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Chern-Simons Model and New Relations between the HOMFLY Coefficients

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Abstract

Using the connection between the vacuum expectation value of the Wilson line operator $W_T(C)$ in the Chern-Simons field theory and the HOMFLY polynomial associated with the knot C , we derive new relations between the HOMFLY coefficients. An explicit analytic expression for the second coefficient of the Alexander-Conway polynomial is presented.

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The role of geometry and topology in quantum field theory is widely recognized. Nowadays it emerges, however, that the implications are not only in one direction. Indeed certain quantum field theories were discovered, which provide valuable tools for understanding the geometry of three- and four-dimensional manifolds. Following some suggestions of Atiyah [1], Witten [2,3] was able to give recently a quantum field theoretical interpretation of Donaldson's invariants of four-manifolds [4], of the Floer homology groups of three-manifolds [5] and of Gromov's invariants of symplectic manifolds [6].

The present paper is concerned with a further development in the above direction, namely the quantum field theory description of the new link invariant found by Jones [7] and its generalization, the HOMFLY polynomial [8,9]. The field theory, which is relevant for these invariants, is the non-Abelian Chern-Simons (C-S) model [10]

$$S_{CS} = \frac{k}{4\pi} \int_{M^3} d^3x \epsilon^{\mu\nu\rho} \text{Tr} (A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho) \quad , \quad (1)$$

where M^3 is a smooth three-manifold and $A_\mu = A_\mu^a T^a$, T^a being the generators in the defining representation of a compact simple Lie group G . We consider in what follows $G = SU(N)$. The coupling constant k is taken to be an integer, which ensures invariance of the model also under large (not connected to the identity) gauge transformations.

The relevance of the C-S theory for describing link invariants has been pointed out by A. S. Schwarz [11] and Atiyah [1]. The basic observation is that the action (1) is independent of the metric on M^3 . Therefore the expectation value of any gauge invariant and metric-independent observable should be a topological invariant of M^3 . Consider for example the Wilson line operator

$$W_R(C) = \text{Tr}_R \text{P exp } i \oint_C A_\mu dx^\mu \quad , \quad (2)$$

where the path ordering is performed along a given oriented closed path without self-intersections (knot) C and the trace is computed in the representation R of G . One expects that the vacuum expectation value $\langle W_R(C) \rangle$ is a knot invariant. More generally,

$$\langle W(L) \rangle \equiv \langle W_{R_1}(C_1) \cdots W_{R_m}(C_m) \rangle \quad , \quad (3)$$

where the link L is the union of non-intersecting knots C_i , should be a link invariant. Investigating this subject, Witten [12] has found that there exists a deep relationship between the C-S model and the two-dimensional conformal invariant Wess-Zumino-Witten (WZW) model. More precisely he has shown that the physical states of the C-S theory describe the conformal blocks of the WZW model. By using this correspondence Witten [12,13] has argued that $\langle W(L) \rangle$, computed in the fundamental representation of $SU(N)$, is related to the HOMFLY polynomial associated with L .

The scope of the present paper is to explore further the existence of a quantum field theory interpretation of the HOMFLY polynomial. We show that by computing $\langle W(L) \rangle$ directly in the three-dimensional field theory one gains a new insight into knot invariants. The main achievement is a systematic method for the derivation of:

- (i) explicit analytic expressions for knot invariants;
- (ii) new relationships between the HOMFLY coefficients.

We start by recalling some results [14,15,16] concerning the calculability of the vacuum expectation value $\langle W(L) \rangle$ in the three-dimensional quantum field theory framework. This problem is twofold. First, one has to construct well-defined correlation functions of the C-S gauge potential. Second, one has to give a definite meaning of the composite operator $W(L)$.

The first aspect of the problem has been discussed in [15]. It has been shown there that in the framework of the BRS quantization the correlation functions $\{ \langle A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) \rangle \}$ have a meaningful perturbative expansion in $\frac{1}{k}$ and that the general covariance is preserved on the physical subspace. The two-loop calculation [15] confirms also that the theory is finite, as it is expected from the constraint on k to be an integer. The Landau gauge turns out to be very convenient for explicit computations. In this gauge, at least in second order of perturbation theory, one has [15]

$$\Pi_{\mu\nu}^{ab}(x-y) = 0 = \Gamma_{\mu\nu\rho}^{abc}(x-y, y-z, z-x) \quad , \quad (4)$$

where $\Pi_{\mu\nu}^{ab}$ and $\Gamma_{\mu\nu\rho}^{abc}$ are the radiative corrections to the functions $\langle A_\mu^a(x) A_\nu^b(y) \rangle$ and $\langle A_\mu^a(x) A_\nu^b(y) A_\rho^c(z) \rangle$ respectively. The Landau gauge in the C-S model is remarkable also because of the presence of a peculiar symmetry [17], relating ghost- and gauge-field

correlation functions. Eqs.(4) are useful for the explicit computation of $\langle W_R(C) \rangle$.

The next step in the construction of $\langle W_R(C) \rangle$ involves integration of correlation functions of the type $\langle A_{\mu_1}(x_1) \cdots A_{\mu_j}(x_j) \rangle$ over the same path C . It is worth stressing that, differently from what one could naively expect, no divergences occur in performing these integrations in the C-S theory [16]. In order to preserve as much as possible the topological character of the classical Wilson line $W_R(C)$, however, one must give an operative meaning to the framing procedure for knots [18] in the context of quantum field theory. Remember that the framing of a knot $C = \{x(t) : 0 \leq t \leq 1\}$ in R^3 consists in introducing a framing contour C_f defined by

$$x^\mu(t) \mapsto y^\mu(t) = x^\mu(t) + \varepsilon n^\mu(t) \quad , \quad (\varepsilon > 0 \quad , \quad |n(t)| = 1) \quad , \quad (5)$$

where n^μ is a vector field orthogonal to C . There are two particular framings playing a distinguished role - the so-called [13] "standard" and "vertical" framings. The standard framing is fixed by the requirement that the Gauss integral (linking number)

$$\chi(C, C_f) = \frac{1}{4\pi} \oint_C dx^\mu \oint_{C_f} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (6)$$

vanishes, whereas in the vertical framing the field n^μ can be identified [19] simply with the principle normal in the Frenet basis associated with C .

The explicit computations at order $\frac{1}{k^2}$ show [16] that in the C-S model the necessity of framing is related to the well-known self-linking problem. This problem has been analyzed some time ago by mathematicians [19,20], who found that the framing procedure assigns a topological meaning to the self-linking for knots in R^3 by fixing the ambiguities due to a certain direction dependent but finite limit.

Summarising, $\langle W_R(C) \rangle$ has a well-defined perturbative expansion in $\frac{1}{k}$ and the classical framing procedure (5) consistently extends [16] to all orders of perturbation theory. As a result, $\langle W_R(C) \rangle$ is entirely expressed in terms of purely topological quantities. Of course, this is precisely the expected result in any generally covariant theory like the C-S model; nevertheless, it is instructive to see how all this is realized in practice. The first few terms, computed in [16], read

$$\langle W_R(C) \rangle = \dim R \left[1 - i \left(\frac{2\pi}{k} \right) c_2(R) \chi(C, C_f) \right]$$

$$-\frac{1}{2} \left(\frac{2\pi}{k} \right)^2 c_2^2(R) \chi^2(C, C_f) + \left(\frac{2\pi}{k} \right)^2 c_v c_2(R) \varrho(C) \Big] + O \left(\frac{1}{k^3} \right) . \quad (7)$$

Here C_f is an arbitrary framing contour,

$$c_2(R)1 = R^a R^a \quad , \quad \delta^{ab} c_v = f^{acd} f^{bcd} \quad , \quad (8)$$

$$\varrho(C) = \varrho_1(C) + \varrho_2(C) \quad , \quad (9)$$

with

$$\varrho_1(C) = -\frac{1}{32\pi^3} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} I^{\sigma\lambda\tau}(x, y, z) \quad , \quad (10)$$

$$I^{\sigma\lambda\tau}(x, y, z) = \int d^3w \frac{(w-x)^\sigma}{|w-x|^3} \frac{(w-y)^\lambda}{|w-y|^3} \frac{(w-z)^\tau}{|w-z|^3} \quad , \quad (11)$$

and

$$\varrho_2(C) = \frac{1}{8\pi^2} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \epsilon_{\sigma\nu\alpha} \epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha}{|w-y|^3} \frac{(z-x)^\beta}{|z-x|^3} . \quad (12)$$

The explicit form of $\varrho(C)$ and its framing-independence have been analyzed in [16], where it has been argued that $\varrho(C)$ is actually a knot invariant. Indeed, we shall see in a moment that $\varrho(C)$ represents the second coefficient of the Alexander-Conway polynomial and consequently is related with the Arf- and Casson-invariants.

Let us introduce now the HOMFLY polynomial. With each oriented link one associates a finite Laurent polynomial $P_L(t, z)$ of two variables with integer coefficients which is invariant under ambient isotopy, i.e., for any link L' which is a continuous deformation of L , one has $P_{L'} = P_L$. The polynomial P_L can be constructed by means of the skein relation

$$tP_{L_+}(t, z) - t^{-1}P_{L_-}(t, z) = zP_{L_0}(t, z) \quad , \quad (13)$$

where the diagrams associated with L_+ , L_- and L_0 differ only at the site of one crossing as shown in Fig.1. Using eq.(13) and the Reidemeister moves [18], one can construct recursively P_L for any L , starting from the polynomial of the unknot U_0 . The latter is usually normalized as follows

$$P_{U_0}(t, z) = 1 \quad . \quad (14)$$

The link invariant $P_L(1, z)$ is the Alexander-Conway polynomial (or Conway potential function) [21], whereas $P_L(t, z = \sqrt{t} - \frac{1}{\sqrt{t}})$ is the Jones polynomial [7]. In the particular case of knots one has [9]

$$P_C(t, z) = \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} b_{ij}(C) z^{2i} t^{2j} . \quad (15)$$

The fundamental relation between the C-S field theory and knot invariants we are going to use below, reads

$$\frac{\langle W_T(C) \rangle_{sf}}{\langle W_T(U_0) \rangle} = P_C(t, z) , \quad (16)$$

where the Wilson lines in the L.H.S. are computed in the fundamental representation $R^a = T^a$ of $SU(N)$, "sf" stands for standard framing and the parameters t and z are expressed in terms of k and N by

$$t = \exp \left(-i \frac{2\pi N}{k} \right) , \quad z = -2i \sin \frac{\pi}{k} . \quad (17)$$

Eqs.(16,17) deserve some comments. In the framework of the three-dimensional C-S model these equations have been proven up to second order in perturbation theory [14,16]. In the approach based on the relationship with the two-dimensional WZW model, eq.(16) has been derived by Witten [12,13]. The values (17) of the parameters t and z are in agreement with Witten's results for

$$k = \ell + N , \quad (18)$$

where ℓ is the level of the WZW model.

The connection between the C-S model and the conformal field theory [12,13,22] has not been fully understood. In particular, the relation between k and ℓ is a priori not obvious and certainly deserves further investigation. Concerning this point it is useful to recall that:

- (a) It is a fact [15] that in the perturbative treatment of the C-S model, there is no quantum mechanics induced shift $k \mapsto k + N$ in the effective action up to two loops at least;

- (b) The explicit computation [16] of $\langle W(L) \rangle$ at second order in perturbation theory confirms eqs.(17);
- (c) The C-S action has a peculiar perturbatively non-anomalous discrete symmetry [16] which implies

$$\langle W_R(C) \rangle|_k = \langle W_R(\tilde{C}) \rangle|_{-k} \quad ,$$

where $\tilde{C}$ is the mirror image of C . From this equation and the well-known relation [9] $P_C(t, z) = P_{\tilde{C}}(t^{-1}, -z)$ it follows that

$$t^{-1}(k) = t(-k) \quad , \quad z(k) = -z(-k) \quad ,$$

which again excludes the shift $k \mapsto k + N$ in $\langle W_R(C) \rangle$;

- (d) The quantum holonomies of A_μ computed in an operator approach [23] to the C-S theory, coincide with the monodromies associated with the correlation functions of primary fields in the WZW model, provided that eq.(18) holds.

Obviously, any redefinition of k ab initio does not invalidate the above arguments and has nothing to do with a real quantum mechanics induced shift of k .

Let us now concentrate on eq.(16). Consider the $\frac{1}{k}$ -expansion (6) of $\langle W_T(C) \rangle$. The contributions at fixed order in $x = \frac{2\pi}{k}$ can be classified by the corresponding group weights. For example at order x^2 one has $c_2^2(T)$ and $c_v c_2(T)$, at order x^3 one finds $c_2^3(T)$, $c_v c_2^2(T)$ and $c_v^2 c_2(T)$ and so on. Not all of these group weights contribute to $\langle W_T(C) \rangle_{sf}$, because some of them multiply $\chi(C, C_f)$, which vanishes in the standard framing. A straightforward computation shows that

$$\begin{aligned} \langle W_T(C) \rangle_{sf} = N \{ & 1 + x^2 c_v c_2(T) \varrho(C) + i x^3 c_v^2 c_2(T) \tau(C) \\ & + x^4 [c_v^3 c_2(T) \sigma_1(C) + c_v^2 c_2^2(T) \sigma_2(C)] \} + O(x^5) \quad , \end{aligned} \quad (19)$$

where

$$c_v = N \quad , \quad c_2(T) = \frac{(N^2 - 1)}{2N} \quad . \quad (20)$$

As in the case of $\varrho(C)$, the C-S perturbation theory provides explicit analytic expressions for the knot invariants $\tau(C)$, $\sigma_1(C)$ and $\sigma_2(C)$. We do not report the rather complicated formulae because we do not need them in the following.

Using eq.(19), the L.H.S. of eq.(16) takes the form

$$\begin{aligned} \frac{\langle W_T(C) \rangle_{sf}}{\langle W_T(U_0) \rangle} &= 1 + x^2 c_v c_2(T) \widehat{\varrho}(C) + ix^3 c_v^2 c_2(T) \widehat{\tau}(C) \\ &+ x^4 [c_v^3 c_2(T) \widehat{\sigma}_1(C) + c_v^2 c_2^2(T) \widehat{\sigma}_2(C)] + O(x^5) \quad , \end{aligned} \quad (21)$$

where

$$\begin{aligned} \widehat{\varrho}(C) &= \varrho(C) - \varrho(U_0) \quad , \quad \widehat{\tau}(C) = \tau(C) - \tau(U_0) \quad , \\ \widehat{\sigma}_1(C) &= \sigma_1(C) - \sigma_1(U_0) \quad , \quad \widehat{\sigma}_2(C) = \sigma_2(C) - \sigma_2(U_0) + \varrho^2(U_0) - \varrho(C)\varrho(U_0) \quad . \end{aligned} \quad (22)$$

On the other hand, inserting eq.(17) in eq.(15), for the R.H.S. of eq.(16) one finds

$$\begin{aligned} P_C(t, z) &= \sum_{j \in \mathbb{Z}} b_{0j} - ixN \sum_{j \in \mathbb{Z}} j b_{0j} - x^2 \sum_{j \in \mathbb{Z}} \left(\frac{1}{2} N^2 j^2 b_{0j} + b_{1j} \right) \\ &+ ix^3 \sum_{j \in \mathbb{Z}} \left(\frac{N^3}{6} j^3 b_{0j} + N j b_{1j} \right) \\ &+ x^4 \sum_{j \in \mathbb{Z}} \left(\frac{N^4}{24} j^4 b_{0j} + \frac{N^2}{2} j^2 b_{1j} + \frac{1}{12} b_{1j} + b_{2j} \right) + \dots \quad . \end{aligned} \quad (23)$$

The relations we are looking for are obtained by comparing the expressions (21) and (23), which must coincide for any $N \geq 2$. In this way, for the first three terms in (23) one gets

$$\sum_{j \in \mathbb{Z}} b_{0j} = 1 \quad , \quad (24)$$

$$\sum_{j \in \mathbb{Z}} j b_{0j} = 0 \quad , \quad (25)$$

$$\frac{1}{2} N^2 \sum_{j \in \mathbb{Z}} j^2 b_{0j} + \sum_{j \in \mathbb{Z}} b_{1j} = \frac{1 - N^2}{2} \widehat{\varrho}(C) \quad . \quad (26)$$

Eq.(26) in turn implies

$$\sum_{j \in \mathbb{Z}} b_{1j} = \frac{1}{2} \widehat{\varrho}(C) \quad , \quad (27)$$

$$\sum_{j \in \mathbb{Z}} j^2 b_{0j} = -\widehat{\varrho}(C) \quad . \quad (28)$$

The L.H.S. of eq.(24) is precisely the first coefficient of the Alexander-Conway polynomial

$$P_C(1, z) = a_0(C) + a_2(C)z^2 + \dots \quad (29)$$

The equality (24) states that $a_0(C) = 1$ for any knot, which is indeed known to be the case. The validity of eq.(25) has been proven recently by Likorish and Millet [9]. The L.H.S. of eq.(27) coincides with the second coefficient of the Alexander-Conway polynomial, i.e.

$$a_2(C) = \frac{1}{2} \widehat{\varrho}(C) \quad (30)$$

Using that for the unknot U_0 one has [16]

$$\varrho(U_0) = -\frac{1}{12} \quad ,$$

eq.(30) can be rewritten in the form

$$a_2(C) = \frac{1}{2} \left[\varrho(C) + \frac{1}{12} \right] \quad (31)$$

Eq.(30) is quite remarkable because, when combined with eqs.(9-12) it provides an analytic expression for $a_2(C)$. This is in our knowledge the first analytic formula for the computation of $a_2(C)$. As is well known, this coefficient plays a distinguished role in knot theory. It is related to the Arf-invariant by [24]

$$a_2(C) \bmod 2 = \text{Arf}(C) \quad , \quad (32)$$

and to the Casson invariant by [25]

$$a_2(C) = \lambda(C_{n+1}) - \lambda(C_n) \quad , \quad (33)$$

where $\lambda(C_n)$ is the Casson invariant of the $\frac{1}{n}$ -th Dehn surgery constructed by C .

Eliminating $\widehat{\varrho}(C)$ from eqs.(27,28) one gets

$$\sum_{j \in \mathbb{Z}} (j^2 b_{0j} + 2b_{1j}) = 0 \quad , \quad (34)$$

which has been proven in [9].

The existing derivation of eqs.(25) and (34) [9] is based on the use of the skein relation and is not quite straightforward. Moreover it is not obvious if there are new relations of this type and how to find them. The perturbative expansion of $\langle W_T(C) \rangle$ in the C-S field theory sheds new light on this problem. Indeed, let us compare the x^3 - and x^4 -contributions to eqs.(21) and (23). One obtains

$$\sum_{j \in \mathbb{Z}} j^3 b_{0j} = 3\hat{\tau}(C) \quad , \quad (35)$$

$$\sum_{j \in \mathbb{Z}} j b_{1j} = -\frac{1}{2}\hat{\tau}(C) \quad , \quad (36)$$

$$\sum_{j \in \mathbb{Z}} j^4 b_{0j} = 12\hat{\sigma}_1(C) + 6\hat{\sigma}_2(C) \quad , \quad (37)$$

$$\sum_{j \in \mathbb{Z}} j^2 b_{1j} = -\hat{\sigma}_1(C) - \hat{\sigma}_2(C) \quad , \quad (38)$$

$$\sum_{j \in \mathbb{Z}} (b_{1j} + 12b_{2j}) = 3\hat{\sigma}_2(C) \quad . \quad (39)$$

Combining eqs.(27) and (39) one obtains the third coefficient of the Alexander-Conway polynomial

$$a_4(C) = \frac{1}{4}\hat{\sigma}_2(C) - \frac{1}{2}\hat{\varrho}(C) \quad . \quad (40)$$

Eliminating $\hat{\tau}(C)$ from eqs.(35,36) and $\hat{\sigma}_1(C)$ and $\hat{\sigma}_2(C)$ from eqs.(37-39) we get

$$\sum_{j \in \mathbb{Z}} (j^3 b_{0j} + 6j b_{1j}) = 0 \quad , \quad (41)$$

$$\sum_{j \in \mathbb{Z}} [j^4 b_{0j} + 2(6j^2 + 1) b_{1j} + 24 b_{2j}] = 0 \quad . \quad (42)$$

These are new relations between the HOMFLY coefficients. Using the table for b_{ij} given in [9], we have verified eqs.(41,42) for all the knots with up to and including 9 crossings. The validity of eqs.(41,42) confirms the values (17) of the parameters t and z .

Summarising, in the framework of the C-S quantum field theory, the relations (24,25), (34) and (41,42) are direct consequences of eq.(16) and the first four orders in the $\frac{1}{k}$ -expansion of $\langle W_T(C) \rangle$. Clearly the investigation of the higher orders gives a

systematic method for the study of the relations between the HOMFLY coefficients. The example of the knot invariant $\varrho(C)$ clearly indicates that the perturbative expansion of $\langle W_T(C) \rangle$ can be also successfully applied in the search of an analytic description of the knot invariants - a major open problem in knot theory [26].

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FIGURE CAPTION

Fig.1 Configurations connected by the skein relation.

L_+

L_-

L_0

Fig. 1