

$$x_{j_1} x_{j_2} = \sum_{l=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} x_l \quad (6.13)$$

and

$$\begin{aligned} \frac{S_{j0}}{S_{00}} &= [2j+1]_q \\ [\lambda]_q &= \frac{q^{\lambda/2} - q^{-\lambda/2}}{q^{1/2} - q^{-1/2}} \quad q = e^{2\pi i/k+2} \end{aligned} \quad (6.14)$$

if the field $\phi = \phi_{1/2}$ then the invariant (6.11) becomes

$$T(g) = q^{3w(g)/4} \sum_j [2j+1]_q \text{tr}_j(g), \quad q \in B_n$$

where j is constrained to be in the range determined by the fusion rules (6.13) in the decomposition of the tensor product $V_{1/2}^{\otimes n}$. Similar computations can be carried out for other groups and it give a derivation in conformal field theory of the Ocneanu decompositions in terms of Young tableaux appearing in [34].

To conclude this section, it is important to realize that regardless of how one wishes to describe a conformal theory, few ingredients are necessary to compute topological invariants : the fusion rules and the matrix S , and the fact that the conformal blocks provide irreducible representations of the braid group. Different descriptions of a RCFT amount to giving different derivations and interpretations of the Markov property of the trace over the representations of B_n (6.9).

7. CONCLUSIONS

We have shown in this paper how the duality properties of a RCFT can be obtained from the representation theory of quantum groups. As specific examples we have presented only the deformations of classical groups related to WZW theories and the rational gaussian models but models based on products of groups can be worked out as well. We have also presented arguments of why one should expect every solution to the polynomial equations to define the structure of a quantum group. This would go a long way towards the reconstruction of a RCFT. Even if one knows that the duality information is contained in a general quantum group, one is still faced with the problem of reconstructing the full chiral algebra. If the only quantum groups appearing in RCFT are groups or products of groups there are very educated guesses that can be made about the chiral algebra. In general however one obtains out of the quantum group the dimensions of the fields modulo integers, the braiding and fusion matrices and the modular transformation properties. With this information we know for example the order of the differential equations satisfied by the blocks and the monodromy and half-monodromies of their solutions. It is at the moment a difficult problem to know whether the set of differential operators which represent the null vectors of the underlying chiral algebra can be reconstructed from this information alone. This is the Riemann-Hilbert problem in a different guise. Nevertheless independently of this important question one would like to know whether it is indeed true that every solution to the polynomial equations provides a quasi-triangular Yang-Baxter algebra. This would transform the classification problem into the problem of classifying quantum groups with some rationality condition on their representation ring. For sequences of RCFT admitting a classical limit, it is known from some general theorems of Deligne (quoted in [11]) that the duality properties in this limit are described in terms of the representation theory of classical groups. The quantization of this classical limit à la Drinfeld-Jimbo should allow one to reconstruct the duality properties of all the elements in the sequence. The problem is to find out whether there are "exceptional" conformal theories which do not admit a classical limit. For ex-

ample it is conceivable that if one looks at deformations of Chevalley algebras over finite fields, some of them may provide exceptional solutions to the polynomial equations.

We have also outlined some of the connections between the quantum group constructions and the Chern-Simons approach [14]. For example the solution of the Gauss law constraint leading to the fusion rule is equivalent to a condition on the co-multiplication. We also find rather directly that the modular transformation matrix S is directly related to a link invariant on S^3 , a result which follows in Chern-Simons from surgery arguments. Both approaches allow us to compute in principle the purely topological properties of a RCFT. In some sense one can think of the computations in the Chern-Simons theory as the Gauss-Bonnet formula for the Euler number of a manifold, while the quantum group provides the combinatorial representation of the same quantity. They yield complementary pictures of the same questions. When one tries to push either approach to attempt a full classification of RCFT one is faced in the three dimensional approach with the classification of all three dimensional topological field theories while in the quantum group point of view one has to classify rational quasi-triangular Yang-Baxter algebras. Another important open problem is the classification of different modular invariant projections for the same theory. For WZW theories, a complete answer is known at the moment only for the $SU(2)$ WZW theories and for the various minimal models [48]. Perhaps more important, is how to extend all these structures to Irrational Conformal Field Theories, in particular to String Theory. The rationality restriction implies that one has to consider only genus zero and genus one information in order to satisfy modular invariance to all orders [11]. It is not difficult to think of examples with an infinite number of primary fields where the first non-trivial problem with modular invariance appears at genus two. These examples are not necessarily derivable from a lagrangian, but one has to check whether the polynomial equations fail to describe their duality properties. It would be quite interesting to know the complete set of equations analogous to the polynomial equations and describing the duality properties of irrational

conformal theories.

We believe that the quantum group approach to the study of conformally invariant theories begun in [24,25] provides a natural language in the study of the structural properties of Conformal Field Theory.

REFERENCES

- [1] D. Friedan and S. Shenker: Nucl. Phys. **B281**(1987)509.
- [2] A. Belavin, A. M. Polyakov and A. Zamolodchikov : Nucl. Phys. **B241** (1984)33.
- [3] D. Friedan, Z. Qiu and S. Shenker : Phys. Rev. Lett. **51** (1984)1575.
- [4] E. Witten : Comm. Math. Phys. **92**(1984)83.
- [5] V. I. Knizhnik and A. Zamolodchikov, Nucl. Phys. **B247** (1984)83.
- [6] D. Gepner and E. Witten : Nucl. Phys. **278** (1986)493.
- [7] A. Tsuchiya and Y. Kanie : Lett. Math. Phys. **13** (1987)303.
- [8] T. Kohno in *Braids* Contemp. Math. vol 78.
- [9] G. Moore and N. Seiberg : Phys. Lett. **212B** (1988)451.
- [10] G. Moore and N. Seiberg *Naturality in Conformal Field Theories* . IASSNS-HEP-88/31.
- [11] G. Moore and N. Seiberg *Classical and Quantum Conformal Field Theories* , IASSNS-HEP-88/35. To appear in Comm. Math. Phys. .
- [12] E. Verlinde : Nucl. Phys. **B300** (1988)360.
- [13] J. Fröhlich : *Statistics of Fields, the Yang-Baxter Equation and the Theory of Knots and Links* . Cargèse Lectures 1987. To appear in Plenum Publ. A. Jaffe et al eds.
- [14] E. Witten : *Quantum Field Theory and the Jones Polynomials* . IASSNS-HEP-88/33.

- [15] M. Bos and V. Nair : *Chern-Simons Approach to $c=1$ Conformal Blocks* University of Columbia preprint. 1989.
- [16] M. Murayama *Explicit Quantization of the Chern-Simons action*. University of Tokyo UT-542/89.
- [17] G. Moore and N. Seiberg *Taming the Conformal Zoo*. IASSNS-HEP-89/6.
- [18] J. Labastida and A. Ramallo : CERN-TH-5334/89.
- [19] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg : *Canonical Quantization of the Chern-Simons-Witten Theory*. IASSNS-HEP-89/20.
- [20] S. Axelrod, S. Della Pietra and E. Witten, paper in preparation.
- [21] K. Rehren and B. Schroer : *Einstein Causality and Artin Braids*. Berlin Preprint FU-88-0439.
- [22] B. Schroer : Nucl. Phys. B295 (1988)4 ; K. H. Rehren : Comm. Math. Phys. 116 (1988)675.
- [23] K. Fredenhagen, R. H. Rehren and B. Schroer : *Superselection Sectors with Braid Group Statistics and Exchange Algebras. I General Theory*. Berlin Freie University Preprint. September 1988.
- [24] L. Alvarez-Gaumé, C. Gomez and G. Sierra : Nucl. Phys. B. To appear.
- [25] L. Alvarez-Gaumé, C. Gomez and G. Sierra : Phys. Lett. 220(1989)142.
- [26] R. Brustein, S. Yankielowicz and J. B. Zuber : TAUP-1647/88.
- [27] P. Goddard, A. Kent and D. I. Olive : Comm. Math. Phys. 103(1986)105.
- [28] V. G. Drinfel'd : Sov. Math. Dokl. 32 (1985)254 ; M. Jimbo : Lett. Math. Phys. 10 (1985)63 ; 11 (1986)247.
- [29] V. G. Drinfel'd *Quantum Groups* Proceedings of the International Congress of Mathematicians. Berkeley Cal. 1986.
- [30] N. Yu. Reshetikhin : *Quantized Universal Enveloping Algebras and Invariants of Links* I, II. LOMI-3-4-87, E-17-87.
- [31] A. N. Kirillov and N. Yu. Reshetikhin : *Representations of the Algebra $U_q(sl(2))$ q -orthogonal Polynomials and Invariants of Links*. LOMI-E-9-88.
- [32] G. Lusztig : *Modular Representations of Quantum Groups* MIT-Preprint 1988.
- [33] V. F. R. Jones : Invent. Math. 72 (1983)1.
- [34] V. F. R. Jones : Ann. Math. 126 (1987)335.
- [35] G. Moore and N. Yu. Reshetikhin : *A Comment in Quantum Group Symmetry in Conformal Field Theory*. IASSNS-HEP-89/18.
- [36] G. Felder, J. Fröhlich and G. Keller : *Braid Matrices and Structure Constants for Minimal Conformal Models*. IAS Princeton Preprint February 1989.
- [37] J. Birman *Links, Braids and Mapping Class Groups*. Princeton U, Press, 1974.
- [38] R. Dijkgraaf and E. Verlinde : *Fusion Algebras and Conformal Field Theories*. Contribution to the Annecy conference on Conformal Field Theory. 1988.
- [39] T. Kawai : *On the Structure of Fusion Algebras*. University of Tokyo Komaba Preprint 21340, 1988.
- [40] H. Wenzl : Invent. Math. 92 (1988)349.
- [41] V. Pasquier and H. Saleur : *Symmetries of the XXZ Chain and Quantum Groups*. Saclay. SPHT/88-187.
- [42] V. Kac : *Infinite Dimensional Lie Algebras*. Cambridge University Press 1985.
- [43] For details and references see R. Dijkgraaf, E. Verlinde and H. Verlinde : Comm. Math. Phys. 115(1988)649.
- [44] J. Lacki and Ph. Zaugg : Paper in preparation

- [45] V. G. Turaev :Invent. Math.**92**(1988)527.
- [46] C. Vafa : Phys. Lett.**B206**(1988)421.
- [47] G. Anderson and G. Moore : Comm. Math. Phys.**117**(198)441.
- [48] A. Cappelli, C. Itzykson and J. B. Zuber : Comm. Math. Phys.**B280**(1987)445, and references therein.

FIGURE CAPTIONS

- Fig. 1. A basis for the space of conformal blocks.
- Fig. 2. The fusion rules. This diagram represents the three point function or the intertwiner for group representation theory.
- Fig. 3. The B and F moves. Braiding and fusing operations on blocks.
- Fig. 4. Braid group generator σ_i .
- Fig. 5. Relation between B and F .
- Fig. 6. Blocks with N identical fields and one spectator field.
- Fig. 7. Rules for constructing the Bratelli diagram.
- Fig. 8. The Bratelli diagram for the theory with two fields $1, \phi$ whose single non-trivial fusion rule is $\phi \times \phi = 1 + \phi$.
- Fig. 9. Bratelli diagram for $SU(2)$.

- Fig. 10. Dynkin-like diagram representation of the Bratelli diagram for the $SU(2)$ level k WZW theory.
- Fig. 11. Constructing the co-multiplication in terms of the intertwiner. This diagram is commutative.
- Fig. 12. Compatibility between braiding and fusing.
- Fig. 13. Bratelli diagram for $SU(2)$ at level k and particular examples for $k = 2, 3$.
- Fig. 14. Young tableaux for $SU(2)$.
- Fig. 15. Young tableaux labelling the representations of $SU(N)$.
- Fig. 16. (a). Integrable representations for $SU(3)$ at level 2. (b). Bratelli diagram for this theory. One can see that the structure is periodic, after a number of steps the diagram repeats
- Fig. 17. Hexagon identity from Yang-Baxter.
- Fig. 18. Action of the generators of the braid group on the blocks with all but one field equal to ϕ .
- Fig. 19. Change of basis defining the $6j$ symbols.
- Fig. 20. Inserting the identity factorized as $k \times \bar{k}$ in the j character.
- Fig. 21. Braid constructing the modular matrix S . As the braid is closed we obtained two linked circles each with an associated representation of the quantum group.
- Fig. 22. Link used in the proof of Verlinde's conjecture.
- Fig. 23. Braid representation of the link in the previous figure.
- Fig. 24. Representation of the Markov move of type II. On the left we have a braid $\alpha \in B_n$ and on the left we consider instead the braid $\alpha\sigma \in B_{n+1}$. Under closure they both produce the same knot or link.
- Fig. 25. A set of n unknotted, unlinked circles.

Fig. 26. If the RCFT is such that all its fields have dimensions multiple of $1/N$, then the conformal theory cannot distinguish the link on the left with the trivial link on the right.

Fig. 27. The same statement as in the previous figure, but in terms of braids.

Fig. 1

Fig. 2

$$\begin{aligned}
 & \text{Diagram 1} = \sum_{p'} \text{Diagram 2} \quad B_{pp'} \left[\begin{matrix} j & k \\ i & l \end{matrix} \right] \\
 & \text{Diagram 1} = \sum_{p'} \text{Diagram 3} \quad F_{pp'} \left[\begin{matrix} j & k \\ i & l \end{matrix} \right]
 \end{aligned}$$

The diagrams in the equations are as follows:

- Diagram 1 (left):** A horizontal line with four vertical segments labeled i , j , k , and l from left to right. Below the line is the label p .
- Diagram 2 (right):** A horizontal line with four vertical segments labeled i , k , j , and l from left to right. Below the line is the label p' .
- Diagram 3 (right):** A horizontal line with four vertical segments labeled i , j , k , and l from left to right. The segments j and k are connected by a Y-shaped structure. Below the line is the label p' .

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

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Fig. 9

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Fig. 11

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Fig. 16

Fig. 17

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Fig. 21

Fig. 22

Fig. 23

Fig. 24

Fig. 25

Fig. 26

Fig. 27